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A Note on the Construction of the Proximate Type of an Entire Dirichlet Series with Index-Pair (p, q).

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SUMMARY. - Considering a natural extension of the notion of Lindelöf's proximate order of an entire function $f(s)$ $(s = \sigma + it)$, A. G. Azpeitia [Trans. Amer. Math. Soc., **104** (1962), pp. 495-501] indicated how one can establish his propositions of existence of linear proximate orders $R(\sigma)$ and linear lower proximate orders $L(\sigma)$ for $f(s)$ defined by a Dirichlet series. Motivated by Azpeitia's work, R. S. L. Srivastava and P. Singh [J. Math. (Jabalpur), **2** (1966), pp. 3-10] proved the corresponding existence theorem for the proximate type $T(\sigma)$ of an entire Dirichlet series representing $f(s)$. For an interesting generalization of this theorem to hold true for an entire Dirichlet series with index-pair (p, q), which is due essentially to H. S. Kasana [J. Math. Anal. Appl., **105** (1985), pp. 445-451], a remarkably simple (and markedly different) construction of the proximate type $T(\sigma)$ is presented here. The main theorem established here applies to a much larger class of entire Dirichlet series with index-pair (p, q) than that considered earlier.

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1. Introduction and definitions.

Let $f(s)$ be an entire function defined by a Dirichlet series

$$(1.1)\qquad f(s)=\sum_{n=1}^{\infty}a_n\exp(s\lambda_n)\,,\qquad \text{with}\qquad \limsup_{n\to\infty}\left\{\frac{\log n}{\lambda_n}\right\}<\infty\,,$$
$$(s=\sigma+it;\ 0\leqq\lambda_1<\lambda_2<\ldots<\lambda_n\uparrow\infty)\,,$$

which is absolutely convergent for all s. Denote by $M(\sigma)$ the maximum modulus of $f(s)$ so that

$$(1.2)\qquad M(\sigma)\equiv M(\sigma,f)=\sup_{-\infty<t<\infty}|f(\sigma+it)|\,.$$

We begin by recalling the concept of the index-pair (p, q), the (p, q)-order, and the (p, q)-type of an entire Dirichlet series $f(s)$ for integers p and q such that

$$(1.3)\qquad p\geqq q+1\geqq 1$$

(see [2] and [3]).

Definition 1. An entire function $f(s)$, defined by (1.1), is said to be of (p, q)-order ϱ if

$$(1.4)\qquad \varrho\equiv\varrho(p,q)=\limsup_{\sigma\to\infty}\left\{\frac{\log^{[p]}M(\sigma)}{\log^{[q]}\sigma}\right\},$$

where $M(\sigma)$ is given by (1.2), and (for convenience)

$$(1.5)\qquad \log^{[m]}x=\exp^{[-m]}x=\log(\log^{[m-1]}x)=\exp(\exp^{[-m-1]}x)$$
$$(m=0,\ \pm1,\ \pm2,\ldots),$$

provided that

$$(1.6)\qquad 0<\log^{[m-1]}x<\infty\qquad (m=1,2,3,\ldots),$$

with, of course,

$$(1.7)\qquad \log^{[0]}x=\exp^{[0]}x=x\,.$$

DEFINITION 2. An entire function $f(s)$, defined by (1.1) and having the (p, q)-order ϱ $(b < \varrho < \infty)$, is said to be of (p, q)-type τ if

$$\tau \equiv \tau(p, q) = \limsup_{\sigma\to\infty} \left\{ \frac{\log^{[p-1]} M(\sigma)}{(\log^{[q-1]} \sigma)^{\varrho}} \right\}, \tag{1.8}$$

where $b = 0$ if $p > q + 1$, and $b = 1$ if $p = q + 1$.

DEFINITION 3. For ϱ and b given by Definitions 1 and 2, respectively, an entire Dirichlet series (1.1) is said to be of index-pair (p, q) for integers p and q constrained by (1.3) if

$$b < \varrho(p, q) < \infty$$

and $\varrho(p-1, q-1)$ is not a non-zero finite number.

The notion of Lindelöf's proximate order of an entire function (*cf.*, *e.g.*, [6, pp. 64-67]) has been extended, in a natural way, to entire Dirichlet series. As a matter of fact, Azpeitia [1, p. 495] has obtained some interesting propositions of existence of linear proximate orders $R(\sigma)$ and linear lower proximate orders $L(\sigma)$ for $f(s)$ defined by (1.1), and Srivastava and Singh [5, p. 5, Theorem 1] have proved the corresponding existence theorem for the proximate type $T(\sigma)$ of the entire Dirichlet series (1.1). Recently, Kasana [4, p. 447, Theorem 1] generalized this theorem by establishing the existence of a real-valued function $T(\sigma)$, called the (p, q)-proximate type, for an entire Dirichlet series $f(s)$ with the index-pair (p, q), the (p, q)-order ϱ $(b < \varrho < \infty)$, and the (p, q)-type τ $(0 < \tau < \infty)$. The object of the present note is to give a remarkably simple (and markedly different) construction of $T(\sigma)$ in this general case. Our proof, presented in Section 3 below, applies to a much wider class of entire Dirichlet series with index-pair (p, q); indeed, for an entire function of (p, q)-type 0, it provides a considerable improvement over Kasana's theorem.

2. The main existence theorem.

The following result of existence of a (p, q)-proximate type of an entire Dirichlet series (1.1) with index-pair (p, q) is an interesting generalization of Kasana's theorem [4, p. 447, Theorem 1]:

THEOREM. *For every entire Dirichlet series* $f(s)$, *defined by* (1.1) *and having the index-pair* (p, q), *the* (p, q)*-order* ϱ $(b < \varrho < \infty)$, *and the* (p, q)*-type* τ $(0 \leqq \tau \leqq \infty)$, *there exists a real-valued function* $T(\sigma)$, *called the* (p, q)*-proximate type of* $f(s)$, *which, for a given number* a $(0 < a < \infty)$, *satisfies each of the following conditions*:

(i) $T(\sigma)$ *is continuous and piecewise differentiable for*

$$\sigma \in [\sigma_0, \infty), \qquad \sigma_0 > \exp^{[q-2]} 1 ;$$

(ii) $T(\sigma) \to \tau$ *as* $\sigma \to \infty$;

(iii) $(\Lambda_{[q-1]}(\sigma) T'(\sigma))/T(\sigma) \to 0$ *as* $\sigma \to \infty$, *where* $T'(\sigma)$ *is to be interpreted as either* $T'(\sigma - 0)$ *or* $T'(\sigma + 0)$ *whenever these derivatives are unequal, and*

(iv) $$\limsup_{\sigma \to \infty} \left\{ \frac{\log^{[p-2]} M(\sigma)}{\exp\{(\log^{[q-1]} \sigma)^{\varrho}\, T(\sigma)\}} \right\} = a ,$$

where, for convenience,

$$\Lambda_{[m]}(x) = \prod_{j=0}^{m} \log^{[j]} x . \tag{2.1}$$

REMARK. For entire Dirichlet series of *finite positive* (p, q)-type, our assertion (iii) is equivalent to the corresponding assertion due to Kasana [4, p. 447, Eq. (1.3)].

Our demonstration of this general theorem, given in Section 3 below, is rather simple and is markedly different from the proof presented earlier by Kasana [4, pp. 447-449]. Moreover, it includes a much wider class of entire Dirichlet series with index-pair (p, q).

3. Construction of the (p, q)-proximate type $T(\sigma)$.

We shall show that the properties asserted by the theorem are possessed by $T(\sigma)$ in each of the following possible alternatives.

If there exists a positive number l such that

$$\log^{[p]} M(\sigma) < \varrho \log^{[q]} \sigma \quad \text{when} \quad \sigma > l , \tag{3.1}$$

then, for a given number a $(0 < a < \infty)$, we define

$$\varphi_q^p(\sigma) = \begin{cases} \sup\limits_{l \leq t \leq \sigma} \left\{ \dfrac{\log^{[3]}(\log^{[p-3]} M(t))^{1/a}}{\log^{[q]} t} \right\} - \varrho & (\sigma > l) \qquad (3.2a) \\ \varphi_q^p(l) & (0 < \sigma \leqq l) \,. \qquad (3.2b) \end{cases}$$

Otherwise we define

$$\varphi_q^p(\sigma) = \sup_{t \geqq \sigma} \left\{ \frac{\log^{[3]}(\log^{[p-3]} M(t))^{1/a}}{\log^{[q]} t} \right\} - \varrho \,. \tag{3.2c}$$

The following cases arise naturally:

Case 1. Let

$$\limsup_{\sigma \to \infty} \{\varphi_q^p(\sigma) \log^{[q]} \sigma\} = -\infty \,, \tag{3.3}$$

where $\varphi_q^p(\sigma)$ is defined by (3.2a). In this case we set

$$T(\sigma) = \sup_{t \geqq \sigma} \{\exp(\varphi_q^p(t) \log^{[q]} t)\} \,. \tag{3.4}$$

Case 2. Let

$$\limsup_{\sigma \to \infty} \{\varphi_q^p(\sigma) \log^{[q]} \sigma\} = \infty \,. \tag{3.5}$$

In this case we set

$$T(\sigma) = \sup_{t \leqq \sigma} \{\exp(\varphi_q^p(t) \log^{[q]} t)\} \,. \tag{3.6}$$

Case 3. Let

$$\limsup_{\sigma \to \infty} \{\varphi_q^p(\sigma) \log^{[q]} \sigma\} = \log \delta \qquad (0 < \delta < \infty) \,. \tag{3.7}$$

In this case we set

$$T(\sigma) = \delta + \sup_{l_1 \leq t \leq \sigma} \left\{ \frac{\log\left(\sup\limits_{x \geqq t} \left\{ \dfrac{a^{-1} \log^{[p-2]} M(x)}{\exp(\delta(\log^{[q-1]} x)^{\varrho})} \right\} \right)}{(\log^{[q-1]} t)^{\varrho}} \right\} , \qquad \text{if } \gamma = 0 \,, \tag{3.8}$$

where l_1 is such that

$$\log^{[p-2]} M(\sigma) < a \exp\left(\delta(\log^{[q-1]}\sigma)^\varrho\right) \tag{3.9}$$

for all $\sigma > l_1$;

$$T(\sigma) = \delta + \frac{\log\gamma}{\log^{[q-1]}\sigma}, \quad \text{if } 0 < \gamma < \infty; \tag{3.10}$$

and

$$T(\sigma) = \delta + \sup_{t \geqq \sigma}\left\{\frac{\log\left(\sup\limits_{x \leqq t}\left\{\dfrac{a^{-1}\log^{[p-2]} M(x)}{\exp\left(\delta(\log^{[q-1]}x)^\varrho\right)}\right\}\right)}{(\log^{[q-1]}t)^\varrho}\right\}, \qquad \text{if } \gamma = \infty, \tag{3.11}$$

where, for convenience,

$$\gamma = \limsup_{\sigma\to\infty}\left\{\frac{\log^{[p-2]} M(\sigma)}{\exp\left(\delta(\log^{[q-1]}\sigma)^\varrho\right)}\right\}. \tag{3.12}$$

We note here that, if $f(s)$ is of *finite positive* (p, q)-type, then $\delta = \tau$.

We now illustrate our general approach by establishing the theorem for the case corresponding to (3.8). For this case, if we let

$$\mu(\sigma) = \sup_{l \geqq t \geqq \sigma}\left\{\frac{a^{-1}\log^{[p-2]} M(t)}{\exp\left(\delta(\log^{[q-1]}t)^\varrho\right)}\right\} \qquad (l > \exp^{[q-2]}1), \tag{3.13}$$

then it is easily verified that $\mu(\sigma)$ is continuous and non-increasing, and that

$$\mu(\sigma) \to 0 \quad \text{as} \quad \sigma \to \infty.$$

Here l is fixed so that $\log\mu(\sigma)$ is always negative. Let $\mathcal{E}$ be the set of $\sigma \in [l, \infty)$ for which $\mu(\sigma) = \mu(t)$ for some $t > \sigma$; then $\mathcal{E}$ is a set of bounded half-open intervals $\{[\alpha_n, \beta_n)\}$ and may be empty, bounded, or unbounded. In the complement of the set of open intervals $\{(\alpha_n, \beta_n)\}$, we have

$$\mu(\sigma) = \frac{a^{-1}\log^{[p-2]} M(\sigma)}{\exp\left(\delta(\log^{[q-1]}\sigma)^\varrho\right)}. \tag{3.14}$$

If $\sigma \in (\alpha_n, \beta_n)$, it follows that

$$(3.15) \qquad 0 > \frac{\log \mu(\sigma)}{(\log^{[q-1]}\sigma)^\varrho} = \frac{\log^{[p-1]} M(\beta_n) - \delta(\log^{[q-1]}\beta_n)^\varrho - \log a}{(\log^{[q-1]}\sigma)^\varrho} \geqq$$
$$\geqq \frac{\log(a^{-1}\log^{[p-2]} M(\beta_n))}{(\log^{[q-1]}\beta_n)^\varrho} - \delta\,,$$

and, if σ lies outside these intervals, we have

$$(3.16) \qquad 0 > \frac{\log \mu(\sigma)}{(\log^{[q-1]}\sigma)^\varrho} = \frac{\log(a^{-1}\log^{[p-2]} M(\sigma))}{(\log^{[q-1]}\sigma)^\varrho} - \delta\,.$$

Hence, by virtue of the definition of δ, we conclude that

$$\frac{\log \mu(\sigma)}{(\log^{[q-1]}\sigma)^\varrho} \to 0 \quad \text{as} \quad \sigma \to \infty\,.$$

Thus

$$(3.17) \qquad \nu(\sigma) = \sup_{l \leqq t \leqq \sigma} \left\{\frac{\log \mu(t)}{(\log^{[q-1]}t)^\varrho}\right\} \qquad (\sigma > l)$$

is a negative and non-decreasing function, and

$$\nu(\sigma) \to 0 \quad \text{as} \quad \sigma \to \infty\,.$$

If, for some $\sigma_n \in (\alpha_n, \beta_n)$, we have

$$(3.18) \qquad \nu(\sigma_n) = \frac{\log \mu(\sigma_n)}{(\log^{[q-1]}\sigma_n)^\varrho} = \frac{\log \mu(\beta_n)}{(\log^{[q-1]}\sigma_n)^\varrho}\,,$$

we must also have

$$(3.19) \qquad \nu(\sigma) = \frac{\log \mu(\beta_n)}{(\log^{[q-1]}\sigma)^\varrho} \quad \text{for} \quad \sigma_n \leqq \sigma \leqq \beta_n\,.$$

Equation (3.19) holds true, in particular, for $\sigma = \beta_n$. Now the set of σ for which

$$(3.20) \qquad \nu(\sigma) = \frac{\log \mu(\sigma)}{(\log^{[q-1]}\sigma)^\varrho}$$

is necessarily unbounded, and we have shown that this set cannot be a subset of the set of open intervals $\{(\alpha_n, \beta_n)\}$. Hence

$$\nu(\sigma) = \frac{\log \mu(\sigma)}{(\log^{[q-1]} \sigma)^{\varrho}} = \frac{\log(a^{-1} \log^{[p-2]} M(\sigma))}{(\log^{[q-1]} \sigma)^{\varrho}} - \delta \tag{3.21}$$

for an unbounded set of σ.

For the case under discussion,

$$T(\sigma) = \nu(\sigma) + \delta\,, \tag{3.22}$$

and hence we have

$$T(\sigma) = \frac{\log(a^{-1} \log^{[p-2]} M(\sigma))}{(\log^{[q-1]} \sigma)^{\varrho}} \tag{3.23}$$

for an unbounded set of σ. Equation (3.23) readily implies the assertions (i) and (ii) of the theorem. Since, by definition,

$$T(\sigma) \geqq \frac{\log(a^{-1} \log^{[p-2]} M(\sigma))}{(\log^{[q-1]} \sigma)^{\varrho}} \quad \text{for all} \quad \sigma > l\,, \tag{3.24}$$

the assertion (iv) of the theorem also follows at once.

The set $\{\sigma\colon l < \sigma < \infty\}$ can be divided into a sequence of intervals in which

$$\nu(\sigma) = \text{a constant}, \tag{3.25}$$

$$\nu(\sigma)(\log^{[q-1]} \sigma)^{\varrho} = \text{a constant}, \tag{3.26}$$

or

$$\nu(\sigma) = \frac{\log(a^{-1} \log^{[p-2]} M(\sigma))}{(\log^{[q-1]} \sigma)^{\varrho}} - \delta\,, \tag{3.27}$$

and since $M(\sigma)$ is differentiable in adjacent intervals, the same is obviously true for $\nu(\sigma)$ and hence also for $T(\sigma)$. If follows from (3.25), (3.26), and (3.27) [or (3.21)] that

$$\nu'(\sigma) = 0\,, \tag{3.28}$$

$$\nu'(\sigma) = -\frac{\varrho \nu(\sigma)}{\Lambda_{[q-1]}(\sigma)}\,, \tag{3.29}$$

or

$$\nu'(\sigma) = -\frac{\varrho\nu(\sigma)}{\Lambda_{[q-1]}(\sigma)} + \frac{\mu'(\sigma)}{\mu(\sigma)(\log^{[q-1]}\sigma)^{\varrho}}, \tag{3.30}$$

where $\Lambda_{[m]}(x)$ is defined by (2.1). Since

$$T'(\sigma) \equiv \nu'(\sigma), \tag{3.31}$$

in view of (3.22), the assertion (iii) of the theorem is easily proved for (3.25) and (3.28), and for (3.26) and (3.29) in which case

$$\frac{\Lambda_{[q-1]}(\sigma)\, T'(\sigma)}{T(\sigma)} = -\frac{\varrho\nu(\sigma)}{T(\sigma)} \to 0 \quad \text{as} \quad \sigma \to \infty, \tag{3.32}$$

by the definition (3.26). The assertion (iii) of the theorem holds true also for (3.27) [or (3.21)] and (3.30), since (3.22) and (3.30) imply that

$$0 \leqq \frac{\Lambda_{[q-1]}(\sigma)\, T'(\sigma)}{T(\sigma)} \leqq -\frac{\varrho\nu(\sigma)}{\nu(\sigma)+\delta} \to 0 \quad \text{as} \quad \sigma \to \infty, \tag{3.33}$$

where we have also used the facts that $\mu'(\sigma) \leqq 0$ and $\nu'(\sigma) \geqq 0$, and that $\nu(\sigma) \to 0$ as $\sigma \to \infty$.

The assertion (iii) of the theorem can similarly be established for $T'(\sigma - 0)$ or $T'(\sigma + 0)$ when these derivatives are unequal.

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