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## Centralizers and Lie Ideals.

LUISA CARINI (\*)

SUMMARY. - Let  $R$  be an associative ring,  $Z(R)$  its center and  $T(U) = \{a \in R \mid au^n = u^n a, n = n(u, a) \geq 1, \text{ all } u \in U\}$ , where  $U$  is a non central Lie ideal of  $R$ . We prove that if  $R$  is a prime ring of characteristic not 2 with no nil right ideals, then either  $T(U) = Z(R)$  or  $R$  is an order in a simple algebra of dimension at most 4 over its center.

Let  $R$  be an associative ring,  $Z(R)$  its center. The hypercenter theorem [4] asserts that in a ring with no nonzero nil ideals an element commuting with a suitable power of every element of the ring must be central.

In this note we want to extend this result to noncentral Lie ideals in case  $R$  is a prime ring of characteristic not 2 with no nil right ideals.

Let  $T(U) = \{a \in R: au^n = u^n a, n = n(u, a) \geq 1, \text{ all } u \in U\}$ , where  $U$  is a noncentral Lie ideal of  $R$ , then one cannot expect the same conclusion of [4], as the following example shows:

EXAMPLE. Let  $R = F_2$ , the  $2 \times 2$  matrices over a field  $F$ ,

$$U = [R, R] = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in F \right\}.$$

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Then  $U$  is a noncentral Lie ideal of  $R$  and  $u^2 \in Z(R)$  for every element  $u \in U$ , therefore  $T(U) = R$ , but  $Z(R) \neq R$ .

Then making use of a result of Felzenszwalb and Giambruno [2], we shall prove the following:

**THEOREM.** *Let  $R$  be a prime ring of characteristic not 2 with no nil right ideals,  $U$  a noncentral Lie ideal of  $R$ . Then either  $T(U) = Z(R)$  or  $R$  is an order in a simple algebra of dimension at most 4 over its center.*

Notice that the conclusion of the theorem is false if one removes the assumption of primeness. In fact, let  $F_k$  be the ring of  $k \times k$  matrices over a field  $F$ . If  $R = \prod_{k=2}^{\infty} F_k$ , then  $R$  is a semisimple ring. Take  $U = \prod_{k=2}^{\infty} U_k$ , where  $U_2 = [F_2, F_2]$  and  $U_k = F_k$  for  $k > 2$ , then  $U$  is a noncentral Lie ideal of  $R$ . Let  $a = (c, 0, 0, \dots)$  with  $c \notin Z(F_2)$ . Then  $a \in T(U)$ , but  $a \notin Z(R)$  and moreover it is clear that  $R$  does not satisfy any polynomial identity.

For  $a, b \in R$  set  $[a, b] = ab - ba$  and for subsets  $U, V \subset R$ , let  $[U, V]$  be the additive subgroup generated by all  $[u, v]$  for  $u \in U$  and  $v \in V$ . We recall that a Lie ideal  $U$  of  $R$  is an additive subgroup of  $R$  such that  $[U, R] \subset U$ .

In all that follows, unless otherwise stated,  $R$  will be a 2-torsion free ring,  $Z = Z(R)$  the center of  $R$ ,  $J(R)$  the Jacobson radical of  $R$ ,  $U$  a noncentral Lie ideal of  $R$  (i.e.  $U \not\subset Z$ ) and

$$T(U) = \{a \in R: au^n = u^n a, n = n(u, a) \geq 1, \text{ all } u \in U\}.$$

We start with

**LEMMA.** *If  $R$  is a primitive ring then either  $T(U) = Z(R)$  or  $R$  is a simple algebra of dimension at most 4 over its center.*

**PROOF.** If  $R$  is primitive, then  $R$  is a dense ring of linear transformations on a vector space  $V$  over a division ring  $D$ . If  $\dim_D V < 2$ , then  $R$  is simple. Since  $U$  is a noncentral Lie ideal of  $R$ , by Theorem 1.5 of [3], we may assume that  $U = [R, R]$ . Therefore

$$T(U) = \{a \in R: a(xy - yx)^n = (xy - yx)^n a, \\ n = n(a, x, y) \geq 1, \text{ all } x, y \in R\}.$$

By a result of Felzenszwalb and Giambruno [2, Theorem 1], then we have the desired conclusion.

Suppose now that  $\dim_D V > 2$ . Since  $R$  is prime of characteristic different from 2, by [1, Lemma 1] there exists a nonzero ideal  $I$  such that  $[I, R] \subset U$  and  $[I, R] \not\subset Z$ . It is also well known that  $I$  acts densely on  $V$  over  $D$  (see [5]).

Let  $a \neq 0$  be an element of  $T(U)$  and suppose that for some  $v \in V$ , the vectors  $v$  and  $va$  are linearly independent over  $D$ . Since  $\dim_D V > 2$ , there exists a vector  $v_3$  in  $V$  such that  $v_1 = v, v_2 = va, v_3$  are linearly independent over  $D$ .

The density of  $R$  and  $I$  on  $V$  gives  $r_2 \in R$  and  $i \in I$  with

$$\begin{aligned} v_1 r_2 &= 0, & v_2 r_2 &= v_3, & v_3 r_2 &= 0, \\ v_1 i &= 0, & v_2 i &= 0, & v_3 i &= v_2. \end{aligned}$$

Clearly  $a$  commutes with  $(ir_2 - r_2 i)^m$ , for a suitable  $m \geq 1$ . Since  $0 = v_1(ir_2 - r_2 i)$  we get:

$$\begin{aligned} 0 &= v_1(ir_2 - r_2 i)^m a = v_1 a(ir_2 - r_2 i)^m = \\ &= v_2(ir_2 - r_2 i)(ir_2 - r_2 i)^{m-1} = -v_2(ir_2 - r_2 i)^{m-1} = \dots = \pm v_2; \end{aligned}$$

a contradiction.

Thus given  $v \in V$ ,  $v$  and  $va$  are linearly dependent over  $D$ . As in [4, Lemma 2] it follows that  $a$  is central. In other words, if  $\dim_D V > 2$ , then  $T(U) = Z$ . With this the lemma is proved.

We recall that a semisimple ring is a subdirect product of primitive rings  $R_\alpha$ . For every  $\alpha$ , let  $P_\alpha$  be a primitive ideal of  $R$  such that  $R_\alpha \cong \cong R/P_\alpha$ . Since  $J(R) = 0$ , then  $\bigcap_\alpha P_\alpha = 0$ . Remark that since  $R$  is 2-torsion free, we may assume that the homomorphic images  $R_\alpha$  are still of characteristic different from 2. In fact, put  $A = \bigcap_{2R \not\subset P_\alpha} P_\alpha$  and  $B = \bigcap_{2R \subset P_\alpha} P_\alpha$  and let  $x \in B$ ; then  $2x \in B$  and also  $2x \in 2R \subset \bigcap_{2R \subset P_\alpha} P_\alpha = A$ , therefore  $2x \in A \cap B = 0$ . Since  $R$  is 2-torsion free  $x = 0$  and so we have proved that  $B = 0$ . In this way  $2R \not\subset P_\alpha$  (and therefore  $\text{char } R/P_\alpha \neq 2$ ) for every  $\alpha$ . Now we are ready to prove the following:

**THEOREM.** *Let  $R$  be a prime ring of characteristic not 2 with no nonzero nil right ideals,  $U$  a noncentral Lie ideal of  $R$ . Then either  $T(U) = Z(R)$  or  $R$  is an order in a simple algebra of dimension at most 4 over its center.*

**PROOF.** Suppose  $R$  is semisimple. If  $U_\alpha$  is the image of  $U$  in  $R_\alpha$ , then  $U_\alpha$  is a Lie ideal of  $R_\alpha$ . Let  $\mathcal{F} = \{P_\alpha: U_\alpha \subset Z(R_\alpha)\}$ . Set  $A = \bigcap_{P_\alpha \in \mathcal{F}} P_\alpha$  and  $B = \bigcap_{P_\alpha \notin \mathcal{F}} P_\alpha$ . Since  $R$  is prime and  $AB \subset A \cap B = 0$ , we must have either  $A = 0$  or  $B = 0$ . If  $A = 0$ , then  $U \subset Z$ , a contradiction. Thus  $B = 0$  and so for every  $\alpha$ ,  $U_\alpha$  is a noncentral Lie ideal of  $R_\alpha$ .

For each  $\alpha$  let  $T_\alpha$  be the image of  $T(U)$  in  $R_\alpha$ . Since  $U_\alpha \not\subset Z(R_\alpha)$ ,  $T_\alpha \subset T(U_\alpha)$  for each  $\alpha$  and by the previous Lemma we get either  $T_\alpha \subset Z(R_\alpha)$  or  $R_\alpha$  satisfies  $S_4$ , the standard identity in four variables.

Put  $I = \{\bigcap P_\alpha: T_\alpha \subset Z(R_\alpha)\}$  and  $J = \{\bigcap P_\alpha: T_\alpha \not\subset Z(R_\alpha)\}$ . Since  $R$  is prime and  $IJ = 0$  we must have either  $I = 0$  or  $J = 0$ .

If  $I = 0$ , we conclude that  $T(U) = Z(R)$ , the desired conclusion. If  $J = 0$  then, for every  $\alpha$ ,  $R_\alpha$  satisfies  $S_4$  and so  $R$  satisfies  $S_4$ ; even in this case we are done.

Therefore we may assume that  $J(R) \neq 0$ . As we remarked before, there exists a nonzero ideal  $I$  of  $R$  such that  $[I, R] \subset U$ . Since  $R$  is prime,  $I \cap J(R)$  is a nonzero ideal of  $R$ .

Let  $T = T([I, R])$ . If  $T$  centralizes  $J(R) \cap I$ , then, since the centralizer of a nonzero ideal in a prime ring is equal to the centre of the ring,  $T \subset C_R(J(R) \cap I) = Z(R)$ .

Suppose then that  $a \in T$ ,  $x \in J \cap I$  and  $ax - xa \neq 0$ . Now

$$0 \neq (ax - xa)(1 + x)^{-1} = a - (1 + x)a(1 + x)^{-1} \in T.$$

Therefore  $0 \neq (ax - xa)(1 + x)^{-1}$  is in  $T \cap I \cap J$  and so  $T \cap I \cap J \neq 0$ .

Consider the following subset of  $R$ :

$$T(I) = \{\alpha \in I: a[x, y]^n = [x, y]^n a, n = n(a, x, y) \geq 1, \text{ all } x, y \in I\}.$$

Since  $I$  as a ring satisfies the same hypotheses placed on  $R$ , by Theorem 1 of [2] either  $T(I) \subset Z(I) \subset Z(R)$  or  $I$  satisfies  $S_4$ .

If the first possibility occurs, since  $0 \neq T \cap J \cap I \subset T(I) \subset Z(R)$  we have  $(ax - xa)(1 + x)^{-1} \in Z$ . Also, if  $b \in T$ , then

$$b(ax - xa)(1 + x)^{-1} \in T \cap J \cap I \subset Z;$$

since both  $0 \neq (ax - xa)(1 + x)^{-1} \in Z$  and  $b(ax - xa)(1 + x)^{-1} \in Z$  and since elements in  $Z$  are not zero divisors in  $R$ , these relations would imply that  $b \in Z$  and we would get  $T = T([I, R]) = Z(R)$  and so  $T(U) \subset Z(R)$ .

Suppose now  $T(U) \neq Z(R)$ . By the above  $T(I) \neq Z(I)$ , then  $I$  and so  $R$  is an order in a simple algebra of dimension at most 4 over its center, the desired conclusion.

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