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A Finiteness Condition on Automorphism Groups.

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1. Introduction.

If $G$ is a group, its automorphism group $\text{Aut} G$ acts on $G$ in a natural way as a permutation group. Should this action be restricted by the imposition of a finiteness condition, there will be repercussions on the structure of the group $G$. The simplest case is where $\text{Aut} G$ is required to be finite and there is a considerable literature dealing with the resulting structure of $G$ (see [3]). Recently groups $G$ for which the automorphism classes (i.e. $\text{Aut} G$-orbits) are finite have been studied ([5]). In the present work we study what is in a sense the dual property, that fixed point subgroups of automorphisms have finite index.

An automorphism $\alpha$ of a group $G$ is said to be virtually trivial if $|G:C_\alpha(\alpha)|$ is finite. (Automorphisms with this property have also been considered in [7] under the name of «bounded automorphisms».) The set of all virtually trivial automorphisms of $G$ is readily seen to be a normal subgroup of $\text{Aut} G$, denoted here by $\text{Aut}_{vt}(G)$.

Should it happen that $\text{Aut} G = \text{Aut}_{vt}(G)$, that is, every automorphism is virtually trivial, we shall say that $G$ is a $VTA$-group.

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Obviously every finite group is VTA and every VTA-group is an $FC$-group. It is not difficult to see that every abelian VTA-group must be finite (by considering the automorphism $x \mapsto x^{-1}$). The simplest example of an infinite VTA-group is

$$G = \langle x, y : y^5 = 1, y^a = y^5 \rangle.$$ 

This group has no outer automorphisms and each conjugacy class has at most 5 elements.

2. Finiteness of the commutator subgroup.

The basic result in the theory of VTA-groups is

**Proposition 1.** If $G$ is a VTA-group, then $G'$ is finite.

**Proof.** Suppose to the contrary that $G'$ is infinite. Choose any element $x_1$ and $X_1$ a finitely generated normal subgroup containing $x_1$, for example $\langle x_1^a \rangle$. Set $C_1 = C_0(X_1)$; then $|G : C_1|$ is finite since $G$ is an $FC$-group. Hence there is a finitely generated normal subgroup $H$ such that $G = HC_1$. Then $G' = H'[H, C_1] C_1'$. By standard results on $FC$-groups (see [2]) $G'$ is a torsion group and the elements of finite order in $H$ form a finite subgroup. Consequently $C_1'$ is infinite and by a well-known theorem of B. H. Neumann [1] there are conjugacy classes of $C_1$ with arbitrarily large finite orders. Hence one can find an element $c$ of $C_1$ such that $|C_1 : C_0(c)|$ exceeds $|X_1 : C_{X_1}(x_1)|$. Put $x_2 = x_1 c$ and observe that $C_0(x_2) = C_0(c)$. Therefore

$$|G : C_0(x_2)| > |C_1 : C_0(c)| > |X_1 : C_{X_1}(x_1)|.$$

Now choose a finitely generated normal subgroup $X_2$ such that $G = X_2 C_0(x_2)$ and $X_1 < X_2$. Then

$$|X_2 : C_{X_1}(x_2)| = |G : C_0(x_2)| > |X_1 : C_{X_1}(x_1)|.$$

Notice that conjugation by $x_1$ and $x_2$ produce the same inner automorphism on $X_1$.

By repetition of this procedure one can construct a sequence of elements $x_1, x_2, \ldots$ and a chain of finitely generated normal subgroups $X_1 < X_2 < \ldots$ with $x_i \in X_i$ such that conjugation by $x_i$ and $x_{i+1}$ have
the same effect on $X_i$ and

$$|X_{i+1}:C_{x_{i+1}}(x_{i+1})| > |X_i:C_{x_i}(x_i)| \quad \text{for all } i = 1, 2, \ldots.$$ 

Let $U = \bigcup_{i=1,2,...} X_i$ and let $\alpha$ be the locally inner automorphism of $U$ whose restriction to $X_i$ is the inner automorphism induced by conjugation by $x_i$. Now it is always true that a locally inner automorphism of a subgroup of an FC-group can be extended to a locally inner automorphism of the group (see [6], Lemma 2.3, but note that the periodicity hypothesis is not essential). It follows that $\alpha$ is a virtually trivial automorphism of $U$ and this yields the contradiction that the $|X_i:C_{x_i}(x_i)|$ are bounded. The proof is now complete.

It is an easy observation that if $G$ is a group with finite derived subgroup, then $G/Z(G)$ has finite exponent. Also a theorem of P. Hall (see [2]) shows that $G/Z_2(G)$ is finite. Hence

**Corollary 1.** If $G$ is a VTA-group, then $G/Z(G)$ is centre-by finite and has finite exponent.

As a result of Proposition 1 one can give an equivalent definition for VTA-groups.

**Lemma 1.** A group $G$ is a VTA-group if and only if $[G, \alpha]$ is finite for every automorphism $\alpha$ of $G$.

**Proof.** It is clear that $|G:C_{\alpha}(\alpha)|$ will be finite if $[G, \alpha]$ is finite. Conversely assume that $G$ is a VTA-group; set $C = Z(G)$. Since $a \mapsto [a, \alpha]$ is an endomorphism of $C$ and $C/C_{\alpha}(\alpha) \cong [C, \alpha]$, the subgroup $[C, \alpha]$ is finite. By Corollary 1 $G/C$ has finite exponent, say $e$, and $[G, \alpha]^e < [G^e, \alpha]G' < [C, \alpha]G'$. Hence $[G, \alpha]^e$ is finite and $[G, \alpha]$ is locally finite. Since $[G, \alpha]$ is certainly finitely generated, it is finite, as required.

**Remark.** In general, $\alpha \in \text{Aut } G$ and $|G:C_{\alpha}(\alpha)| < \alpha$ do not imply that $[G, \alpha]$ is finite. For let $G = \langle x, y | x^2 = 1, y^x = y^{-1} \rangle$ be the infinite dihedral group and let $\alpha \in \text{Aut } G$ be defined by $x \mapsto xy$, $y \mapsto y$.

The following lemma is a sharper form of [7], Theorem 1:

**Lemma 2.** Let $G$ be an arbitrary group and let $X$ be a finitely generated subgroup of $\text{Aut}_{vt}(G)$. Then $X/Z(X) \cap \text{Inn } G$ is finite. In particular $\text{Aut}_{vt}(G)$ is locally FC and $\text{Aut}_{vt}(G)/\text{Aut}_{vt}(G) \cap \text{Inn } G$ is locally finite.
PROOF. We know that $|G:C_0(X)|$ is finite. Let $N$ be the core of $C_0(X)$ in $G$, so that $|G:N|$ is finite. Put $X_1 = C_2(G/N)$. Then $X/X_1$ is finite and there is an obvious injection

$$\mu : X_1 \to \text{Der}(G/N, Z(N)).$$

Of course $X_1$ is finitely generated. Now $H^1(G/N, Z(N))$ is a bounded abelian group, being annihilated by $|G|N|$. Hence $X_1/X_2$ is finite where $X_2$ is the inverse image of $\text{Inn}(G/N, Z(N))$ under $\mu$. But $X_2$ consists of inner automorphisms induced by elements of $Z(N)$, and such automorphisms commute with $X$. Hence $X_2 < Z(X) \cap \text{Inn} G$.

COROLLARY 2. If $G$ is a VTA-group, then Aut $G$ is locally finite.

3. Structure of the centre.

LEMMA 3. If $G$ is a VTA-group, then $Z(G)$ is reduced and its primary components are all finite.

PROOF. Let $C = Z(G)$ and $Q = G/C$. By Corollary 1 it is possible to express $Q$ as a direct product $Q_1 \times Q_2$ where $Q_1$ is finite and $Q_2$ is abelian. By the Universal Coefficients Theorem $H^2(Q, C)$ is bounded, say $l \cdot H^2(Q, C) = 0$ with $l > 0$.

Suppose that $C$ is not reduced and so $C = D \times E$ where $D$ is either a $p^\infty$-group or $Q$. Then there is an automorphism of $C$ in which $d \mapsto d^k$ and $e \mapsto e$ ($d \in D$, $e \in E$); here $k > 1$ and $k \equiv 1$ (mod $l$ or $pl$) according as $D \simeq Q$ or $p^\infty$. Since $l \cdot H^2(Q, C) = 0$ this $\alpha$ extends to an automorphism of $G$. Hence $|D:C_\alpha(\alpha)| < \alpha$ which is of course impossible.

Next assume that the $p$-component $C_p$ is infinite for some prime $p$. Then $C[p] = \{a \in C : a^p = 1\}$ is infinite since $C$ is reduced. By constructing central automorphisms of $G$ corresponding to elements of $\text{Hom}(Q_n, C[p])$ one can see that $Q_n/Q_n^p$ must be finite. Since $Q_n$ has finite exponent, this implies that $(Q_n)_{\alpha}$ is finite. There is nothing to be lost in supposing $Q_n$ to be a $p'$-group. Since $C_p/C_p^\alpha$ is infinite, so is $C/C_p^\alpha$. Also $G'$ is finite; thus $G/C_p^\alpha$ has an infinite elementary abelian quotient. Therefore $C_p[p]$ is necessarily finite, which shows that $(C_p)_{\alpha}$ is finite. It follows that $C_p$ has finite exponent. Hence it is possible to write $C = C_\alpha \times F$ for some subgroup $F$. 
Since \(Q_2 = G_2/F\) is a \(p'\)-group, \(G_2/F\) splits over \(C/F\) and
\[
G_2/F = X/F \times C/F,
\]
say. Thus \(G_2 = X \times C_p\). Because \(Q_1\) is finite, there is a finitely generated subgroup \(Y\) such that \(G = YG_2\) and \(Y \cap G_2 < C\). But \(C_p\) has finite exponent, so one can write \(C_p = C_{p,0} \times C_{p,1}\) where \(C_{p,0}\) is finite and \(Y \cap G_2 = X \cap C_{p,0}\). Hence
\[
G = YG_2 = Y \times F C_{p,0} C_{p,1} = (YXC_{p,0})C_{p,1}.
\]

If \(yx_{c_0} = c_1\) where \(y \in Y, x \in X, c_i \in C_{p,i}\), then \(y \in Y \cap G_2 \leq X \times C_{p,0}\). Hence one can assume that \(y = 1\) and \(xc_0 = c_1\). But now \(x \in X \cap C_p = 1\) and \(c_0 = c_1 = 1\). Consequently \(G = (Y \times C_{p,0}) \times C_{p,1}\). Since \(G\) is a VTA-group and \(C_{p,1}\) has finite exponent, \(C_{p,1}\) is finite. This gives the contradiction that \(C_p\) is finite.

**Lemma 4.** If \(G\) is a torsion VTA-group, then \(Z(G)\) is finite and \(G\) has finite exponent.

**Proof.** By Lemma 3 and Corollary 1, it is sufficient to prove that the set \(\pi\) of prime divisors of orders of elements of \(G\) is finite. Let \(L\) denote the second centre of \(G\). Then \(|G:L|\) is finite and there is a finite normal subgroup \(N\) such that \(G = NL\); since \(G'\) is finite, one can assume that \(G' \leq N\). If \(\pi_0\) is the set of prime divisors of \(|N|\), then \(G = N(L_{\pi_0} \times L_{\pi_0}') = (NL_{\pi_0}) \times L_{\pi_0}'.\) Now \((L_{\pi_0}')' \leq L_{\pi_0}' \cap G' = 1\), so \(L_{\pi_0}'\) is abelian; therefore it is finite by the VTA-property.

However in general the torsion subgroup of the centre of a VTA-group can be infinite, as will be shown in \$5\$.

**4. Necessary and sufficient conditions.**

Let \(Q\) be a group and \(C\) an abelian group regarded as a trivial \(Q\)-module. Then there are natural left and right actions of \(\text{Aut}\ Q\) and \(\text{Aut}\ C\) respectively on \(H^2(Q, C)\). If \(C \mapsto G \mapsto Q\) is a central extension with cohomology class \(\Delta\), then a necessary and sufficient condition for there to exist an automorphism of \(G\) inducing in \(Q\) and \(C\) automorphisms \(\alpha\) and \(\gamma\) is that \(\alpha \Delta = \Delta \gamma\). If we make \(H^2(Q, C)\) into a right \(\text{Aut}\ Q \times \text{Aut}\ C\)-module by means of the rule \(\Delta(\alpha, \gamma) = \alpha^{-1} \Delta \gamma\),
then the above condition is equivalent to

$$(\alpha, \gamma) \in C_{\operatorname{Aut}_Q \times \operatorname{Aut}_C}(\Lambda) .$$

These observations, the background to which may be found in [4], may be used to give necessary and sufficient conditions for a group to be a VTA-group.

**Theorem 1.** Let $G$ be a group with centre $C$ and central quotient group $Q$. Let $\Lambda$ be the cohomology class of the extension $C \hookrightarrow G \twoheadrightarrow Q$. Then $G$ is a VTA-group if and only if the following hold:

(i) $G'$ is finite;

(ii) each primary component of $C$ is finite;

(iii) $C_{\operatorname{Aut}_Q \times \operatorname{Aut}_C}(\Lambda)$ is contained in $\operatorname{Aut}_{\text{et}}(Q) \times \operatorname{Aut}_{\text{et}}(C)$.

**Proof.** Let $G$ be a VTA-group; then (i) and (ii) hold by Proposition 1 and Lemma 3. If $(x, y)$ belongs to $C_{\operatorname{Aut}_Q \times \operatorname{Aut}_C}(\Lambda)$, then $x$ and $y$ are induced by an automorphism of $G$. Hence

$$x \in \operatorname{Aut}_{\text{et}}(Q) \quad \text{and} \quad y \in \operatorname{Aut}_{\text{et}}(C).$$

Conversely, assume that the three conditions hold. Let $\alpha$ be an automorphism of $G$ inducing $x$ in $Q$ and $\gamma$ in $C$; then $x \in \operatorname{Aut}_{\text{et}}(Q)$ and $\gamma \in \operatorname{Aut}_{\text{et}}(C)$ by (iii). Set $K/C = C_{\alpha}$ and $L = C_{\gamma}$. Then $|G/K|$ and $|C/L|$ are finite. The mapping $x \mapsto [x, \alpha]$ is a homomorphism from $K$ onto $[K, \alpha]$ whose kernel contains $L$. Now condition (i) implies that $G/C$ has finite exponent, whence so does $G/L$. Consequently $[K, \alpha]$ has finite exponent. Since $[K, \alpha] \leq C$, it follows that $[K, \alpha]$ is contained in $T$, the torsion subgroup of $C$. Since $T$ has finite primary components by (ii), the subgroup $[K, \alpha]$ is finite. This shows that $|K:C_{\alpha}(\alpha)|$ is finite, therefore $|G:C_{\alpha}(\alpha)|$ is finite and $G$ is a VTA-group, as claimed.

It is of course the third condition which is difficult to deal with. For torsion groups this can be simplified slightly.

**Theorem 2.** Let $G$ be a torsion group with centre $C$ and central quotient group $Q$. Let $\Lambda$ be the cohomology class of the extension $C \hookrightarrow G \twoheadrightarrow Q$. Then $G$ is a VTA-group if and only if the following con-
ditions hold:

(i) $G'$ is finite;

(ii) $C$ is finite;

(iii) $St_{\text{Aut} Q}(\Delta_{\text{Aut} C})$, the stabilizer in $\text{Aut} Q$ of the $\text{Aut} C$-orbit containing $\Delta$, is contained in $\text{Aut}_{vt}(Q)$.

The point to note here is that $\alpha$ in $\text{Aut} Q$ is induced by an automorphism of $G$ if and only if there exists a $\gamma$ in $\text{Aut} C$ such that $\alpha \Delta = \Delta \gamma$, that is, $\alpha$ stabilizes $\Delta_{\text{Aut} C}$. Theorem 2 is now a consequence of Theorem 1 and Lemma 4. (An example of an infinite torsion VTA-group is given in § 5.)

Finitely generated VTA-group admit a precise characterization.

THEOREM 3. A finitely generated group $G$ is a VTA-group if and only if it is either finite or the split extension of a finite group $F$ by an infinite cyclic group $\langle x \rangle$ such that $\hat{x}$, the image of $x$ in $\text{Out} F$, is not conjugate to its inverse.

PROOF. Let $G$ be a finitely generated infinite VTA-group, and write $C = Z(G)$ and $Q = G/C$. By Corollary 1 the group $Q$ is finite, so $C$ is finitely generated. Suppose that $C$ had a free abelian direct factor of rank 2, say $\langle a \rangle \times \langle b \rangle$. If $|Q| = m$, then it is easy to see that $a \mapsto abm$, $b \mapsto b$ extends to an automorphism of $G$; but this cannot be virtually trivial. Consequently $C$ has torsion-free rank 1. Since $G'$ is finite, the elements of finite order form a finite subgroup $F$ and $G/F$ is infinitely cyclic. Write $G = \langle x \rangle F$ with $\langle x \rangle \cap F = 1$. If $\alpha \in \text{Aut} G$ induces the inversion automorphism in $G/F$, then $C_\alpha(x) F$ and $|G:C_\alpha(x)|$ is infinite. Therefore every automorphism of $G$ acts trivially on $G/F$.

Conversely let $G = \langle x \rangle \times F$ have the property that all automorphisms of $G$ act trivially on $G/F$. Let $\alpha \in \text{Aut} G$. Now $G/C$ is finite where $C = Z(G)$, and $C = D \times E$ where $D$ is finite of order $d$, say, and $E$ is infinite cyclic. Hence there is an $m > 0$ such that $x^m \in C \subseteq E$. Now $C$ is characteristic in $G$ and infinite cyclic, so $\langle x^m \rangle$ is $\alpha$-invariant. Since $\alpha$ clearly cannot map $x^m$ to its inverse, it follows that $x^m \in C_\alpha(x)$ and $|G:C_\alpha(x)|$ is finite. Thus $G$ is a VTA-group.

It remains to decide when the inversion automorphism of $Q$ lifts to $G$. A simple argument shows that this occurs if and only if $\hat{x}$ is conjugated to its inverse in $\text{Out} F$ by some $\varphi \in \text{Aut} F$. 
REMARK. In order to construct all finitely generated VTA-groups one must select a finite group $F$ with the property that not every element of $\text{Out } F$ is conjugate to its inverse (or, equivalently, such that some irreducible C-character of $F$ is not real). If $\phi$ is such an element, lift $\phi$ in any way to an automorphism $\varphi$ of $F$ and form the semidirect product $\langle x \rangle \times F$ of $F$ with an infinite cyclic group $\langle x \rangle$ where $x$ induces $\varphi$ in $F$. The smallest possible $F$ is the cyclic group of order 5; this gives rise to the VTA-group mentioned in § 1.

Hence the isomorphism classes of finitely generated VTA-groups are in one-one correspondence with pairs $(F, \phi)$ where $F$ is a finite group as above and $\phi$ is a conjugacy class of $\text{Out } F$ not equal to its inverse.

5. Examples.

We shall now construct the examples of infinite VTA-group promised in earlier sections.

$(A)$ There is an infinite VTA-group which is a p-group with nilpotency class 2.

For $p = 2$ the group constructed in [5], Prop. 3 will do. Let $p > 2$ and let $G$ be the group with generators $a, b, x_1, x_2, \ldots$ and relations

\[
[x_{2i-1}, x_{2i}] = a, \quad [x_{2i}, x_{2i+1}] = b, \quad [x_i, x_j] = 1 \quad \text{if} \quad i < j - 1,
\]

\[
[a, x_i] = 1 = [b, x_i] = [a, b] = a^p = b^p,
\]

\[
x_i^p = ab, \quad x_2^p = a, \quad x_i^p = 1 \quad \text{if} \quad i > 2.
\]

One quickly sees that $G' = Z(G) = \langle a \rangle \times \langle b \rangle$ and $Q = G/Z(G)$ is an infinite elementary abelian p-group. For $i = 1, 2, \ldots$ put $M_i = \langle x_i \rangle G'$ and $C_i = \langle x_{i+2}, x_{i+3}, \ldots \rangle G'$. It is easily seen that $M_i$ is the set of all $y \in G$ such that $[G:C_0(y)] = p$. Thus $M_1$ is characteristic in $G$, as is $C_1 = \Omega_1(C_0(M_1))$. The elements $y$ of $G$ for which $|M_1:C_1(y)| = p = |C_1:C_0(y)|$ are those of the form $x_i^s x_2 z$ where $s \equiv 0 \pmod p$ and $z \in G'$. From this it follows that: $\langle x_1, x_2 \rangle G'$ is characteristic in $G$. Among these $y$ those that generate a cyclic normal subgroup of $\langle x_1, x_2 \rangle G'$ are of the form $x_i^s z$; hence $M_2$ is characteristic in $G$. 
Now suppose that $M_1, \ldots, M_i$ are characteristic in $G$ (where $i \geq 2$); then also $C_{i-1} = \Omega_1(C_0(M_1 \ldots M_{i-1}))$ is characteristic. A little calculation shows that the elements $y$ of $C_{i-1}$ such that $[y, C_{i-1}] = p$ are those of the form $x_i^{r}z$ ($r \not\equiv 0 \pmod{p}$, $z \in G'$). It follows that $M_{i+1}$ is characteristic in $G$.

Next let $\alpha \in \text{Aut } G$. We have just shown that $x_i^{\alpha} \equiv x_i^{n_i} \pmod{G'}$ for all $i$ and some integers $n_i$. Our relations imply $n_1 \equiv n_p \equiv n_{2i-1} \cdot \cdot n_{2i} \equiv n_{2i} n_{2i+1}$, ($i = 1, 2, \ldots$), i.e. $n_j \equiv 1 \pmod{p}$ for all $j$. Thus every automorphism of $G$ acts trivially on $Z(G)$ and on $Q$. Hence $|G:C_\alpha(\alpha)| < p^2$ for every automorphism $\alpha$. Notice also that Aut $G \cong \text{Hom} (Q, Z(G))$, an elementary abelian $p$-group of cardinality $2^{N_0}$.

(B) There exists a VTA-group with finite torsion-free rank whose centre has infinite torsion subgroup.

The construction falls into two parts; first we assign the central quotient, then we construct the centre.

Let $Q$ be a finite group satisfying the conditions

(i) $Q_{ab} = Q/Q'$ is not an elementary abelian 2-group;

(ii) no automorphism of $Q$ induces an automorphism of order $2$ in $Q_{ab}$;

(iii) $Z(Q) = 1$.

Of course such groups abound; the simplest example is the holomorph of a cyclic group of order 5. Let $\sigma$ be the set of all primes which do not divide the order of $Q$.

The centre of our group has to be chosen with some care. Let $\langle t_p \rangle$ be a cyclic group of order $p$, written additively, and define

$$T = \bigoplus_{p \in \sigma} \langle t_p \rangle \quad \text{and} \quad T^* = \bigoplus_{p \in \sigma} \langle t_p \rangle,$$

the direct and cartesian sums. We shall construct a group $C$ such that

(i)' $T \triangleleft C \triangleleft T^*$;

(ii)' $\bar{C} = C/T$ has automorphism group of order 2;

(iii)' Aut $C = \text{Aut}_{\sigma}(C) \times \langle -1 \rangle$.

Here of course $\langle -1 \rangle$ refers to the automorphism $c \mapsto -c$ of $C$. 
Assuming this $C$ to be constructed, one regards it as a trivial $Q$-module. Then by the Universal Coefficients Theorem

\[ H^2(Q, C) \cong \text{Ext}(Q_{ab}, \bar{C}) . \]

This group has an element $\Delta$ with order $> 2$ by properties (i) and (ii)'. Let be a central extension with cohomology class $\Delta$. Then $Z(G) = C$ since $Z(Q) = 1$. Let $\alpha$ in $\text{Aut } G$ induce automorphisms $\kappa$ and $\gamma$ in $Q$ and $C$ respectively; then $\kappa \Delta = \Delta \gamma$. If $\gamma \notin \text{Aut}_{\text{triv}}(C)$, then by (iii)' $\gamma$ must induce $-1$ on $\bar{C}$; in this case $\kappa \Delta = - \Delta$ by the above (natural) isomorphism. But $\kappa$ cannot induce an automorphism of even order in $Q_{ab}$, so we reach a contradiction. It follows that $\gamma \in \text{Aut}_{\text{triv}}(C)$ and $\alpha \in \text{Aut}_{\text{triv}}(G)$ since $Q$ is finite.

**Construction of $C$.** It remains to find an abelian group $C$ satisfying (i)', (ii)', (iii)'. First we specify $\bar{C}$. Let

\[ \sigma = \pi \cup \varrho \]

be a partition of $\sigma$ into two infinite sets of primes. Define $x^*, y^* \in T^*$ by

\[
(x^*)_p = t_p, \quad (y^*)_p = 0 \quad \text{if } p \in \pi \\
(x^*)_p = 0, \quad (y^*) = t_p \quad \text{if } p \in \varrho.
\]

Also let $q$ be a prime not in $\sigma$. Define $A_1$, $A_2$, $A_3$ to be the subrings of $Q$ generated by $\pi^{-1}$, $\varrho^{-1}$, $q^{-1}$ respectively.

Now $u = x^* + T$, $v = y^* + T$ are independent vectors in the $Q$-vector space $\bar{T} = T^*/T$, and we may define our group $C$ by requiring $T \leq C \leq T^*$ and

\[
\bar{C} \equiv C/T = A_1 u + A_2 v + A_3 (u + v) .
\]

It is straightforward to check that $\bar{C}$ has automorphism group of order 2. It remains to check property (iii)' Let $\gamma \in \text{Aut } \bar{C}$; replacing $\gamma$ by $-\gamma$ if necessary, we can assume that $\gamma$ acts trivially on $\bar{C}$. The task is now to show that $\gamma \in \text{Aut}_{\text{triv}}(C)$. Let $p \in \pi$. There is a unique
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coset \( b_\tau + T \in \bar{C} \) such that \( x^* - \tau b_\tau \in T \); notice that, since \( \tau b_\tau \) obviously has trivial \( p \)-component, the \( p \)-component of \( x^* - \tau b_\tau \) is \( t_\tau \), and we may as well assume that \( x^* - \tau b_\tau = t_\tau \). Now \( b_\tau (\gamma - 1) = a_\tau \) (say) is in \( T \), and one obtains

\[
t_\tau (\gamma - 1) = x^*(\gamma - 1) - p a_\tau.
\]

Looking at \( p \)-components one gets

\[
t_\tau (\gamma - 1) \in \langle x^*(\gamma - 1) \rangle \text{ for all } p \in \pi.
\]

This means that \( \gamma \) fixes almost all \( t_\tau \) with \( p \) in \( \pi \); a similar conclusion holds for \( \tau \). Modifying \( \gamma \) by a virtually trivial automorphism of \( C \), one can assume that \( \gamma \) operates trivially on \( T \). However it is easily checked that \( \text{Hom}(\bar{C}, T) \) is periodic, and this implies that \( C(\gamma - 1) \) is finite and \( |C: C_c(\gamma)| \) is finite. Therefore \( \gamma \in \text{Aut}_\tau (C) \).

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