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FEDERICO MENEGAZZO
DEREK J. S. ROBINSON

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A Finiteness Condition on Automorphism Groups.

Federico Menegazzo - Derek J. S. Robinson (*)

1. Introduction.

If G is a group, its automorphism group Aut G acts on G in a natural way as a permutation group. Should this action be restricted by the imposition of a finiteness condition, there will be repercussions on the structure of the group G. The simplest case is where Aut G is required to be finite and there is a considerable literature dealing with the resulting structure of G (see [3]). Recently groups G for which the automorphism classes (i.e. Aut G-orbits) are finite have been studied ([5]). In the present work we study what is in a sense the dual property, that fixed point subgroups of automorphisms have finite index.

An automorphism α of a group G is said to be *virtually trivial* if $|G:C_G(\alpha)|$ is finite. (Automorphisms with this property have also been considered in [7] under the name of « bounded automorphisms ».) The set of all virtually trivial automorphisms of G is readily seen to be a normal subgroup of Aut G, denoted here by $\mathrm{Aut}_{vt}(G)$.

Should it happen that $\mathrm{Aut}\, G = \mathrm{Aut}_{vt}(G)$, that is, every automorphism is virtually trivial, we shall say that G is a *VTA-group.*

(*) Indirizzo degli AA.: F. Menegazzo: Seminario Matematico, Università di Padova, Padova, Italy; D. J. S. Robinson: Department of Mathematics, University of Illinois, Urbana, Illinois, U.S.A.

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Obviously every finite group is VTA and every VTA-group is an FC-group. It is not difficult to see that every abelian VTA-group must be finite (by considering the automorphism $x \mapsto x^{-1}$). The simplest example of an infinite VTA-group is

$$G = \langle x, y : y^5 = 1, y^x = y^2 \rangle .$$

This group has no outer automorphisms and each conjugacy class has at most 5 elements.

2. Finiteness of the commutator subgroup.

The basic result in the theory of VTA-groups is

PROPOSITION 1. *If G is a VTA-group, then G' is finite.*

PROOF. Suppose to the contrary that G' is infinite. Choose any element x_1 and X_1 a finitely generated normal subgroup containing x_1, for example $\langle x_1^G \rangle$. Set $C_1 = C_G(X_1)$; then $|G : C_1|$ is finite since G is an FC-group. Hence there is a finitely generated normal subgroup H such that $G = HC_1$. Then $G' = H'[H, C_1]C_1'$. By standard results on FC-groups (see [2]) G' is a torsion group and the elements of finite order in H form a finite subgroup. Consequently C_1' is infinite and by a well-known theorem of B. H. Neumann [1] there are conjugacy classes of C_1 with arbitrarily large finite orders. Hence one can find an element c of C_1 such that $|C_1 : C_{C_1}(c)|$ exceeds $|X_1 : C_{X_1}(x_1)|$. Put $x_2 = x_1 c$ and observe that $C_{C_1}(x_2) = C_{C_1}(c)$. Therefore

$$|G : C_G(x_2)| \geqslant |C_1 : C_{C_1}(c)| > |X_1 : C_{X_1}(x_1)| .$$

Now choose a finitely generated normal subgroup X_2 such that $G = X_2 C_G(x_2)$ and $X_1 \leqslant X_2$. Then

$$|X_2 : C_{X_2}(x_2)| = |G : C_G(x_2)| > |X_1 : C_{X_1}(x_1)| .$$

Notice that conjugation by x_1 and x_2 produce the same inner automorphism on X_1.

By repetition of this procedure one can construct a sequence of elements $x_1, x_2, \ldots$ and a chain of finitely generated normal subgroups $X_1 < X_2 < \ldots$ with $x_i \in X_i$ such that conjugation by x_i and x_{i+1} have the same effect on X_i and

$$|X_{i+1}:C_{X_{i+1}}(x_{i+1})| > |X_i:C_{X_i}(x_i)| \quad \text{for all } i = 1, 2, \ldots .$$

Let $U = \bigcup_{i=1,2,\ldots} X_i$ and let α be the locally inner automorphism of U whose restriction to X_i is the inner automorphism induced by conjugation by x_i. Now it is always true that a locally inner automorphism of a subgroup of an FC-group can be extended to a locally inner automorphism of the group (see [6], Lemma 2.3, but note that the periodicity hypothesis is not essential). It follows that α is a virtually trivial automorphism of U and this yields the contradiction that the $|X_i:C_{X_i}(x_i)|$ are bounded. The proof is now complete.

It is an easy observation that if G is a group with finite derived subgroup, then $G/Z(G)$ has finite exponent. Also a theorem of P. Hall (see [2]) shows that $G/Z_2(G)$ is finite. Hence

COROLLARY 1. *If G is a VTA-group, then $G/Z(G)$ is centre-by finite and has finite exponent.*

As a result of Proposition 1 one can give an equivalent definition for VTA-groups.

LEMMA 1. *A group G is a VTA-group if and only if $[G, \alpha]$ is finite for every automorphism α of G.*

PROOF. It is clear that $|G:C_G(\alpha)|$ will be finite if $[G, \alpha]$ is finite. Conversely assume that G is a VTA-group; set $C = Z(G)$. Since $a \mapsto [a, \alpha]$ is an endomorphism of C and $C/C_C(\alpha) \simeq [C, \alpha]$, the subgroup $[C, \alpha]$ is finite. By Corollary 1 G/C has finite exponent, say e, and $[G, \alpha]^e \leqslant [G^e, \alpha]G' \leqslant [C, \alpha]G'$. Hence $[G, \alpha]^e$ is finite and $[G, \alpha]$ is locally finite. Since $[G, \alpha]$ is certainly finitely generated, it is finite, as required.

REMARK. In general, $\alpha \in \operatorname{Aut} G$ and $|G:C_G(\alpha)| < \alpha$ do not imply that $[G, \alpha]$ is finite. For let $G = \langle x, y | x^2 = 1, y^x = y^{-1}\rangle$ be the infinite dihedral group and let $\alpha \in \operatorname{Aut} G$ be defined by $x \mapsto xy$, $y \mapsto y$.

The following lemma is a sharper form of [7], Theorem 1:

LEMMA 2. *Let G be an arbitrary group and let X be a finitely generated subgroup of $\operatorname{Aut}_{vt}(G)$. Then $X/Z(X) \cap \operatorname{Inn} G$ is finite. In particular $\operatorname{Aut}_{vt}(G)$ is locally FC and $\operatorname{Aut}_{vt}(G)/\operatorname{Aut}_{vt}(G) \cap \operatorname{Inn} G$ is locally finite.*

PROOF. We know that $|G:C_G(X)|$ is finite. Let N be the core of $C_G(X)$ in G, so that $|G:N|$ is finite. Put $X_1 = C_X(G/N)$. Then X/X_1 is finite and there is an obvious injection

$$\mu : X_1 \to \mathrm{Der}\,(G/N,\, Z(N))\,.$$

Of course X_1 is finitely generated. Now $H^1(G/N, Z(N))$ is a bounded abelian group, being annihilated by $|G:N|$. Hence X_1/X_2 is finite where X_2 is the inverse image of $\mathrm{Inn}\,(G/N, Z(N))$ under μ. But X_2 consists of inner automorphisms induced by elements of $Z(N)$, and such automorphisms commute with X. Hence $X_2 \leqslant Z(X) \cap \mathrm{Inn}\, G$.

COROLLARY 2. *If G is a VTA-group, then* Aut G *is locally finite.*

3. Structure of the centre.

LEMMA 3. *If G is a VTA-group, then $Z(G)$ is reduced and its primary components are all finite.*

PROOF. Let $C = Z(G)$ and $Q = G/C$. By Corollary 1 it is possible to express Q as a direct product $Q_1 \times Q_2$ where Q_1 is finite and Q_2 is abelian. By the Universal Coefficients Theorem $H^2(Q, C)$ is bounded, say $l \cdot H^2(Q, C) = 0$ with $l > 0$.

Suppose that C is not reduced and so $C = D \times E$ where D is either a p^∞-group or $\mathbb{Q}$. Then there is an automorphism of C in which $d \mapsto d^k$ and $e \mapsto e$ $(d \in D,\, e \in E)$; here $k > 1$ and $k \equiv 1 \pmod{l \text{ or } pl}$ according as $D \simeq \mathbb{Q}$ or p^∞. Since $l \cdot H^2(Q, C) = 0$ this α extends to an automorphism of G. Hence $|D : C_D(\alpha)| < \alpha$ which is of course impossible.

Next assume that the p-component C_p is infinite for some prime p. Then $C[p] \equiv \{a \in C : a^p = 1\}$ is infinite since C is reduced. By constructing central automorphisms of G corresponding to elements of $\mathrm{Hom}\,(Q_2, C[p])$ one can see that Q_2/Q_2^p must be finite. Since Q_2 has finite exponent, this implies that $(Q_2)_p$ is finite. There is nothing to be lost in supposing Q_2 to be a p'-group. Since C_p/C_p^p is infinite, so is C/C^p. Also G' is finite; thus G/C^p has an infinite elementary abelian quotient. Therefore $C^p[p]$ is necessarily finite, which shows that $(C^p)_p$ is finite. It follows that C_p has finite exponent. Hence it is possible to write $C = C_p \times F$ for some subgroup F.

Since $Q_2 \equiv G_2/F$ is a p'-group, G_2/F splits over C/F and

$$G_2/F = X/F \times C/F\,,$$

say. Thus $G_2 = X \times C_p$. Because Q_1 is finite, there is a finitely generated subgroup Y such that $G = YG_2$ and $Y \cap G_2 \leqslant C$. But C_p has finite exponent, so one can write $C_p = C_{p,0} \times C_{p,1}$ where $C_{p,0}$ is finite and $Y \cap G_2 = X \cap C_{p,0}$. Hence

$$G = YG_2 = Y \times FC_{p,0}\,C_{p,1} = (YXC_{p,0})\,C_{p,1}\,.$$

If $yxc_0 = c_1$ where $y \in Y$, $x \in X$, $c_i \in C_{p,i}$, then $y \in Y \cap G_2 \leqslant X \times C_{p,0}$. Hence one can assume that $y = 1$ and $xc_0 = c_1$. But now $x \in X \cap \cap C_p = 1$ and $c_0 = c_1 = 1$. Consequently $G = (Y \times C_{p,0}) \times C_{p,1}$. Since G is a VTA-group and $C_{p,1}$ has finite exponent, $C_{p,1}$ is finite. This gives the contradiction that C_p is finite.

LEMMA 4. *If G is a torsion VTA-group, then $Z(G)$ is finite and G has finite exponent.*

PROOF. By Lemma 3 and Corollary 1, it is sufficient to prove that the set π of prime divisors of orders of elements of G is finite. Let L denote the second centre of G. Then $|G\colon L|$ is finite and there is a finite normal subgroup N such that $G = NL$; since G' is finite, one can assume that $G' \leqslant N$. If π_0 is the set of prime divisors of $|N|$, then $G = N(L_{\pi_0} \times L_{\pi_0'}) = (NL_{\pi_0}) \times L_{\pi_0'}$. Now $(L_{\pi_0'})' \leqslant L_{\pi_0'} \cap G' = 1$, so $L_{\pi_0'}$ is abelian; therefore it is finite by the VTA-property.

However in general the torsion subgroup of the centre of a VTA-group can be infinite, as will be shown in § 5.

4. Necessary and sufficient conditions.

Let Q be a group and C an abelian group regarded as a trivial Q-module. Then there are natural left and right actions of $\operatorname{Aut} Q$ and $\operatorname{Aut} C$ respectively on $H^2(Q, C)$. If $C \rightarrowtail G \twoheadrightarrow Q$ is a central extension with cohomology class Δ, then a necessary and sufficient condition for there to exist an automorphism of G inducing in Q and C automorphisms $\varkappa$ and γ is that $\varkappa\Delta = \Delta\gamma$. If we make $H^2(Q, C)$ into a right $\operatorname{Aut} Q \times \operatorname{Aut} C$-module by means of the rule $\Delta(\varkappa, \gamma) = \varkappa^{-1}\Delta\gamma$, then the above condition is equivalent to

$$(\varkappa, \gamma) \in C_{\mathrm{Aut}\, Q \times \mathrm{Aut}\, C}(\Delta)\,.$$

These observations, the background to which may be found in [4], may be used to give necessary and sufficient conditions for a group to be a VTA-group.

THEOREM 1. *Let G be a group with centre C and central quotient group Q. Let Δ be the cohomology class of the extension $C \hookrightarrow G \twoheadrightarrow Q$. Then G is a VTA-group if and only if the following hold*:

(i) *G' is finite*;

(ii) *each primary component of C is finite*;

(iii) *$C_{\mathrm{Aut}\, Q \times \mathrm{Aut}\, C}(\Delta)$ is contained in* $\mathrm{Aut}_{vt}(Q) \times \mathrm{Aut}_{vt}(C)$.

PROOF. Let G be a VTA-group; then (i) and (ii) hold by Proposition 1 and Lemma 3. If $(\varkappa, \gamma)$ belongs to $C_{\mathrm{Aut}\, Q \times \mathrm{Aut}\, C}(\Delta)$, then $\varkappa$ and γ are induced by an automorphism of G. Hence

$$\varkappa \in \mathrm{Aut}_{vt}(Q) \quad \text{and} \quad \gamma \in \mathrm{Aut}_{vt}(C)\,.$$

Conversely, assume that the three conditions hold. Let α be an automorphism of G inducing $\varkappa$ in Q and γ in C; then $\varkappa \in \mathrm{Aut}_{vt}(Q)$ and $\gamma \in \mathrm{Aut}_{vt}(C)$ by (iii). Set $K/C = C_Q(\varkappa)$ and $L = C_C(\gamma)$. Then $|G:K|$ and $|C:L|$ are finite. The mapping $x \mapsto [x, \alpha]$ is a homomorphism from K onto $[K, \alpha]$ whose kernel contains L. Now condition (i) implies that G/C has finite exponent, whence so does G/L. Consequently $[K, \alpha]$ has finite exponent. Since $[K, \alpha] \leqslant C$, it follows that $[K, \alpha]$ is contained in T, the torsion subgroup of C. Since T has finite primary components by (ii), the subgroup $[K, \alpha]$ is finite. This shows that $|K: C_K(\alpha)|$ is finite, therefore $|G: C_G(\alpha)|$ is finite and G is a VTA-group, as claimed.

It is of course the third condition which is difficult to deal with. For torsion groups this can be simplified slightly.

THEOREM 2. *Let G be a torsion group with centre C and central quotient group Q. Let Δ be the cohomology class of the extension $C \hookrightarrow G \twoheadrightarrow Q$. Then G is a VTA-group if and only if the following con-*

ditions hold:

(i) *G' is finite;*

(ii) *C is finite;*

(iii) $St_{\mathrm{Aut}\,Q}(\Delta^{\mathrm{Aut}\,C})$, *the stabilizer in* Aut Q *of the* Aut C*-orbit containing Δ, is contained in* $\mathrm{Aut}_{vt}(Q)$.

The point to note here is that $\varkappa$ in Aut Q is induced by an automorphism of G if and only if there exists a γ in Aut C such that $\varkappa\Delta = \Delta\gamma$, that is, $\varkappa$ stabilizes $\Delta^{\mathrm{Aut}\,C}$. Theorem 2 is now a consequence of Theorem 1 and Lemma 4. (An example of an infinite torsion VTA-group is given in § 5.)

Finitely generated VTA-group admit a precise characterization.

THEOREM 3. *A finitely generated group G is a VTA-group if and only if it is either finite or the split extension of a finite group F by an infinite cyclic group $\langle x\rangle$ such that $\hat{x}$, the image of x in* Out F, *is not conjugate to its inverse.*

PROOF. Let G be a finitely generated infinite VTA-group, and write $C = Z(G)$ and $Q = G/C$. By Corollary 1 the group Q is finite, so C is finitely generated. Suppose that C had a free abelian direct factor of rank 2, say $\langle a\rangle\times\langle b\rangle$. If $|Q| = m$, then it is easy to see that $a\mapsto ab^m$, $b\mapsto b$ extends to an automorphism of G; but this cannot be virtually trivial. Consequently C has torsion-free rank 1. Since G' is finite, the elements of finite order form a finite subgroup F and G/F is infinitely cyclic. Write $G = \langle x\rangle F$ with $\langle x\rangle\cap F = 1$. If $\alpha\in\mathrm{Aut}\,G$ induces the inversion automorphism in G/F, then $C_G(\alpha)\leqslant F$ and $|G\colon C_G(\alpha)|$ is infinite. Therefore every automorphism of G acts trivially on G/F.

Conversely let $G = \langle x\rangle\ltimes F$ have the property that all automorphisms of G act trivially on G/F. Let $\alpha\in\mathrm{Aut}\,G$. Now G/C is finite where $C = Z(G)$, and $C = D\times E$ where D is finite of order d, say, and E is infinite cyclic. Hence there is an $m>0$ such that $x^m\in C^d = E^d$. Now C^d is characteristic in G and infinite cyclic, so $\langle x^m\rangle$ is α-invariant. Since α clearly cannot map x^m to its inverse, it follows that $x^m\in C_G(\alpha)$ and $|G\colon C_G(\alpha)|$ is finite. Thus G is a VTA-group.

It remains to decide when the inversion automorphism of Q lifts to G. A simple argument shows that this occurs if and only if $\hat{x}$ is conjugated to its inverse in Out F by some $\varphi\in\mathrm{Aut}\,F$.

REMARK. In order to construct all finitely generated VTA-groups one must select a finite group F with the property that not every element of $\operatorname{Out} F$ is conjugate to its inverse (or, equivalently, such that some irreducible $\mathbb{C}$-character of F is not real). If $\hat{\varphi}$ is such an element, lift $\hat{\varphi}$ in any way to an automorphism φ of F and form the semidirect product $\langle x\rangle \ltimes F$ of F with an infinite cyclic group $\langle x\rangle$ where x induces φ in F. The smallest possible F is the cyclic group of order 5; this gives rise to the VTA-group mentioned in § 1.

Hence the isomorphism classes of finitely generated VTA-groups are in one-one correspondence with pairs $(F, \hat{\varphi})$ where F is a finite group as above and $\hat{\varphi}$ is a conjugacy class of $\operatorname{Out} F$ not equal to its inverse.

5. Examples.

We shall now construct the examples of infinite VTA-group promised in earlier sections.

(A) *There is an infinite* VTA-*group which is a p-group with nilpotency class* 2.

For $p = 2$ the group constructed in [5], Prop. 3 will do. Let $p > 2$ and let G be the group with generators $a, b, x_1, x_2, \dots$ and relations

$$[x_{2i-1}, x_{2i}] = a\,, \qquad [x_{2i}, x_{2i+1}] = b\,, \qquad [x_i, x_j] = 1 \quad \text{if } i < j-1\,,$$

$$[a, x_i] = 1 = [b, x_i] = [a, b] = a^p = b^p\,,$$

$$x_1^p = ab\,, \qquad x_2^p = a\,, \qquad x_i^p = 1 \quad \text{if } i > 2\,.$$

One quickly sees that $G' = Z(G) = \langle a\rangle \times \langle b\rangle$ and $Q \equiv G/Z(G)$ is an infinite elementary abelian p-group. For $i = 1, 2, \dots$ put $M_i = \langle x_i\rangle G'$ and $C_i = \langle x_{i+2}, x_{i+3,\dots}\rangle G'$. It is easily seen that M_1 is the set of all $y \in G$ such that $|G : C_G(y)| = p$. Thus M_1 is characteristic in G, as is $C_1 = \Omega_1(C_G(M_1))$. The elements y of G for which $|M_1 : C_{M_1}(y)| = p = |C_1 : C_{C_1}(y)|$ are those of the form $x_1^r x_2^s z$ where $s \not\equiv 0 \pmod p$ and $z \in G'$. From this it follows that: $\langle x_1, x_2\rangle G'$ is characteristic in G. Among these y those that generate a cyclic normal subgroup of $\langle x_1, x_2\rangle G'$ are of the form $x_2^s z$; hence M_2 is characteristic in G.

Now suppose that $M_1, \dots, M_i$ are characteristic in G (where $i \geqslant 2$); then also $C_{i-1} = \Omega_1(C_G(M_1 \dots M_{i-1}))$ is characteristic. A little calculation shows that the elements y of C_{i-1} such that $|[y, C_{i-1}]| = p$ are those of the form $x_{i+1}^r z$ $(r \not\equiv 0 \pmod p),\ z \in G')$. It follows that M_{i+1} is characteristic in G.

Next let $\alpha \in \operatorname{Aut} G$. We have just shown that $x_i^\alpha \equiv x_i^{n_i} \pmod{G'}$ for all i and some integers n_i. Our relations imply $n_1 \equiv n_2 \equiv n_{2i-1} \cdot n_{2i} \equiv n_{2i} n_{2i+1}$, $(i = 1, 2, \dots)$, i.e. $n_j \equiv 1 \pmod p$ for all j. Thus every automorphism of G acts trivially on $Z(G)$ and on Q. Hence $|G : C_G(\alpha)| \leqslant p^2$ for every automorphism α. Notice also that $\operatorname{Aut} G \simeq \operatorname{Hom}(Q, Z(G))$, an elementary abelian p-group of cardinality $2^{\aleph_0}$.

(B) *There exists a* VTA*-group with finite torsion-free rank whose centre has infinite torsion subgroup.*

The construction falls into two parts; first we assign the central quotient, then we construct the centre.

Let Q be a finite group satisfying the conditions

(i) $Q_{ab} \equiv Q/Q'$ is not an elementary abelian 2-group;

(ii) no automorphism of Q induces an automorphism of order 2 in Q_{ab};

(iii) $Z(Q) = 1$.

Of course such groups abound; the simplest example is the holomorph of a cyclic group of order 5. Let σ be the set of all primes which do not divide the order of Q.

The centre of our group has to be chosen with some care. Let $\langle t_p \rangle$ be a cyclic group of order p, written additively, and define

$$T = \operatorname*{Dr}_{p \in \sigma} \langle t_p \rangle \qquad \text{and} \qquad T^* = \operatorname*{Cr}_{p \in \sigma} \langle t_p \rangle ,$$

the direct and cartesian sums. We shall construct a group C such that

(i)′ $T \leqslant C \leqslant T^*$;

(ii)′ $\bar{C} = C/T$ has automorphism group of order 2;

(iii)′ $\operatorname{Aut} C = \operatorname{Aut}_{vt}(C) \times \langle -1 \rangle$.

Here of course «-1» refers to the automorphism $c \mapsto -c$ of C.

Assuming this C to be constructed, one regards it as a trivial Q-module. Then by the Universal Coefficients Theorem

$$H^2(Q, C) \cong \mathrm{Ext}\,(Q_{ab}, \bar{C})\,.$$

This group has an element Δ with order > 2 by properties (i) and (ii)$'$. Let

$$C \rightarrowtail G \twoheadrightarrow Q$$

be a central extension with cohomology class Δ. Then $Z(G) = C$ since $Z(Q) = 1$. Let α in Aut G induce automorphisms $\varkappa$ and γ in Q and C respectively; then $\varkappa\Delta = \Delta\gamma$. If $\gamma \notin \mathrm{Aut}_{vt}(C)$, then by (iii)$'$ γ must induce -1 on $\bar{C}$; in this case $\varkappa\Delta = -\Delta$ by the above (natural) isomorphism. But $\varkappa$ cannot induce an automorphism of even order in Q_{ab}, so we reach a contradiction. It follows that $\gamma \in \mathrm{Aut}_{vt}(C)$ and $\alpha \in \mathrm{Aut}_{vt}(G)$ since Q is finite.

Construction of C. It remains to find an abelian group C satisfying (i)$'$, (ii)$'$, (iii)$'$. First we specify $\bar{C}$. Let

$$\sigma = \pi \cup \varrho$$

be a partition of σ into two infinite sets of primes. Define $x^*, y^* \in T^*$ by

$$(x^*)_p = t_p\,, \quad (y^*)_p = 0 \quad \text{if } p \in \pi$$

$$(x^*)_p = 0\,, \quad (y^*) = t_p \quad \text{if } p \in \varrho\,.$$

Also let q be a prime not in σ. Define A_1, A_2, A_3 to be the subrings of $\mathbb{Q}$ generated by π^{-1}, ϱ^{-1}, q^{-1} respectively.

Now $u = x^* + T$, $v = y^* + T$ are independent vectors in the $\mathbb{Q}$-vector space $\bar{T} = T^*/T$, and we may define our group C by requiring $T \leqslant C \leqslant T^*$ and

$$\bar{C} \equiv C/T = A_1 u + A_2 v + A_3(u + v)\,.$$

It is straightforward to check that $\bar{C}$ has automorphism group of order 2. It remains to check property (iii)$'$. Let $\gamma \in \mathrm{Aut}\, C$; replacing γ by $-\gamma$ if necessary, we can assume that γ acts trivially on $\bar{C}$. The task is now to show that $\gamma \in \mathrm{Aut}_{vt}(C)$. Let $p \in \pi$. There is a unique coset $b_p + T \in \bar{C}$ such that $x^* - pb_p \in T$; notice that, since pb_p obviously has trivial p-component, the p-component of $x^* - pb_p$ is t_p, and we may as well assume that $x^* - pb_p = t_p$. Now $b_p(\gamma - 1) = a_p$ (say) is in T, and one obtains

$$t_p(\gamma - 1) = x^*(\gamma - 1) - pa_p .$$

Looking at p-components one gets

$$t_p(\gamma - 1) \in \langle x^*(\gamma - 1)\rangle \quad \text{for all } p \in \pi .$$

This means that γ fixes almost all t_p with p in π; a similar conclusion holds for ϱ. Modifying γ by a virtually trivial automorphism of C, one can assume that γ operates trivially on T. However it is easily checked that Hom $(\bar{C}, T)$ is periodic, and this implies that $C(\gamma - 1)$ is finite and $|C : C_C(\gamma)|$ is finite. Therefore $\gamma \in \mathrm{Aut}_{vt}(C)$.

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