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A Note on the Construction of the Proximate Type of an Entire Dirichlet Series with Index-Pair (p, q) .

H. M. SRIVASTAVA (*) - H. S. KASANA (*)

SUMMARY. - Considering a natural extension of the notion of Lindelöf's proximate order of an entire function $f(s)$ ($s = \sigma + it$), A. G. Azpeitia [Trans. Amer. Math. Soc., **104** (1962), pp. 495-501] indicated how one can establish his propositions of existence of linear proximate orders $R(\sigma)$ and linear lower proximate orders $L(\sigma)$ for $f(s)$ defined by a Dirichlet series. Motivated by Azpeitia's work, R. S. L. Srivastava and P. Singh [J. Math. (Jabalpur), **2** (1966), pp. 3-10] proved the corresponding existence theorem for the proximate type $T(\sigma)$ of an entire Dirichlet series representing $f(s)$. For an interesting generalization of this theorem to hold true for an entire Dirichlet series with index-pair (p, q) , which is due essentially to H. S. Kasana [J. Math. Anal. Appl., **105** (1985), pp. 445-451], a remarkably simple (and markedly different) construction of the proximate type $T(\sigma)$ is presented here. The main theorem established here applies to a much larger class of entire Dirichlet series with index-pair (p, q) than that considered earlier.

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1. Introduction and definitions.

Let $f(s)$ be an entire function defined by a Dirichlet series

$$(1.1) \quad f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n), \quad \text{with} \quad \limsup_{n \rightarrow \infty} \left\{ \frac{\log n}{\lambda_n} \right\} < \infty,$$

$$(s = \sigma + it; 0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_n \uparrow \infty),$$

which is absolutely convergent for all s . Denote by $M(\sigma)$ the maximum modulus of $f(s)$ so that

$$(1.2) \quad M(\sigma) \equiv M(\sigma, f) = \sup_{-\infty < t < \infty} |f(\sigma + it)|.$$

We begin by recalling the concept of the index-pair (p, q) , the (p, q) -order, and the (p, q) -type of an entire Dirichlet series $f(s)$ for integers p and q such that

$$(1.3) \quad p \geq q + 1 \geq 1$$

(see [2] and [3]).

DEFINITION 1. An entire function $f(s)$, defined by (1.1), is said to be of (p, q) -order ρ if

$$(1.4) \quad \rho \equiv \rho(p, q) = \limsup_{\sigma \rightarrow \infty} \left\{ \frac{\log^{[p]} M(\sigma)}{\log^{[q]} \sigma} \right\},$$

where $M(\sigma)$ is given by (1.2), and (for convenience)

$$(1.5) \quad \log^{[m]} x = \exp^{[-m]} x = \log(\log^{[m-1]} x) = \exp(\exp^{[-m-1]} x)$$

$$(m = 0, \pm 1, \pm 2, \dots),$$

provided that

$$(1.6) \quad 0 < \log^{[m-1]} x < \infty \quad (m = 1, 2, 3, \dots),$$

with, of course,

$$(1.7) \quad \log^{[0]} x = \exp^{[0]} x = x.$$

DEFINITION 2. An entire function $f(s)$, defined by (1.1) and having the (p, q) -order ϱ ($b < \varrho < \infty$), is said to be of (p, q) -type τ if

$$(1.8) \quad \tau \equiv \tau(p, q) = \limsup_{\sigma \rightarrow \infty} \left\{ \frac{\log^{[p-1]} M(\sigma)}{(\log^{[q-1]} \sigma)^e} \right\},$$

where $b = 0$ if $p > q + 1$, and $b = 1$ if $p = q + 1$.

DEFINITION 3. For ϱ and b given by Definitions 1 and 2, respectively, an entire Dirichlet series (1.1) is said to be of index-pair (p, q) for integers p and q constrained by (1.3) if

$$b < \varrho(p, q) < \infty$$

and $\varrho(p-1, q-1)$ is not a non-zero finite number.

The notion of Lindelöf's proximate order of an entire function (*cf.*, *e.g.*, [6, pp. 64-67]) has been extended, in a natural way, to entire Dirichlet series. As a matter of fact, Azpeitia [1, p. 495] has obtained some interesting propositions of existence of linear proximate orders $R(\sigma)$ and linear lower proximate orders $L(\sigma)$ for $f(s)$ defined by (1.1), and Srivastava and Singh [5, p. 5, Theorem 1] have proved the corresponding existence theorem for the proximate type $T(\sigma)$ of the entire Dirichlet series (1.1). Recently, Kasana [4, p. 447, Theorem 1] generalized this theorem by establishing the existence of a real-valued function $T(\sigma)$, called the (p, q) -proximate type, for an entire Dirichlet series $f(s)$ with the index-pair (p, q) , the (p, q) -order ϱ ($b < \varrho < \infty$), and the (p, q) -type τ ($0 < \tau < \infty$). The object of the present note is to give a remarkably simple (and markedly different) construction of $T(\sigma)$ in this general case. Our proof, presented in Section 3 below, applies to a much wider class of entire Dirichlet series with index-pair (p, q) ; indeed, for an entire function of (p, q) -type 0, it provides a considerable improvement over Kasana's theorem.

2. The main existence theorem.

The following result of existence of a (p, q) -proximate type of an entire Dirichlet series (1.1) with index-pair (p, q) is an interesting generalization of Kasana's theorem [4, p. 447, Theorem 1]:

THEOREM. For every entire Dirichlet series $f(s)$, defined by (1.1) and having the index-pair (p, q) , the (p, q) -order ρ ($b < \rho < \infty$), and the (p, q) -type τ ($0 \leq \tau \leq \infty$), there exists a real-valued function $T(\sigma)$, called the (p, q) -proximate type of $f(s)$, which, for a given number a ($0 < a < \infty$), satisfies each of the following conditions:

(i) $T(\sigma)$ is continuous and piecewise differentiable for

$$\sigma \in [\sigma_0, \infty), \quad \sigma_0 > \exp^{[q-2]1};$$

(ii) $T(\sigma) \rightarrow \tau$ as $\sigma \rightarrow \infty$;

(iii) $(A_{[q-1]}(\sigma)T'(\sigma))/T(\sigma) \rightarrow 0$ as $\sigma \rightarrow \infty$, where $T'(\sigma)$ is to be interpreted as either $T'(\sigma-0)$ or $T'(\sigma+0)$ whenever these derivatives are unequal, and

$$(iv) \quad \limsup_{\sigma \rightarrow \infty} \left\{ \frac{\log^{[p-2]} M(\sigma)}{\exp\{(\log^{[q-1]}\sigma)^e T(\sigma)\}} \right\} = a,$$

where, for convenience,

$$(2.1) \quad A_{[m]}(x) = \prod_{j=0}^m \log^{[j]} x.$$

REMARK. For entire Dirichlet series of *finite positive* (p, q) -type, our assertion (iii) is equivalent to the corresponding assertion due to Kasana [4, p. 447, Eq. (1.3)].

Our demonstration of this general theorem, given in Section 3 below, is rather simple and is markedly different from the proof presented earlier by Kasana [4, pp. 447-449]. Moreover, it includes a much wider class of entire Dirichlet series with index-pair (p, q) .

3. Construction of the (p, q) -proximate type $T(\sigma)$.

We shall show that the properties asserted by the theorem are possessed by $T(\sigma)$ in each of the following possible alternatives.

If there exists a positive number l such that

$$(3.1) \quad \log^{[p]} M(\sigma) < \rho \log^{[q]} \sigma \quad \text{when} \quad \sigma > l,$$

then, for a given number a ($0 < a < \infty$), we define

$$(3.2a) \quad \varphi_a^p(\sigma) = \begin{cases} \sup_{t \leq t \leq \sigma} \left\{ \frac{\log^{[3]}(\log^{[p-3]} M(t))^{1/a}}{\log^{[a]} t} \right\} - \varrho & (\sigma > l) \end{cases}$$

$$(3.2b) \quad \varphi_a^p(\sigma) = \begin{cases} \varphi_a^p(l) & (0 < \sigma \leq l) . \end{cases}$$

Otherwise we define

$$(3.2c) \quad \varphi_a^p(\sigma) = \sup_{t \geq \sigma} \left\{ \frac{\log^{[3]}(\log^{[p-3]} M(t))^{1/a}}{\log^{[a]} t} \right\} - \varrho .$$

The following cases arise naturally:

Case 1. Let

$$(3.3) \quad \limsup_{\sigma \rightarrow \infty} \{ \varphi_a^p(\sigma) \log^{[a]} \sigma \} = -\infty ,$$

where $\varphi_a^p(\sigma)$ is defined by (3.2a). In this case we set

$$(3.4) \quad T(\sigma) = \sup_{t \geq \sigma} \{ \exp(\varphi_a^p(t) \log^{[a]} t) \} .$$

Case 2. Let

$$(3.5) \quad \limsup_{\sigma \rightarrow \infty} \{ \varphi_a^p(\sigma) \log^{[a]} \sigma \} = \infty .$$

In this case we set

$$(3.6) \quad T(\sigma) = \sup_{t \leq \sigma} \{ \exp(\varphi_a^p(t) \log^{[a]} t) \} .$$

Case 3. Let

$$(3.7) \quad \limsup_{\sigma \rightarrow \infty} \{ \varphi_a^p(\sigma) \log^{[a]} \sigma \} = \log \delta \quad (0 < \delta < \infty) .$$

In this case we set

$$(3.8) \quad T(\sigma) = \delta + \sup_{t_1 \leq t \leq \sigma} \left\{ \frac{\log \left(\sup_{x \geq t} \left\{ \frac{\alpha^{-1} \log^{[p-2]} M(x)}{\exp(\delta(\log^{[a-1]} x)^e)} \right\} \right)}{(\log^{[a-1]} t)^e} \right\} , \quad \text{if } \gamma = 0 ,$$

where l_1 is such that

$$(3.9) \quad \log^{[p-2]} M(\sigma) < a \exp(\delta(\log^{[\alpha-1]} \sigma)^e)$$

for all $\sigma > l_1$;

$$(3.10) \quad T(\sigma) = \delta + \frac{\log \gamma}{\log^{[\alpha-1]} \sigma}, \quad \text{if } 0 < \gamma < \infty;$$

and

$$(3.11) \quad T(\sigma) = \delta + \sup_{t \geq \sigma} \left\{ \frac{\log \left(\sup_{x \leq t} \left\{ \frac{a^{-1} \log^{[p-2]} M(x)}{\exp(\delta(\log^{[\alpha-1]} x)^e)} \right\} \right)}{(\log^{[\alpha-1]} t)^e} \right\}, \quad \text{if } \gamma = \infty,$$

where, for convenience,

$$(3.12) \quad \gamma = \limsup_{\sigma \rightarrow \infty} \left\{ \frac{\log^{[p-2]} M(\sigma)}{\exp(\delta(\log^{[\alpha-1]} \sigma)^e)} \right\}.$$

We note here that, if $f(s)$ is of *finite positive* (p, q) -type, then $\delta = \tau$.

We now illustrate our general approach by establishing the theorem for the case corresponding to (3.8). For this case, if we let

$$(3.13) \quad \mu(\sigma) = \sup_{t \geq \sigma} \left\{ \frac{a^{-1} \log^{[p-2]} M(t)}{\exp(\delta(\log^{[\alpha-1]} t)^e)} \right\} \quad (l > \exp^{[\alpha-2]} 1),$$

then it is easily verified that $\mu(\sigma)$ is continuous and non-increasing, and that

$$\mu(\sigma) \rightarrow 0 \quad \text{as } \sigma \rightarrow \infty.$$

Here l is fixed so that $\log \mu(\sigma)$ is always negative. Let \mathfrak{E} be the set of $\sigma \in [l, \infty)$ for which $\mu(\sigma) = \mu(t)$ for some $t > \sigma$; then \mathfrak{E} is a set of bounded half-open intervals $\{[\alpha_n, \beta_n)\}$ and may be empty, bounded, or unbounded. In the complement of the set of open intervals $\{(\alpha_n, \beta_n)\}$, we have

$$(3.14) \quad \mu(\sigma) = \frac{a^{-1} \log^{[p-2]} M(\sigma)}{\exp(\delta(\log^{[\alpha-1]} \sigma)^e)}.$$

If $\sigma \in (\alpha_n, \beta_n)$, it follows that

$$(3.15) \quad 0 > \frac{\log \mu(\sigma)}{(\log^{[q-1]} \sigma)^e} = \frac{\log^{[p-1]} M(\beta_n) - \delta(\log^{[q-1]} \beta_n)^e - \log a}{(\log^{[q-1]} \sigma)^e} \geq \\ \geq \frac{\log(a^{-1} \log^{[p-2]} M(\beta_n))}{(\log^{[q-1]} \beta_n)^e} - \delta,$$

and, if σ lies outside these intervals, we have

$$(3.16) \quad 0 > \frac{\log \mu(\sigma)}{(\log^{[q-1]} \sigma)^e} = \frac{\log(a^{-1} \log^{[p-2]} M(\sigma))}{(\log^{[q-1]} \sigma)^e} - \delta.$$

Hence, by virtue of the definition of δ , we conclude that

$$\frac{\log \mu(\sigma)}{(\log^{[q-1]} \sigma)^e} \rightarrow 0 \quad \text{as} \quad \sigma \rightarrow \infty.$$

Thus

$$(3.17) \quad \nu(\sigma) = \sup_{t \leq t \leq \sigma} \left\{ \frac{\log \mu(t)}{(\log^{[q-1]} t)^e} \right\} \quad (\sigma > l)$$

is a negative and non-decreasing function, and

$$\nu(\sigma) \rightarrow 0 \quad \text{as} \quad \sigma \rightarrow \infty.$$

If, for some $\sigma_n \in (\alpha_n, \beta_n)$, we have

$$(3.18) \quad \nu(\sigma_n) = \frac{\log \mu(\sigma_n)}{(\log^{[q-1]} \sigma_n)^e} = \frac{\log \mu(\beta_n)}{(\log^{[q-1]} \sigma_n)^e},$$

we must also have

$$(3.19) \quad \nu(\sigma) = \frac{\log \mu(\beta_n)}{(\log^{[q-1]} \sigma)^e} \quad \text{for} \quad \sigma_n \leq \sigma \leq \beta_n.$$

Equation (3.19) holds true, in particular, for $\sigma = \beta_n$. Now the set of σ for which

$$(3.20) \quad \nu(\sigma) = \frac{\log \mu(\sigma)}{(\log^{[q-1]} \sigma)^e}$$

is necessarily unbounded, and we have shown that this set cannot be a subset of the set of open intervals $\{(\alpha_n, \beta_n)\}$. Hence

$$(3.21) \quad \nu(\sigma) = \frac{\log \mu(\sigma)}{(\log^{[\alpha-1]}\sigma)^e} = \frac{\log (a^{-1} \log^{[p-2]} M(\sigma))}{(\log^{[\alpha-1]}\sigma)^e} - \delta$$

for an unbounded set of σ .

For the case under discussion,

$$(3.22) \quad T(\sigma) = \nu(\sigma) + \delta,$$

and hence we have

$$(3.23) \quad T(\sigma) = \frac{\log (a^{-1} \log^{[p-2]} M(\sigma))}{(\log^{[\alpha-1]}\sigma)^e}$$

for an unbounded set of σ . Equation (3.23) readily implies the assertions (i) and (ii) of the theorem. Since, by definition,

$$(3.24) \quad T(\sigma) \geq \frac{\log (a^{-1} \log^{[p-2]} M(\sigma))}{(\log^{[\alpha-1]}\sigma)^e} \quad \text{for all } \sigma > l,$$

the assertion (iv) of the theorem also follows at once.

The set $\{\sigma: l < \sigma < \infty\}$ can be divided into a sequence of intervals in which

$$(3.25) \quad \nu(\sigma) = \text{a constant},$$

$$(3.26) \quad \nu(\sigma)(\log^{[\alpha-1]}\sigma)^e = \text{a constant},$$

or

$$(3.27) \quad \nu(\sigma) = \frac{\log (a^{-1} \log^{[p-2]} M(\sigma))}{(\log^{[\alpha-1]}\sigma)^e} - \delta,$$

and since $M(\sigma)$ is differentiable in adjacent intervals, the same is obviously true for $\nu(\sigma)$ and hence also for $T(\sigma)$. It follows from (3.25), (3.26), and (3.27) [or (3.21)] that

$$(3.28) \quad \nu'(\sigma) = 0,$$

$$(3.29) \quad \nu'(\sigma) = -\frac{e\nu(\sigma)}{A_{[\alpha-1]}(\sigma)},$$

or

$$(3.30) \quad \nu'(\sigma) = -\frac{\varrho\nu(\sigma)}{A_{[\alpha-1]}(\sigma)} + \frac{\mu'(\sigma)}{\mu(\sigma)(\log^{[\alpha-1]}\sigma)^e},$$

where $A_{[m]}(x)$ is defined by (2.1). Since

$$(3.31) \quad T'(\sigma) \equiv \nu'(\sigma),$$

in view of (3.22), the assertion (iii) of the theorem is easily proved for (3.25) and (3.28), and for (3.26) and (3.29) in which case

$$(3.32) \quad \frac{A_{[\alpha-1]}(\sigma) T'(\sigma)}{T(\sigma)} = -\frac{\varrho\nu(\sigma)}{T(\sigma)} \rightarrow 0 \quad \text{as } \sigma \rightarrow \infty,$$

by the definition (3.26). The assertion (iii) of the theorem holds true also for (3.27) [or (3.21)] and (3.30), since (3.22) and (3.30) imply that

$$(3.33) \quad 0 \leq \frac{A_{[\alpha-1]}(\sigma) T'(\sigma)}{T(\sigma)} \leq -\frac{\varrho\nu(\sigma)}{\nu(\sigma) + \delta} \rightarrow 0 \quad \text{as } \sigma \rightarrow \infty,$$

where we have also used the facts that $\mu'(\sigma) \leq 0$ and $\nu'(\sigma) \geq 0$, and that $\nu(\sigma) \rightarrow 0$ as $\sigma \rightarrow \infty$.

The assertion (iii) of the theorem can similarly be established for $T'(\sigma - 0)$ or $T'(\sigma + 0)$ when these derivatives are unequal.

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