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A Calculation of Injective Dimension over Valuation Domains.

PAUL C. EKLOF - SAHARON SHELAH (*)

This paper takes up a problem which was posed in a paper by S. Bazzoni [B], about the injective dimension of certain direct sums of divisible modules over a valuation domain. We refer the reader to that paper for the motivation for the problem. We shall make use of the same notation as in [B], which we now proceed to review.

Let R be a valuation domain of global dimension $n + 1$, where $n \geq 2$. Let $\{L_\alpha: \alpha \in A\}$ be a family of archimedean ideals of R , where A is a set of cardinality $\geq \aleph_{n-2}$. For each α let I_α be the injective envelope of R/L_α . Let $I = \prod_{\alpha \in A} I_\alpha$, and for each $1 \leq k \leq n$, let D_{n-k} be the submodule of I consisting of those elements having support of cardinality $< \aleph_{n-k}$, i.e. for all $y \in I$, y belongs to D_{n-k} if and only if the cardinality of

$$\{\alpha \in A: y(\alpha) \neq 0\}$$

is strictly less than \aleph_{n-k} .

Bazzoni proves in [B] that the injective dimension of D_{n-k} is at most k . She also shows that the injective dimension of D_{n-1} is exactly 1 and that it is consistent with ZFC that the injective dimension of

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D_{n-2} is exactly 2. It is the main purpose of this paper to prove this latter result in ZFC. In fact we prove:

THEOREM. *The injective dimension of D_{n-k} is ≥ 2 if $2 \leq k \leq n$.*

Before proving the theorem we prove some lemmas. The first of these is a combinatorial fact. (Compare [Sh; § 6].)

LEMMA 1. *Let κ be a regular cardinal. There exists a family $\{w_\nu^\alpha: \alpha < \kappa^+, \nu < \kappa\}$ of subsets of κ^+ satisfying for all $\alpha < \kappa^+$:*

$$(1) \alpha = \bigcup_{\nu < \kappa} w_\nu^\alpha;$$

$$(2) \text{ for all } \nu < \mu < \kappa, w_\nu^\alpha \subseteq w_\mu^\alpha;$$

$$(3) \text{ for all } \nu < \kappa \text{ and all } \beta < \alpha, \beta \in w_\nu^\alpha \Rightarrow w_\nu^\beta = w_\nu^\alpha \cap \beta;$$

$$(4) \text{ for all } \nu < \kappa, \text{ the cardinality of } w_\nu^\alpha \text{ is } < \kappa.$$

PROOF. We shall define the w_ν^α for all ν by induction on α . Let $w_\nu^0 = \emptyset$ for all ν . Now suppose that w_ν^β has been defined for all $\beta < \alpha$. If α is a successor ordinal, say $\alpha = \gamma + 1$, then let $w_\nu^\alpha = w_\nu^\gamma \cup \{\gamma\}$ for all ν . It is easy to see that (1)-(4) hold for α if they hold for γ .

If α is a limit ordinal, let $\lambda =$ the cofinality of α , and let $\eta: \lambda \rightarrow \alpha$ be a strictly increasing function such that the supremum of its range is α . Define a function $f: \lambda \rightarrow \kappa$ by the rule:

$$f(\mu) = \text{the least } \nu < \kappa \text{ such that}$$

$$\text{for all } \tau < \sigma \leq \mu, \eta(\tau) \in w_\nu^{\eta(\sigma)}.$$

It is easy to see that f is well-defined because of (1) and (2) and because κ is regular and $\geq \lambda > |\mu|$. Now for each $\nu < \kappa$ let

$$w_\nu^\alpha = \bigcup \{w_\nu^{\eta(\mu)}: \mu < \nu \text{ and } f(\mu) < \nu\}.$$

Conditions (2) and (4) are easily verified. To see that (1) holds, suppose $\gamma < \alpha$ and choose μ such that $\eta(\mu) > \gamma$. Then $\gamma \in w_\nu^{\eta(\mu)}$ for some ν , so if $\nu > \max \{\tau, \mu, f(\mu)\}$, then $\gamma \in w_\nu^\alpha$. To prove (3), let us fix α and ν and let $Y = \{\mu < \nu: f(\mu) < \nu\}$. Thus

$$w_\nu^\alpha = \bigcup_{\mu \in Y} w_\nu^{\eta(\mu)}.$$

Notice first that if $\tau < \mu$ and $\mu \in Y$, then $\eta(\tau) \in w_v^{\eta(\mu)}$; so by induction $w_v^{\eta(\tau)} = w_v^{\eta(\mu)} \cap \eta(\tau)$. Now if $\beta \in w_v^\alpha$ then $\beta \in w_v^{\eta(\mu)}$ for some $\mu \in Y$; in this case it is easy to see, using the previous observation, that $\beta \in w_v^{\eta(\tau)}$ for any $\tau \in Y$ such that $\beta < \eta(\tau)$. Clearly

$$w_v^\beta = w_v^{\eta(\mu)} \cap \beta \subseteq w_v^\alpha \cap B,$$

so we are left with proving the opposite inclusion. Suppose $\gamma \in w_v^\alpha \cap \beta$; then $\gamma \in w_v^{\eta(\tau)}$ for some $\tau \in Y$. As above, $\gamma \in w_v^{\eta(\sigma)}$ for any $\sigma \in Y$ such that $\gamma < \eta(\sigma)$, so without loss of generality $\beta < \eta(\tau)$. But then $\gamma \in w_v^{\eta(\tau)} \cap \beta = w_v^\beta$, since $\beta \in w_v^{\eta(\tau)}$.

The second lemma will be used to show that for certain submodules $K' \supseteq K$ of I_α , the quotient K'/K has sufficiently large cardinality. (K and K' will have the form $\{u \in I_\alpha: ru = 0\}$ for an appropriate r .) Here $\mathcal{F}(\gamma)$ is the set of all subsets of γ .

LEMMA 2. *Let $\{r_\nu: \nu < \gamma\}$ be a sequence of elements of R , and let N be a pure-injective module such that for all $\mu < \gamma$ there exists an element $a_\mu \in N$ such that $r_\mu a_\mu = 0$ and $r_{\mu+1} a_\mu \neq 0$. Then for each $S \in \mathcal{F}(\gamma)$ there exist an element x_S of N such that*

$$(*) \quad \text{for all } \beta < \gamma \text{ and all } S, T \in \mathcal{F}(\gamma), \text{ if } S \cap \beta = T \cap \beta, \text{ then } r_{\beta+1}(x_S - x_T) = 0 \text{ if and only if } S \cap (\beta + 1) = T \cap (\beta + 1).$$

PROOF. The idea of the construction is that x_S should « be » $\sum_{\mu \in S} a_\mu$.

The actual construction is by induction on γ . If γ is finite and $S \subseteq \gamma$, let $x_S = \sum_{\mu \in S} a_\mu$. (We let $x_\emptyset = 0$.) Now suppose that for all $\delta < \gamma$ and all $S \subseteq \delta$ we have defined x_S so that $(*)$ holds. We consider two cases.

Case 1: $\gamma = \delta + 1$ for some δ . We let $x_S = x_{S \cap \delta}$ if $\delta \notin S$, and we let $x_S = x_{S \cap \delta} + a_\delta$ if $\delta \in S$. It is easy to check, using the inductive hypothesis, that $(*)$ holds.

Case 2: $\gamma = \lambda$, a limit ordinal. Here we use the fact that since N is pure-injective it is algebraically compact: see, for example, [FS; p. 215]. For any $S \subseteq \lambda$ we let x_S be a solution of the set of equations

$$\{r_{\beta+1}(x - x_{S \cap (\beta+1)}) = 0: \beta < \lambda\}$$

in the single unknown x . (The elements $x_{S \cap (\beta+1)}$ of N have been defined by induction.) This system of equations is finitely solvable

in N : indeed, for any finite subset F of λ , if $\delta > \sup(F)$, then $x_{S \cap \delta}$ is a solution of

$$\{_{\beta+1}(x - x_{S \cap (\beta+1)}) = 0 : \beta \in F\}.$$

Hence by algebraic compactness there is a global solution, x_S . It remains to check that $(*)$ is satisfied. So suppose that S and T are subsets of λ , and $\beta < \lambda$ such that $S \cap \beta = T \cap \beta$. We have:

$$x_S - x_T = (x_S - x_{S \cap (\beta+1)}) + (x_{S \cap (\beta+1)} - x_{T \cap (\beta+1)}) + (x_{T \cap (\beta+1)} - x_T)$$

so $r_{\beta+1}(x_S - x_T) = 0 + r_{\beta+1}(x_{S \cap (\beta+1)} - x_{T \cap (\beta+1)}) + 0$; hence we are done by induction. \square

The third lemma will guarantee us the existence of the elements a_μ in Lemma 2 provided that $r_{\mu+1} \notin r_\mu R$. (Of course, over a valuation domain, injective = pure-injective + divisible.)

LEMMA 3. *Suppose L is an archimedean ideal and N is a divisible module containing R/L . Suppose also that r, s, t are elements of R such that t is a non-unit and $r = st$. Then there exists $a \in N$ such that $ra = 0$ and $sa \neq 0$.*

PROOF. We shall let \bar{b} denote the coset, $b + L$ of $b \in R$ in $R/L \subseteq N$. Since L is archimedean there is an element $b \in L \setminus tL$. If $bt^{-1} \in R$, let $a \in N$ such that $sa = bt^{-1} + L$. Then $ra = \bar{b} = 0$, but $sa \neq 0$ since $bt^{-1} \notin L$ (because $b \notin tL$). If $tb^{-1} \in R$, let $a \in N$ such that $s(tb^{-1})a = \bar{1}$. Then $ra = \bar{b} = 0$, but $sa \neq 0$ since $tb^{-1}(sa) = \bar{1}$. \square

We are now ready to give the:

PROOF OF THE THEOREM. Let $D = D_{n-k}$. As Bazzoni observes, we can assume that $|A| = \aleph_{n-k}$ since we can replace D by the direct summand of D consisting of elements whose support lies in a fixed subset of A of size \aleph_{n-k} . It suffices to prove that $\text{Ext}^1(J, D) \neq 0$ for some ideal J of R , for then $\text{Ext}^2(R/J, D) \neq 0$ (cf. [FS; VI.5.2]). For this it suffices to prove that the canonical map: $\text{Hom}(J, I) \rightarrow \text{Hom}(J, I/D)$ is not surjective. In fact we shall show that this map is not surjective whenever J is an ideal of R which is not generated by a set of size \aleph_{n-k} but is generated by a set of size \aleph_{n-k+1} ; there is such an ideal because $\text{gl. dim } R > n - k + 2$ (cf. [0] or [FS; IV.2.3].)

Let $\{j_{\alpha+1} : \alpha < \aleph_{n-k+1}\}$ be a set of generators of J ordered so that for all $\beta < \alpha$, $j_{\beta+1} \in Rj_{\alpha+1}$ and $j_{\alpha+1} \notin Rj_{\beta+1}$. Thus for every pair of

ordinals $\beta < \alpha$ we have a non-unit r_β^α of R such that $r_\beta^\alpha j_{\alpha+1} = j_{\beta+1}$. Moreover, for all $\beta < \gamma < \alpha$ we have $r_\beta^\alpha = r_\beta^\gamma r_\gamma^\alpha$.

Let $\kappa = \aleph_{n-k}$. We may as well suppose that $\lambda = \kappa$. So defining $f: J \rightarrow I/D$ amounts to choosing, for each $\nu < \kappa$, elements $x_\nu^\alpha \in I_\nu$ ($\alpha < \kappa^+ = \aleph_{n-k+1}$) so that for all $\beta < \alpha$, $|\{\nu < \kappa: r_\beta^\alpha x_\nu^\alpha \neq x_\nu^\beta\}| < \kappa$; for then we can define $f(j_{\alpha+1}) = x^\alpha + D$, where $x^\alpha = \langle x_\nu^\alpha: \nu < \kappa \rangle \in I$. We are going to use the sets w_ν^α ($\alpha < \kappa^+$, $\nu < \kappa$) constructed in Lemma 1 in order to define the x_ν^α 's; in fact, we shall construct them so that $r_\beta^\alpha x_\nu^\alpha = x_\nu^\beta$ if $\beta \in w_\nu^\alpha$. Then f will be defined because, by (1) of Lemma 1, for any $\beta < \alpha$ there exists $\mu < \kappa$ so that $\beta \in w_\mu^\alpha$, and hence by (2), the set of ν such that $r_\beta^\alpha x_\nu^\alpha \neq x_\nu^\beta$ is contained in μ , and thus has cardinality less than κ .

In order to make f not liftable to a homomorphism into I we shall also require that the x_ν^α be chosen so that if $\text{sup}(w_\nu^\alpha) + \kappa < \beta < \alpha$, then $r_\beta^\alpha x_\nu^\alpha \neq x_\nu^\beta$. (The sum is ordinal addition.) Indeed, if there were a $g: J \rightarrow I$ which lifted f , then we would have $g(j_\alpha) = y^\alpha$ for some $y^\alpha \in I$ such that $y^\alpha = x^\alpha + d^\alpha$ for some $d^\alpha \in D$, for all $\alpha < \kappa^+$. For each $\mu < \kappa$, let

$$Y_\mu \stackrel{\text{def}}{=} \{\alpha < \kappa^+: \mu \notin \text{supp}(d^\alpha)\};$$

then for some $\nu < \kappa$, Y_ν is a stationary subset of κ^+ since $\bigcup Y_\mu = \kappa^+$ (cf. [J; Lemma 7.4]). Now by (4), $\text{sup}(w_\nu^\alpha) < \alpha$ if $\text{cf}(\alpha) = \kappa$, so by Fodor's Lemma ([J; p. 59]) there is a stationary subset Y' of Y_ν , and an ordinal γ such that for all $\alpha \in Y'$ $\text{sup}(w_\nu^\alpha) = \gamma$. Hence there are elements β, α of Y' such that $\gamma + \kappa < \beta < \alpha$. But then $y^\alpha(\nu) = x_\nu^\alpha$ and $y^\beta(\nu) = x_\nu^\beta$, and by construction $r_\beta^\alpha x_\nu^\alpha \neq x_\nu^\beta$, which means that g is not a homomorphism.

Thus it remains only to construct for each ν the elements x_ν^α of I , so that for all $\beta < \alpha < \kappa^+$:

- (i) $r_\beta^\alpha x_\nu^\alpha = x_\nu^\beta$ if $\beta \in w_\nu^\alpha$;
- (ii) $r_\beta^\alpha x_\nu^\alpha \neq x_\nu^\beta$ if $\beta > \text{sup}(w_\nu^\alpha) + \kappa$.

We shall do this for each fixed ν by induction on α . Let $x_\nu^0 = \bar{1}$. Suppose now that x_ν^β has been defined for all $\beta < \alpha$ so that (i) and (ii) hold where defined. In order to satisfy (i) it is enough to choose x_ν^α to be a solution, z , of the system of equations

$$(\dagger) \quad \{r_\beta^\alpha z = x_\nu^\beta: \beta \in w_\nu^\alpha\}.$$

Since I is pure-injective, it suffices to show that this system is finitely solvable in I_ν . If F is a finite subset of w_ν^α and $\sigma = \max(F)$, we claim that any z such that $r_\sigma^\alpha z = x_\nu^\sigma$ will be a solution of

$$\{r_\beta^\alpha z = x_\nu^\beta : \beta \in F\}.$$

In fact, if $\beta \in F$ and $\beta < \sigma$, then since $\sigma, \beta \in w_\nu^\alpha$, (3) implies that $\beta \in w_\nu^\sigma$, so $r_\beta^\sigma x_\nu^\sigma = x_\nu^\beta$ and hence $r_\beta^\alpha z = r_\beta^\sigma r_\sigma^\alpha z = r_\beta^\sigma x_\nu^\sigma = x_\nu^\beta$.

Now consider (ii). Let $\delta = \sup(w_\nu^\alpha)$. Let z be a fixed solution of (†). Then (i) will hold if x_ν^α is of the form $z + u$ where $r_\delta^\alpha u = 0$. Let $\beta = \delta + \kappa + 1$. It suffices to choose u so that $r_\delta^\alpha u = 0$ and for each γ such that $\beta \leq \gamma < \alpha$, $r_\beta^\alpha u \neq r_\beta^\gamma x_\nu^\gamma - r_\beta^\alpha z$. (We let $r_\beta^\beta = 1$.) For then, since $r_\beta^\alpha = r_\beta^\gamma r_\gamma^\alpha$, we have that $r_\gamma^\alpha(z + u) \neq x_\nu^\gamma$. But Lemma 2 (with $r_\nu = r_{\delta+\nu}^\alpha$ for $\nu < \kappa$) in conjunction with Lemma 3 implies that the quotient group

$$\{u \in I_\nu : r_\delta^\alpha u = 0\} / \{u \in I_\nu : r_\beta^\alpha u = 0\}$$

has cardinality $\geq 2^\kappa$. Thus there certainly is a u with the desired properties. This completes the inductive step of the construction, and hence completes the proof of the theorem. \square

COROLLARY. *If gI , $\dim(R) \geq 3$, and for each $n \in \omega$, I_n is an injective module containing R/L_n for some archimedean ideal L_n of R , then the injective dimension of $\bigoplus_{n \in \omega} I_n$ is ≥ 2 . \square*

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