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A note on the construction of the proximate type of an entire Dirichlet series with index-pair \((p, q)\)


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A Note on the Construction of the Proximate Type of an Entire Dirichlet Series with Index-Pair \((p, q)\).

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SUMMARY. - Considering a natural extension of the notion of Lindelöf's proximate order of an entire function \(f(s) = \sigma + it\), A. G. Azpeitia [Trans. Amer. Math. Soc., 104 (1962), pp. 495-501] indicated how one can establish his propositions of existence of linear proximate orders \(R(\sigma)\) and linear lower proximate orders \(L(\sigma)\) for \(f(s)\) defined by a Dirichlet series. Motivated by Azpeitia's work, R. S. L. Srivastava and P. Singh [J. Math. (Jabalpur), 2 (1966), pp. 3-10] proved the corresponding existence theorem for the proximate type \(T(\sigma)\) of an entire Dirichlet series representing \(f(s)\). For an interesting generalization of this theorem to hold true for an entire Dirichlet series with index-pair \((p, q)\), which is due essentially to H. S. Kasana [J. Math. Anal. Appl., 105 (1985), pp. 445-451], a remarkably simple (and markedly different) construction of the proximate type \(T(\sigma)\) is presented here. The main theorem established here applies to a much larger class of entire Dirichlet series with index-pair \((p, q)\) than that considered earlier.

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1. Introduction and definitions.

Let \( f(s) \) be an entire function defined by a Dirichlet series

\[
f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n), \quad \text{with} \quad \limsup_{n \to \infty} \left\{ \frac{\log n}{\lambda_n} \right\} < \infty, \quad (s = \sigma + it; 0 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_n \uparrow \infty),
\]

which is absolutely convergent for all \( s \). Denote by \( M(\sigma) \) the maximum modulus of \( f(s) \) so that

\[
M(\sigma) = M(\sigma, f) = \sup_{-\infty < t < \infty} |f(\sigma + it)|.
\]

(1.3) \[ p \geq q + 1 \geq 1 \]

(see [2] and [3]).

**DEFINITION 1.** An entire function \( f(s) \), defined by \( 1.1 \), is said to be of \((p, q)\)-order \( \rho \) if

\[
\rho \equiv \rho(p, q) = \limsup_{\sigma \to \infty} \left\{ \frac{\log^{(p)} M(\sigma)}{\log^{(q)} \sigma} \right\},
\]

where \( M(\sigma) \) is given by \( 1.2 \), and (for convenience)

\[
\log^{(m)} x = \exp^{(-m)} x = \log (\log^{(m-1)} x) = \exp (\exp^{(-m-1)} x)
\]

\[
(m = 0, \pm 1, \pm 2, \ldots),
\]

provided that

\[
0 < \log^{(m-1)} x < \infty \quad (m = 1, 2, 3, \ldots),
\]

with, of course,

\[
\log^{(0)} x = \exp^{(0)} x = x.
\]
DEFINITION 2. An entire function $f(s)$, defined by (1.1) and having the $(p, q)$-order $q$ ($b < q < \infty$), is said to be of $(p, q)$-type $\tau$ if

$$
\tau = \tau(p, q) = \limsup_{\sigma \to \infty} \left\{ \frac{\log^{(q-1)} M(\sigma)}{(\log^{(q-1)} \sigma)^b} \right\},
$$

where $b = 0$ if $p > q + 1$, and $b = 1$ if $p = q + 1$.

DEFINITION 3. For $\rho$ and $b$ given by Definitions 1 and 2, respectively, an entire Dirichlet series (1.1) is said to be of index-pair $(p, q)$ for integers $p$ and $q$ constrained by (1.3) if

$$b < \rho(p, q) < \infty$$

and $\rho(p - 1, q - 1)$ is not a non-zero finite number.

The notion of Lindelöf's proximate order of an entire function (cf., e.g., [6, pp. 64-67]) has been extended, in a natural way, to entire Dirichlet series. As a matter of fact, Azpeitia [1, p. 495] has obtained some interesting propositions of existence of linear proximate orders $B(\sigma)$ and linear lower proximate orders $L(\sigma)$ for $f(s)$ defined by (1.1), and Srivastava and Singh [5, p. 5, Theorem 1] have proved the corresponding existence theorem for the proximate type $T(\sigma)$ of the entire Dirichlet series (1.1). Recently, Kasana [4, p. 447, Theorem 1] generalized this theorem by establishing the existence of a real-valued function $T(\sigma)$, called the $(p, q)$-proximate type, for an entire Dirichlet series $f(s)$ with the index-pair $(p, q)$, the $(p, q)$-order $\rho$ ($b < \rho < \infty$), and the $(p, q)$-type $\tau$ ($0 < \tau < \infty$). The object of the present note is to give a remarkably simple (and markedly different) construction of $T(\sigma)$ in this general case. Our proof, presented in Section 3 below, applies to a much wider class of entire Dirichlet series with index-pair $(p, q)$; indeed, for an entire function of $(p, q)$-type 0, it provides a considerable improvement over Kasana's theorem.

2. The main existence theorem.

The following result of existence of a $(p, q)$-proximate type of an entire Dirichlet series (1.1) with index-pair $(p, q)$ is an interesting generalization of Kasana's theorem [4, p. 447, Theorem 1]:
THEOREM. For every entire Dirichlet series \( f(s) \), defined by (1.1) and having the index-pair \((p, q)\), the \((p, q)\)-order \( \varrho \) \((0 < \varrho < \infty)\), and the \((p, q)\)-type \( \tau \) \((0 \leq \tau \leq \infty)\), there exists a real-valued function \( T(\sigma) \), called the \((p, q)\)-proximate type of \( f(s) \), which, for a given number \( a \) \((0 < a < \infty)\), satisfies each of the following conditions:

(i) \( T(\sigma) \) is continuous and piecewise differentiable for 
\[ \sigma \in [\sigma_0, \infty), \quad \sigma_0 > \exp^{\log 2} 1; \]

(ii) \( T(\sigma) \to \tau \) as \( \sigma \to \infty \);

(iii) \( (A_{t_{n-1}}(\sigma)T'(\sigma))/T(\sigma) \to 0 \) as \( \sigma \to \infty \), where \( T'(\sigma) \) is to be interpreted as either \( T'(\sigma - 0) \) or \( T'(\sigma + 0) \) whenever these derivatives are unequal, and

(iv) 
\[ \limsup_{\sigma \to \infty} \left[ \frac{\log^{t_{n-1}} M(\sigma)}{\exp \{ (\log^{t_{n-1}} \sigma)^a T(\sigma) \}} \right] = a, \]

where, for convenience,

\[ A_{t_{m1}}(x) = \prod_{j=0}^{m} \log^{t_{j1}} x. \]

REMARK. For entire Dirichlet series of finite positive \((p, q)\)-type, our assertion (iii) is equivalent to the corresponding assertion due to Kasana [4, p. 447, Eq. (1.3)].

Our demonstration of this general theorem, given in Section 3 below, is rather simple and is markedly different from the proof presented earlier by Kasana [4, pp. 447-449]. Moreover, it includes a much wider class of entire Dirichlet series with index-pair \((p, q)\).

3. Construction of the \((p, q)\)-proximate type \( T(\sigma) \).

We shall show that the properties asserted by the theorem are possessed by \( T(\sigma) \) in each of the following possible alternatives.

If there exists a positive number \( l \) such that

\[ \log^{t_1} M(\sigma) < \varrho \log^{t_0} \sigma \quad \text{when} \quad \sigma > l, \]
then, for a given number \( a \) (\( 0 < a < \infty \)), we define

\[
(3.2a) \quad \varphi^{p}_a(\sigma) = \begin{cases} 
\sup_{i \leq t \leq \sigma} \left\{ \frac{\log^{[3]}(\log^{[p-3]} M(t))^{1/a}}{\log^{[q]} t} \right\} - q & (\sigma > l) \\
\varphi^{p}_a(l) & (0 < \sigma \leq l).
\end{cases}
\]

Otherwise we define

\[
(3.2b) \quad \varphi^{p}_a(\sigma) = \sup_{t \geq \sigma} \left\{ \frac{\log^{[3]}(\log^{[p-3]} M(t))^{1/a}}{\log^{[q]} t} \right\} - q.
\]

The following cases arise naturally:

**Case 1.** Let

\[
(3.3) \quad \lim_{\sigma \to \infty} \sup_{\sigma} \{ \varphi^{p}_a(\sigma) \log^{[q]} \sigma \} = -\infty,
\]

where \( \varphi^{p}_a(\sigma) \) is defined by (3.2a). In this case we set

\[
(3.4) \quad T(\sigma) = \sup_{t \geq \sigma} \{ \exp(\varphi^{p}_a(t) \log^{[q]} t) \}.
\]

**Case 2.** Let

\[
(3.5) \quad \lim_{\sigma \to \infty} \sup_{\sigma} \{ \varphi^{p}_a(\sigma) \log^{[q]} \sigma \} = \infty.
\]

In this case we set

\[
(3.6) \quad T(\sigma) = \sup_{t \leq \sigma} \{ \exp(\varphi^{p}_a(t) \log^{[q]} t) \}.
\]

**Case 3.** Let

\[
(3.7) \quad \lim_{\sigma \to \infty} \sup_{\sigma} \{ \varphi^{p}_a(\sigma) \log^{[q]} \sigma \} = \log \delta \quad (0 < \delta < \infty).
\]

In this case we set

\[
(3.8) \quad T(\sigma) = \delta + \sup_{i_1 \leq i \leq \sigma} \left\{ \log \left( \sup_{z \geq i} \left\{ \frac{a^{-1} \log^{[p-3]} M(x)}{\exp(\delta(\log^{[p-3]} x)^{q})} \right\} \right) \right\}, \quad \text{if } \gamma = 0,
\]

\[
(3.9) \quad T(\sigma) = \delta \quad \text{if } \gamma = 0.
\]
where $l_1$ is such that

$$\log^{(p-2)} M(\sigma) < a \exp \left( \delta \log^{(q-1)} \sigma \right)$$

for all $\sigma > l_1$;

$$T(\sigma) = \delta + \frac{\log \gamma}{\log^{(q-1)} \sigma}, \quad \text{if } 0 < \gamma < \infty;$$

and

$$T(\sigma) = \delta + \sup_{t \geq \sigma} \left\{ \log \left( \sup_{x \leq l} \left\{ \frac{a^{-1} \log^{(p-2)} M(x)}{\exp \left( \delta \log^{(q-1)} x \right)^\sigma} \right\} \right) \right\} \left( \log^{(q-1)} l \right)^\sigma,$$

if $\gamma = \infty$,

where, for convenience,

$$\gamma = \limsup_{\sigma \to \infty} \left\{ \frac{\log^{(p-2)} M(\sigma)}{\exp \left( \delta \log^{(q-1)} \sigma \right)^\sigma} \right\}.$$

We note here that, if $f(s)$ is of finite positive $(p, q)$-type, then $\delta = \tau$.

We now illustrate our general approach by establishing the theorem for the case corresponding to (3.8). For this case, if we let

$$\mu(\sigma) = \sup_{t \geq \sigma} \left\{ \frac{a^{-1} \log^{(p-2)} M(t)}{\exp \left( \delta \log^{(q-1)} t \right)^\sigma} \right\} \left( l > \exp(t-1) \right),$$

then it is easily verified that $\mu(\sigma)$ is continuous and non-increasing, and that

$$\mu(\sigma) \to 0 \quad \text{as} \quad \sigma \to \infty.$$

Here $l$ is fixed so that $\log \mu(\sigma)$ is always negative. Let $\delta$ be the set of $\sigma \in [l, \infty)$ for which $\mu(\sigma) = \mu(t)$ for some $t > \sigma$; then $\delta$ is a set of bounded half-open intervals $\{[\alpha_n, \beta_n]\}$ and may be empty, bounded, or unbounded. In the complement of the set of open intervals $\{[\alpha_n, \beta_n]\}$, we have

$$\mu(\sigma) = \frac{a^{-1} \log^{(p-2)} M(\sigma)}{\exp \left( \delta \log^{(q-1)} \sigma \right)^\sigma}.$$
If \( \sigma \in (\alpha_n, \beta_n) \), it follows that

\[
(3.15) \quad 0 > \frac{\log \mu(\sigma)}{(\log^{(p-1)} \sigma)^e} = \frac{\log^{(p-1)} M(\beta_n) - \delta (\log^{(p-1)} \beta_n)^e - \log a}{(\log^{(p-1)} \beta_n)^e} \geq \frac{\log (\alpha^{-1} \log^{(p-2)} M(\beta_n))}{(\log^{(p-1)} \beta_n)^e} - \delta,
\]

and, if \( \sigma \) lies outside these intervals, we have

\[
(3.16) \quad 0 > \frac{\log \mu(\sigma)}{(\log^{(p-1)} \sigma)^e} = \frac{\log (\alpha^{-1} \log^{(p-2)} M(\sigma))}{(\log^{(p-1)} \sigma)^e} - \delta.
\]

Hence, by virtue of the definition of \( \delta \), we conclude that

\[
\frac{\log \mu(\sigma)}{(\log^{(p-1)} \sigma)^e} \to 0 \quad \text{as} \quad \sigma \to \infty.
\]

Thus

\[
(3.17) \quad \nu(\sigma) = \sup_{t \leq t < \sigma} \left\{ \frac{\log \mu(t)}{(\log^{(p-1)} t)^e} \right\} \quad (\sigma > l)
\]

is a negative and non-decreasing function, and

\[
\nu(\sigma) \to 0 \quad \text{as} \quad \sigma \to \infty.
\]

If, for some \( \sigma_n \in (\alpha_n, \beta_n) \), we have

\[
(3.18) \quad \nu(\sigma_n) = \frac{\log \mu(\sigma_n)}{(\log^{(p-1)} \sigma_n)^e} = \frac{\log \mu(\beta_n)}{(\log^{(p-1)} \beta_n)^e},
\]

we must also have

\[
(3.19) \quad \nu(\sigma) = \frac{\log \mu(\beta_n)}{(\log^{(p-1)} \sigma)^e} \quad \text{for} \quad \sigma_n \leq \sigma \leq \beta_n.
\]

Equation (3.19) holds true, in particular, for \( \sigma = \beta_n \). Now the set of \( \sigma \) for which

\[
(3.20) \quad \nu(\sigma) = \frac{\log \mu(\sigma)}{(\log^{(p-1)} \sigma)^e}
\]
is necessarily unbounded, and we have shown that this set cannot
be a subset of the set of open intervals \{(\alpha_n, \beta_n)\}. Hence

\[
(3.21) \quad \nu(\sigma) = \frac{\log \mu(\sigma)}{(\log^{\alpha-1} \sigma)^\varepsilon} = \frac{\log \left( a^{-1} \log^{(p-2)} M(\sigma) \right)}{(\log^{\alpha-1} \sigma)^\varepsilon} - \delta
\]

for an unbounded set of \( \sigma \).

For the case under discussion,

\[
(3.22) \quad T(\sigma) = \nu(\sigma) + \delta ,
\]

and hence we have

\[
(3.23) \quad T(\sigma) = \frac{\log \left( a^{-1} \log^{(p-2)} M(\sigma) \right)}{(\log^{\alpha-1} \sigma)^\varepsilon}
\]

for an unbounded set of \( \sigma \). Equation (3.23) readily implies the assertions (i) and (ii) of the theorem. Since, by definition,

\[
(3.24) \quad T(\sigma) \geq \frac{\log \left( a^{-1} \log^{(p-2)} M(\sigma) \right)}{(\log^{\alpha-1} \sigma)^\varepsilon} \quad \text{for all } \sigma > l ,
\]

the assertion (iv) of the theorem also follows at once.

The set \( \{\sigma: l < \sigma < \infty\} \) can be divided into a sequence of intervals in which

\[
(3.25) \quad \nu(\sigma) = \text{a constant},
\]

\[
(3.26) \quad \nu(\sigma)(\log^{\alpha-1} \sigma)^\varepsilon = \text{a constant},
\]

or

\[
(3.27) \quad \nu(\sigma) = \frac{\log \left( a^{-1} \log^{(p-2)} M(\sigma) \right)}{(\log^{\alpha-1} \sigma)^\varepsilon} - \delta ,
\]

and since \( M(\sigma) \) is differentiable in adjacent intervals, the same is obviously true for \( \nu(\sigma) \) and hence also for \( T(\sigma) \). If follows from (3.25), (3.26), and (3.27) [or (3.21)] that

\[
(3.28) \quad \nu'(\sigma) = 0 ,
\]

\[
(3.29) \quad \nu'(\sigma) = - \frac{\rho \nu(\sigma)}{A_{\alpha-1}(\sigma)} ,
\]
or

\[(3.30) \quad \nu'(\sigma) = -\frac{\nu v(\sigma)}{A_{(2-1)}(\sigma)} + \frac{\mu'(\sigma)}{\mu(\sigma)(\log^{(2-1)} \sigma)^{\delta}},\]

where \(A_{(2-1)}(x)\) is defined by (2.1). Since

\[(3.31) \quad T''(\sigma) \equiv \nu'(\sigma),\]

in view of (3.22), the assertion (iii) of the theorem is easily proved for (3.25) and (3.28), and for (3.26) and (3.29) in which case

\[(3.32) \quad \frac{A_{(2-1)}(\sigma)}{T'(\sigma)} = -\frac{\nu v(\sigma)}{T'(\sigma)} \to 0 \quad \text{as} \quad \sigma \to \infty,\]

by the definition (3.26). The assertion (iii) of the theorem holds true also for (3.27) [or (3.21)] and (3.30), since (3.22) and (3.30) imply that

\[(3.33) \quad 0 \leq \frac{A_{(2-1)}(\sigma)}{T(\sigma)} \leq \frac{\nu v(\sigma)}{\nu(\sigma) + \delta} \to 0 \quad \text{as} \quad \sigma \to \infty,\]

where we have also used the facts that \(\mu'(\sigma) \leq 0\) and \(\nu'(\sigma) \geq 0\), and that \(\nu(\sigma) \to 0\) as \(\sigma \to \infty\).

The assertion (iii) of the theorem can similarly be established for \(T'(\sigma - 0)\) or \(T'(\sigma + 0)\) when these derivatives are unequal.

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