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Centralizers and Lie ideals

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Centralizers and Lie Ideals.

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SUMMARY. - Let R be an associative ring, $Z(R)$ its center and $T(U) = \{a \in R | au^n = u^n a,\ n = n(u, a) \geq 1, \text{ all } u \in U\}$, where U is a non central Lie ideal of R. We prove that if R is a prime ring of characteristic not 2 with no nil right ideals, then either $T(U) = Z(R)$ or R is an order in a simple algebra of dimension at most 4 over its center.

Let R be an associative ring, $Z(R)$ its center. The hypercenter theorem [4] asserts that in a ring with no nonzero nil ideals an element commuting with a suitable power of every element of the ring must be central.

In this note we want to extend this result to noncentral Lie ideals in case R is a prime ring of characteristic not 2 with no nil right ideals.

Let $T(U) = \{a \in R: au^n = u^n a,\ n = n(u, a) \geq 1, \text{ all } u \in U\}$, where U is a noncentral Lie ideal of R, then one cannot expect the same conclusion of [4], as the following example shows:

EXAMPLE. Let $R = F_2$, the 2×2 matrices over a field F,

$$U = [R, R] = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in F \right\}.$$

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Then U is a noncentral Lie ideal of R and $u^2 \in Z(R)$ for every element $u \in U$, therefore $T(U) = R$, but $Z(R) \neq R$.

Then making use of a result of Felzenszwalb and Giambruno [2], we shall prove the following:

THEOREM. *Let R be a prime ring of characteristic not 2 with no nil right ideals, U a noncentral Lie ideal of R. Then either $T(U) = Z(R)$ or R is an order in a simple algebra of dimension at most 4 over its center.*

Notice that the conclusion of the theorem is false if one removes the assumption of primeness. In fact, let F_k be the ring of $k \times k$ matrices over a field F. If $R = \prod_{k=2}^{\infty} F_k$, then R is a semisimple ring. Take $U = \prod_{k=2}^{\infty} U_k$, where $U_2 = [F_2, F_2]$ and $U_k = F_k$ for $k > 2$, then U is a noncentral Lie ideal of R. Let $a = (c, 0, 0, \ldots)$ with $c \notin Z(F_2)$. Then $a \in T(U)$, but $a \notin Z(R)$ and moreover it is clear that R does not satisfy any polynomial identity.

For $a, b \in R$ set $[a, b] = ab - ba$ and for subsets U, $V \subset R$, let $[U, V]$ be the additive subgroup generated by all $[u, v]$ for $u \in U$ and $v \in V$. We recall that a Lie ideal U of R is an additive subgroup of R such that $[U, R] \subset U$.

In all that follows, unless otherwise stated, R will be a 2-torsion free ring, $Z = Z(R)$ the center of R, $J(R)$ the Jacobson radical of R, U a noncentral Lie ideal of R (i.e. $U \not\subset Z$) and

$$T(U) = \{a \in R \colon au^n = u^n a,\ n = n(u, a) \geqslant 1,\ \text{all } u \in U\}.$$

We start with

LEMMA. *If R is a primitive ring then either $T(U) = Z(R)$ or R is a simple algebra of dimension at most 4 over its center.*

PROOF. If R is primitive, then R is a dense ring of linear transformations on a vector space V over a division ring D. If $\dim_D V \leqslant 2$, then R is simple. Since U is a noncentral Lie ideal of R, by Theorem 1.5 of [3], we may assume that $U = [R, R]$. Therefore

$$T(U) = \{a \in R \colon a(xy - yx)^n = (xy - yx)^n a,\quad n = n(a, x, y) \geqslant 1,\ \text{all } x, y \in R\}.$$

By a result of Felzenszwalb and Giambruno [2, Theorem 1], then we have the desired conclusion.

Suppose now that $\dim_D V > 2$. Since R is prime of characteristic different from 2, by [1, Lemma 1] there exists a nonzero ideal I such that $[I, R] \subset U$ and $[I, R] \not\subset Z$. It is also well known that I acts densely on V over D (see [5]).

Let $a \neq 0$ be an element of $T(U)$ and suppose that for some $v \in V$, the vectors v and va are linearly independent over D. Since $\dim_D V > 2$, there exists a vector v_3 in V such that $v_1 = v$, $v_2 = va$, v_3 are linearly independent over D.

The density of R and I on V gives $r_2 \in R$ and $i \in I$ with

$$v_1 r_2 = 0\,, \qquad v_2 r_2 = v_3\,, \qquad v_3 r_2 = 0\,,$$

$$v_1 i = 0\,, \qquad v_2 i = 0\,, \qquad v_3 i = v_2\,.$$

Clearly a commutes with $(ir_2 - r_2 i)^m$, for a suitable $m \geqslant 1$. Since $0 = v_1(ir_2 - r_2 i)$ we get:

$$0 = v_1(ir_2 - r_2 i)^m a = v_1 a (ir_2 - r_2 i)^m =$$

$$= v_2(ir_2 - r_2 i)(ir_2 - r_2 i)^{m-1} = - v_2(ir_2 - r_2 i)^{m-1} = \ldots = \pm v_2\,;$$

a contradiction.

Thus given $v \in V$, v and va are linearly dependent over D. As in [4, Lemma 2] it follows that a is central. In other words, if $\dim_D V > 2$, then $T(U) = Z$. With this the lemma is proved.

We recall that a semisimple ring is a subdirect product of primitive rings R_α. For every α, let P_α be a primitive ideal of R such that $R_\alpha \cong R/P_\alpha$. Since $J(R) = 0$, then $\bigcap_\alpha P_\alpha = 0$. Remark that since R is 2-torsion free, we may assume that the homomorphic images R_α are still of characteristic different from 2. In fact, put $A \bigcap_{2R \subset P_\alpha} P_\alpha$ and $B = \bigcap_{2R \not\subset P_\alpha} P_\alpha$ and let $x \in B$; then $2x \in B$ and also $2x \in 2R \subset \bigcap_{2R \subset P_\alpha} P_\alpha = A$, therefore $2x \in A \cap B = 0$. Since R is 2-torsion free $x = 0$ and so we have proved that $B = 0$. In this way $2R \not\subset P_\alpha$ (and therefore char $R/P_\alpha \neq 2$) for every α. Now we are ready to prove the following:

THEOREM. *Let R be a prime ring of characteristic not 2 with no nonzero nil right ideals, U a noncentral Lie ideal of R. Then either $T(U) = Z(R)$ or R is an order in a simple algebra of dimension at most 4 over its center.*

PROOF. Suppose R is semisimple. If U_α is the image of U in R_α, then U_α is a Lie ideal of R_α. Let $\mathcal{F} = \{P_\alpha : U_\alpha \subset Z(R_\alpha)\}$. Set $A = \bigcap_{P_\alpha \in \mathcal{F}} P_\alpha$ and $B = \bigcap_{P_\alpha \notin \mathcal{F}} P_\alpha$. Since R is prime and $AB \subset A \cap B = 0$, we must have either $A = 0$ or $B = 0$. If $A = 0$, then $U \subset Z$, a contradiction. Thus $B = 0$ and so for every α, U_α is a noncentral Lie ideal of R_α.

For each α let T_α be the image of $T(U)$ in R_α. Since $U_\alpha \not\subset Z(R_\alpha)$, $T_\alpha \subset T(U_\alpha)$ for each α and by the previous Lemma we get either $T_\alpha \subset$ $\subset Z(R_\alpha)$ or R_α satisfies S_4, the standard identity in four variables.

Put $I = \{\cap P_\alpha : T_\alpha \subset Z(R_\alpha)\}$ and $J = \{\cap P_\alpha : T_\alpha \not\subset Z(R_\alpha)\}$. Since R is prime and $IJ = 0$ we must have either $I = 0$ or $J = 0$.

If $I = 0$, we conclude that $T(U) = Z(R)$, the desired conclusion. If $J = 0$ then, for every α, R_α satisfies S_4 and so R satisfies S_4; even in this case we are done.

Therefore we may assume that $J(R) \neq 0$. As we remarked before, there exists a nonzero ideal I of R such that $[I, R] \subset U$. Since R is prime, $I \cap J(R)$ is a nonzero ideal of R.

Let $T = T([I, R])$. If T centralizes $J(R) \cap I$, then, since the centralizer of a nonzero ideal in a prime ring is equal to the centre of the ring, $T \subset C_R(J(R) \cap I) = Z(R)$.

Suppose then that $a \in T$, $x \in J \cap I$ and $ax - xa \neq 0$. Now

$$0 \neq (ax - xa)(1 + x)^{-1} = a - (1 + x)\, a(1 + x)^{-1} \in T\,.$$

Therefore $0 \neq (ax - xa)(1 + x)^{-1}$ is in $T \cap I \cap J$ and so $T \cap I \cap J \neq 0$.

Consider the following subset of R:

$$T(I) = \{\alpha \in I : a[x, y]^n = [x, y]^n a,\ n = n(a, x, y) \geqslant 1,\ \text{all } x, y \in I\}\,.$$

Since I as a ring satisfies the same hypotheses placed on R, by Theorem 1 of [2] either $T(I) \subset Z(I) \subset Z(R)$ or I satisfies S_4.

If the first possibility occurs, since $0 \neq T \cap J \cap I \subset T(I) \subset Z(R)$ we have $(ax - xa)(1 + x)^{-1} \in Z$. Also, if $b \in T$, then

$$b(ax - xa)(1 + x)^{-1} \in T \cap J \cap I \subset Z\,;$$

since both $0 \neq (ax - xa)(1 + x)^{-1} \in Z$ and $b(ax - xa)(1 + x)^{-1} \in Z$ and since elements in Z are not zero divisors in R, these relations would imply that $b \in Z$ and we would get $T = T([I, R]) = Z(R)$ and so $T(U) \subset Z(R)$.

Suppose now $T(U) \neq Z(R)$. By the above $T(I) \neq Z(I)$, then I and so R is an order in a simple algebra of dimension at most 4 over its center, the desired conclusion.

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