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Nonlinear Stability of a Spatially Symmetric Solution of the Relativistic Poisson-Vlasov Equation.

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SUMMARY - We prove that the distribution functions $f(\boldsymbol{x}, \boldsymbol{p})$, spatially homogeneous, possibly nonregular, nonincreasing in $\|\boldsymbol{p}\|$, stationary solutions of the relativistic Poisson-Vlasov equation, are nonlinear (Liapunov) stable.

1. Introduction and description of the problem.

A relativistic plasma, when the collisions are neglected and the particles interact only via a medium field, is described by the relativistic Vlasov equation [8], that, when there is one only type of particles, for simplicity, assumes the form:

$$p^\mu \frac{\partial \hat{f}}{\partial x^\mu}(x, p) + mF^\mu(x, p) \frac{\partial \hat{f}}{\partial p^\mu}(x, p) = 0 \tag{1.1}$$

in which $x = x^\mu = (ct, \boldsymbol{x})$ and $p = p^\mu = (p^0, \boldsymbol{p})$ are the position 4-vector and the momentum 4-vector of a particle respectively; m is the rest

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mass of the particles, that enters into the mass shell condition

$$p^\mu p_\mu = g_{\mu\nu} p^\mu p^\nu = (p_0)^2 - \boldsymbol{p}^2 = m^2 c^2 \; (*)$$

or

$$p_0 = \sqrt{m^2 c^2 + \boldsymbol{p}^2}$$

and c is the speed of light.

The external 4-force $F^\mu = (F^0_*, \boldsymbol{F}_*)$ satisfies the condition

$$F^\mu p_\mu = 0 \;\Rightarrow\; F^0_* = \frac{\boldsymbol{F}_* \cdot \boldsymbol{p}}{p^0} \tag{1.2}$$

i.e. it is purely mechanical (it does not modify the rest mass of the particles). This requirement is fulfilled by the Lorentz 4-force, defined by

$$F^\mu(x, p) = \frac{-q}{mc} F^{\mu\nu}(x)\, p_\nu$$

where $F^{\mu\nu}$ is the electromagnetic tensor, related to the electric and magnetic fields $\boldsymbol{E}, \boldsymbol{B}$ by the equations:

$$F^{0i} = E^i \quad \text{when } i = 1, 2, 3$$

$$F^{ij} = B^k \quad \text{when } i, j, k \text{ is an even permutation of } 1, 2, 3 .$$

The function $\hat{f} = \hat{f}(x, p)$ is called the distribution function, and is defined on the relativistic phase space (x, p considered as 8 independent scalar variables).

The absence of « collision terms » into the right hand side of eq. (1.1) is related to the assumption that the particles interact only via a medium field; this is admissible when the gas is sufficiently rarefied.

Rewriting (1.1) in term of spatial quantities (with respect to an inertial reference system), we have:

$$\frac{\partial \hat{f}}{\partial t} + \frac{c\boldsymbol{p}}{p^0} \cdot \frac{\partial \hat{f}}{\partial \boldsymbol{x}} + \frac{mc}{p^0} F^0_* \frac{\partial \hat{f}}{\partial p^0} + \frac{mc\boldsymbol{F}_*}{p^0} \cdot \frac{\partial \hat{f}}{\partial \boldsymbol{p}} = 0$$

(*) Greek indices run from 0 to 3, and have space-time meaning; latin indices run from 1 to 3, and have space meaning; $g_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$.

that, substituting (1.2), introducing the relation between the (spatial) velocity $\boldsymbol{v}$ and the (spatial) momentum $\boldsymbol{p}$

$$\boldsymbol{v} = \frac{c\boldsymbol{p}}{p^0}$$

and replacing $\hat{f}$ with f through the definition

$$f(x, \boldsymbol{p}) = \hat{f}(x, p^0 = \sqrt{m^2c^2 + \boldsymbol{p}^2}, \boldsymbol{p})$$

becomes

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \frac{mc}{p^0}\boldsymbol{F}_*\cdot\frac{\partial f}{\partial \boldsymbol{p}} = 0$$

or

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \boldsymbol{F}\cdot\frac{\partial f}{\partial \boldsymbol{p}} = 0$$

in which $\boldsymbol{F} = (mc/p^0)\boldsymbol{F}_*$ is the ordinary (spatial) force, equal to the time derivative of the (spatial) momentum, evaluated in the inertial reference system defined above.

In the electromagnetic case, that interests us,

$$F^i = \frac{mc}{p^0}\left(-\frac{q}{mc}F^{i\nu}P_\nu\right) = qE^i - \frac{q}{c}F^{ij}v_j$$

or

$$\boldsymbol{F} = q\boldsymbol{E} + \frac{q}{c}\,\boldsymbol{v}\wedge\boldsymbol{B}\,.$$

In plasma physics are relevant the Maxwell-Vlasov model, based on the equations

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + q\left(\boldsymbol{E} + \frac{1}{c}\,\boldsymbol{v}\wedge\boldsymbol{B}\right)\cdot\frac{\partial f}{\partial \boldsymbol{p}} = 0$$

$$\operatorname{div}\boldsymbol{B} = 0\,, \qquad \operatorname{div}\boldsymbol{E} = \varrho = \varrho_+ + \int_P q f(x, \boldsymbol{p})\,d\boldsymbol{p}$$

$$\operatorname{curl}\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = 0\,, \qquad \operatorname{curl}\boldsymbol{B} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{c}\boldsymbol{J} = \frac{q}{c}\int_P \boldsymbol{v} f(x, \boldsymbol{p})\,d\boldsymbol{p}$$

and the Poisson-Vlasov model

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + q\boldsymbol{E}\cdot\frac{\partial f}{\partial \boldsymbol{p}} = 0$$

$$\operatorname{div} \boldsymbol{E} = \varrho = \varrho_+ + \int_{\boldsymbol{P}} q\, f(x, \boldsymbol{p})\, d\boldsymbol{p}\,.$$

In either models, $\boldsymbol{v} = c\boldsymbol{p}/p^0$ in the relativistic case, and $\boldsymbol{v} = \boldsymbol{p}/m$ in the classical one.

As is clear from the equations, in the Poisson-Vlasov model, the magnetic field is assumed to be null, and this approximation, that originates from an obvious requirement of simplicity, is acceptable in many quasi-stationary phoenomena.

The charge density ϱ is the sum of a constant (in time and space) term ϱ_+, and of another one, depending on the distribution function f, so that the global charge is null. In a plasma model with one only type of particles (electrons), ϱ_+ may be identified with the constant and uniform charge density of the fixed positive ions.

Concerning the mathematical problem of the existence of the solutions of the above problem, we say that, for the non relativistic Maxwell-Vlasov model, the local-in-time existence and uniqueness of the solution is proved in [26, 28]. In [28] is also proved that the classical solutions of the non-relativistic Maxwell-Vlasov equation converge to the solutions of the Poisson-Vlasov equation, when the speed of light goes to infinity. In one space dimension, the problem of the existence is treated in [7]. The Hamiltonian structure of the equation of motion is discussed in [18, 12] and in many other papers referred in [12].

Concerning the non-relativistic Poisson-Vlasov model, the associated Cauchy problem has been completely solved in 1 and 2 spatial dimensions [6, 22, 27]. In 3 space dimensions, the existence of global weak solutions is proved in [1, 14, 15], and, when the initial data are small enough, the existence of global classical solutions [3] is also proved. In 3 spatial dimensions, there exist classical global solutions for symmetric initial data [4, 13, 25]. The existence of classical solutions corresponding to any initial data is an open problem, as well as the uniqueness problem of weak solutions. For reader's utility, we refer also the papers [2, 5, 11] and the review papers [19, 20].

As discussed in [23, 24], when the speeds of the particles approach the speed of light, the classical Maxwell and Poisson-Vlasov models become inadequate, and must be substituted by the relativistic ones.

Concerning the Cauchy problem we refer [10, 29]. In [10] the 3-dimensional relativistic Poisson-Vlasov equation is treated, and the existence of global-in-time classical solutions is proved, assuming spherical symmetry of the initial data. The relativistic Maxwell-Vlasov equation in 3-space dimensions is treated in [29], and the local-in-time existence of classical solutions is proved, starting from regular initial data. Moreover, it is proved that the solutions of the Relativistic Maxwell-Vlasov equation converge in a pointwise sense to the solutions of the non-relativistic Poisson-Vlasov equation, when the speed of light goes to infinity.

In this paper we treat the relativistic Poisson-Vlasov model, with a plasma made of electrons, whose density in phase space is determined by the distribution function $f(\boldsymbol{x}, t, \boldsymbol{p})$ and a background of positive fixed ions, uniformly distributed on the domain, so that the global charge is null.

We assume that the domain is a flat ν-dimensional torus T^ν, ($\nu = 1, 2, 3$), having dimensions L_x, L_y, L_z respectively, or, equivalently, we assume that the distribution function and all the other physical quantities of the system are spatially periodic.

In order to simplify the notations, we assume the charge, the mass of the particles, the speed of light and the charge density of the positive background equal to 1, and so we obtain the following system of equations:

$$\text{(1.3)} \qquad \begin{cases} \dfrac{\partial f}{\partial t} + \boldsymbol{v}\cdot\dfrac{\partial f}{\partial \boldsymbol{x}} + \boldsymbol{E}\cdot\dfrac{\partial f}{\partial \boldsymbol{p}} = 0 \\[2ex] \boldsymbol{v} = \dfrac{\boldsymbol{p}}{\sqrt{1+\boldsymbol{p}^2}}, \qquad \boldsymbol{E}(\boldsymbol{x}, t) = -\nabla_x \varphi(\boldsymbol{x}, t) \\[2ex] -\Delta_x \varphi(\boldsymbol{x}, t) = \varrho(\boldsymbol{x}, t) = 1 - \displaystyle\int_{P\in\mathbf{R}^\nu} f(\boldsymbol{x}, t, \boldsymbol{p})\, d\boldsymbol{p}\,. \end{cases}$$

This system of equations admits a first integral, which is the total energy of the system:

$$\text{(1.4)} \qquad \int_{x\in T^\nu} d\boldsymbol{x} \int_{P\in\mathbf{R}^\nu} d\boldsymbol{p}\sqrt{1+\boldsymbol{p}^2}\, f(\boldsymbol{x}, t, \boldsymbol{p}) + \frac{1}{2}\int_{x\in T^\nu} d\boldsymbol{x}\, \boldsymbol{E}^2(\boldsymbol{x}) \overset{\text{def}}{=} T + U \overset{\text{def}}{=} E\,.$$

In this work we prove that the class of spatially homogeneous distribution functions $f(\boldsymbol{x}, \boldsymbol{p})$, possibly non regular, non increasing in $\|\boldsymbol{p}\|$, stationary solutions of the relativistic Poisson-Vlasov equation (1.3) is non-linearly (Liapunov) stable. We stress that the proof does not use the regularity property of the solution, and this fact may be physically interesting in some special case. By non regular solution, we mean a solution of the weak formulation of eq. (1.3)

$$\frac{d}{dt} f_t[g] - f_t[\boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} g] + f_t\left[\boldsymbol{E} \cdot \frac{\partial g}{\partial \boldsymbol{p}}\right] = 0$$

where

$$f_t[g] = \int_{\boldsymbol{x} \in T^\nu} d\boldsymbol{x} \int_{\boldsymbol{P} \in \mathbf{R}^\nu} d\boldsymbol{p}\, f(\boldsymbol{x}, t, \boldsymbol{p})\, g(\boldsymbol{x}, \boldsymbol{p})$$

g being any test function.

The proof of our result is organized into 3 Lemmata (whose proofs, rather technical, will be given later), and into a Theorem. The formulation of the Lemmata and of the Theorem, and their proofs, when it is possible, will not depend on the number of space dimensions.

The proof relies on the application to the relativistic Poisson-Vlasov equation of a technique, suggested by Marchioro and Pulvirenti [16, 17] that is based on the observation that the kinetic energy corresponding to a stable stationary solution is an extremum, constrained to the orbits of the coadjoint representation [12] of the measure preserving diffeomorphysms group acting on a domain D (when the system is an incompressible ideal fluid occupying a domain D, and so the configuration space of the system is the group of diffeomorphysms of D), and the measure preserving diffeomorphysms group acting on the phase space, (when the system is an ideal plasma). By a classical viewpoint, these orbits are linked with the conservation of the vorticity integral, in two space dimensions, and with the Liouville theorem. See also, for this subject [12, 9, 21].

In this work we assume that when the initial data is « near » the stationary solution, the Cauchy problem admits a global in time weak solution.

We finally remark that similar methods could be used for the relativistic Maxwell-Vlasov equation.

1. Proof of the main result.

DEFINITION. Let S be the set of the stationary solutions of the eq. (1.3) satisfying the following conditions:

$$S = \Big\{ \bar{f} \,\Big|\, \bar{f}\colon T^\nu \times \mathbf{R}^\nu \to R_+,\ \bar{f}(\boldsymbol{x}, \boldsymbol{p}) = \bar{f}(\|\boldsymbol{p}\|),\ \bar{f}(\cdot)\colon R^+ \to R^+ \\ \bar{f}(\cdot) \text{ nonincreasing},\ \bar{f}(\cdot,\cdot) \in L_\infty(T^\nu \times \mathbf{R}^\nu) \\ \int\limits_{\boldsymbol{P} \in \mathbf{R}^\nu} \sqrt{1 + \boldsymbol{p}^2}\, \bar{f}(\boldsymbol{x}, \boldsymbol{p})\, d\boldsymbol{p} < \infty \qquad \forall \boldsymbol{x} \in T^\nu \\ \bar{f}(\cdot) \text{ with compact support when } \nu = 2, 3 \Big\}.$$

DEFINITION. $\forall M > 0$, given a stationary solution $\bar{f} \in S$, we define the class

$$\mathfrak{I}(\bar{f}, M) = \Big\{ f \,\Big|\, f\colon T^\nu \times \mathbf{R}^\nu \to R^+, \\ \int\limits_{\boldsymbol{P} \in \mathbf{R}^\nu} |f(\boldsymbol{x}, \boldsymbol{p}) - \bar{f}(\boldsymbol{x}, \boldsymbol{p})| \sqrt{1 + \boldsymbol{p}^2}\, d\boldsymbol{p} \leqslant M,\ \forall \boldsymbol{x} \in T^\nu \Big\}$$

of distribution functions whose kinetic energy density is «near» the kinetic energy density of $\bar{f}$, $\forall \boldsymbol{x} \in T^\nu$.

DEFINITION. Given a distribution function $F\colon T^\nu \times \mathbf{R}^\nu \to R^+$, we define the class

$$I(F) = \{ f | f\colon T^\nu \times \mathbf{R}^\nu \to R^+,\ \forall \lambda \in \mathbf{R}^+,\ \text{meas}\, \{(\boldsymbol{x}, \boldsymbol{p}) | f(\boldsymbol{x}, \boldsymbol{p}) > \lambda = \\ = \text{meas}\, \{(\boldsymbol{x}, \boldsymbol{p}) | F(\boldsymbol{x}, \boldsymbol{p}) > \lambda\}\}\,.$$

The functions of $I(F)$ assume the same values assumed by F, but in different points of the domain, with the property that this rearrangement preserves the measure.

LEMMA 1. Let $\bar{f} \in S$, and $f_0\colon T^\nu \times \mathbf{R}^\nu \to R^+$ be the initial datum of the problem (1.3). There exists

$$\tilde{f}\colon T^\nu \times \mathbf{R}^\nu \to R^+\,, \quad \tilde{f}(\boldsymbol{x}, \boldsymbol{p}) = \tilde{f}(\|\boldsymbol{p}\|)\,, \quad \tilde{f}(\cdot)\colon R^+ \to R^+\,, \\ \tilde{f}(\cdot) \text{ non-increasing}$$

such that:

1) $f_0 \in I(\tilde{f})$

2) $\|f_0 - \bar{f}\|_1 < \delta \to \|\tilde{f} - \bar{f}\|_1 < \delta$ when δ is sufficiently small.

LEMMA 2. Let $\tilde{f} \in S$, $f \in I(\tilde{f}) \cap L_\infty(T^\nu \times \mathbb{R}^\nu)$, and defining

$$T(f) \stackrel{\text{def}}{=} \int_{\boldsymbol{x}} \int_{\boldsymbol{p}} \sqrt{1 + \boldsymbol{p}^2}\, f(\boldsymbol{x}, \boldsymbol{p})\, d\boldsymbol{x}\, d\boldsymbol{p}$$

we have the results:

1) when $\nu = 1$, there exist constants C_1, C_2 such that

$$C_1[C_2\gamma\|f - \tilde{f}\|_1 - \gamma^3]^2 \leqslant T(f) - T(\tilde{f}) \qquad \text{for any } \gamma^2 < C_2\|f - \tilde{f}\|_1$$

2) when $\nu = 2, 3$, and $f, \tilde{f}$ have compact supports, there exists a constant C_3, such that

$$C_3\|f - \tilde{f}\|_1^2 \leqslant T(f) - T(\tilde{f})\,.$$

LEMMA 3. Let $\tilde{f}$ be a function satisfying all the properties stated in Lemma 1, and let $f_0 \in I(\tilde{f}) \cap \mathfrak{I}(\tilde{f}, M)$, (and, by Liouville theorem, $f_t \in I(\tilde{f})$). We have the result:

$$T(f_t) - T(\tilde{f}) \leqslant g(\|f_0 - \tilde{f}\|_1)$$

where $g(x) \to 0$ when $x \to 0$.

THEOREM. Let $\bar{f} \in S$ and $f_0 \in L_\infty(T^\nu \times R^\nu) \cap \mathfrak{I}(\bar{f}, M)$ be the initial datum of the problem (1.3). Then

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \quad \text{such that} \quad \|f_0 - \bar{f}\|_1 < \delta \;\Rightarrow\; \sup_{t \geqslant 0} \|f_t - \bar{f}\|_1 < \varepsilon\,.$$

PROOF. Let the initial datum f_0 such that $\|f_0 - \bar{f}\|_1 < \delta$. By triangular inequality

$$\|f_t - \bar{f}\|_1 \leqslant \|f_t - \tilde{f}\|_1 + \|\tilde{f} - \bar{f}_1\|_1 \tag{2.1}$$

Using Lemma 1, there exists $\tilde{f}$ such that

1) f_0 (and f_t, by Liouville theorem) $\in I(\tilde{f})$;

2) $\|\tilde{f}-\bar{f}\|_1<\delta$.

Now we estimate the first term in the right hand side of (2.1) in the following way:

$$\begin{aligned}
\|f_t-\tilde{f}\|_1 &\leqslant K\sqrt{T(f_t)-T(\tilde{f})} && (f_t\in I(\tilde{f}) \text{ so Lemma 2 applies})\\
&\leqslant K\sqrt{g(\|f_0-\tilde{f}\|_1)} && (\text{by Lemma 3})\\
&\leqslant K\sqrt{g(\|f_0-\bar{f}\|_1+\|\tilde{f}-\bar{f}\|_1)} && (\text{by triangular inequality}).
\end{aligned}$$

Now, $\|f_0-\bar{f}\|_1<\delta$, and $\|\tilde{f}-\bar{f}\|_1<\delta$ using Lemma 1.
In concluding □

$$\|f_t-\bar{f}\|_1\leqslant K\sqrt{g(2\delta)}+\delta\,. \quad \square$$

PROOF OF LEMMA 1. $\forall\lambda\in R^+$, we define the family of sets (see [16, 17]):

$$A_\lambda=\{(\boldsymbol{x},\boldsymbol{p})\in T^\nu\times\mathbf{R}^\nu|f_0(\boldsymbol{x},\boldsymbol{p})>\lambda\}$$

and the functions:

$$p(\lambda)=\frac{\text{meas}\,A_\lambda}{2L_x} \qquad \text{when } \nu=1$$

$$p(\lambda)=\left(\frac{\text{meas}\,A_\lambda}{\pi L_x L_y}\right)^{\frac{1}{2}} \qquad \text{when } \nu=2$$

$$p(\lambda)=\left(\frac{3\ \text{meas}\,A_\lambda}{4\pi L_x L_y L_z}\right)^{\frac{1}{3}} \qquad \text{when } \nu=3$$

$$\tilde{f}(p)=\sup\{\lambda|p(\lambda)\geqslant p,\ \lambda\in(\text{es. inf. } f_0,\ \text{es. sup. } f_0)\}$$
$$\tilde{f}(\boldsymbol{x},\boldsymbol{p})=\tilde{f}(\|\boldsymbol{p}\|)\,.$$

Obviously, the function $\tilde{f}(\cdot)$ is nonincreasing, and satisfies $f_0\in I(\tilde{f})$. To prove the second step of the Theorem 1, we define the sets:

$$S(\lambda,f)=\{(\boldsymbol{x},\boldsymbol{p})\in T^\nu\times\mathbf{R}^\nu|f(\boldsymbol{x},\boldsymbol{p})>\lambda\}\,,\qquad \lambda\in\mathbf{R}^+$$

and the quantities

$$\sigma(\lambda) = \text{meas}\left(S(\lambda, \tilde{f}) \,\Delta\, S(\lambda, \bar{f})\right)$$

$$\Sigma(\lambda) = \text{meas}\left(S(\lambda, \bar{f}) \,\Delta\, S(\lambda, f_0)\right)$$

in which $\bar{f}$, f_0 and $\tilde{f}$ are the stationary solution, the initial datum, and the function built above, respectively. Denoting by χ the set function, we have:

$$\sigma(\lambda) = \int d\boldsymbol{x}\int d\boldsymbol{p}\,|\chi_{S(\lambda,\tilde{f})} - \chi_{S(\lambda,\bar{f})}| = \left|\int d\boldsymbol{x}\int d\boldsymbol{p}(\chi_{S(\lambda,\tilde{f})} - \chi_{S(\lambda,\bar{f})})\right| =$$

$$= \left|\int d\boldsymbol{x}\int d\boldsymbol{p}\; \chi_{S(\lambda,\tilde{f})} - \int d\boldsymbol{x}\int d\boldsymbol{p}\; \chi_{S(\lambda,\bar{f})}\right|$$

and, by definition of $\tilde{f}$

$$\sigma(\lambda) = \left|\int d\boldsymbol{x}\int d\boldsymbol{p}\; \chi_{S(\lambda,f_0)} - \int d\boldsymbol{x}\int d\boldsymbol{p}\; \chi_{S(\lambda,\bar{f})}\right| \leqslant \int d\boldsymbol{x}\int d\boldsymbol{p}\,|\chi_{S(\lambda,f_0)} - \chi_{S(\lambda,\bar{f})}| = \Sigma(\lambda)$$

In concluding:

$$\|\tilde{f} - \bar{f}\|_1 = \int_{\mathbf{R}^+} d\lambda\,\sigma(\lambda) \leqslant \int_{\mathbf{R}^+} d\lambda\,\Sigma(\lambda) = \|\bar{f} - f_0\|_1$$

and this completes the proof of Lemma 1. $\square$

Proof of Lemma 2. Let $\|f\|_\infty = \|\tilde{f}\|_\infty = \alpha$ and N be an integer. We define the sets

$$A_k = \left\{(\boldsymbol{x}, \boldsymbol{p}) \in T^\nu \times \mathbb{R}^\nu \,\middle|\, f(\boldsymbol{x}, \boldsymbol{p}) > k\,\frac{\alpha}{N}\right\} \qquad k = 1, \ldots, N-1$$

$$A_k = \left\{(\boldsymbol{x}, \boldsymbol{p}) \in T^\nu \times \mathbb{R}^\nu \,\middle|\, f(\boldsymbol{x}, \boldsymbol{p}) > k\,\frac{\alpha}{N}\right\}$$

and the step functions

$$f^N = \sum_{k=1}^{N-1} \frac{\alpha}{N}\,\chi(A_k)\,, \qquad \tilde{f}^N = \sum_{k=1}^{N-1} \frac{\alpha}{N}\,\chi(\tilde{A}_k)$$

χ being the set function.

Now we consider the difference between the kinetic energies associated with the functions f^N and $\tilde{f}^N$:

$$T(f^N) - T(\tilde{f}^N) = \int d\boldsymbol{x} \int d\boldsymbol{p} \sqrt{1 + \boldsymbol{p}^2}\,(f^N - \tilde{f}^N) = \tag{2.2}$$

$$= \frac{\alpha}{N} \sum_{k=1}^{N-1} \left\{ \iint_{A_k} d\boldsymbol{x}\, d\boldsymbol{p} \sqrt{1 + \boldsymbol{p}^2} - \iint_{\tilde{A}_k} d\boldsymbol{x}\, d\boldsymbol{p} \sqrt{1 + \boldsymbol{p}^2} \right\} =$$

$$= \frac{\alpha}{N} \sum_{k=1}^{N-1} \left\{ \iint_{A_k \setminus \tilde{A}_k} d\boldsymbol{x}\, d\boldsymbol{p} \sqrt{1 + \boldsymbol{p}^2} - \iint_{\tilde{A}_k \setminus A_k} d\boldsymbol{x}\, d\boldsymbol{p} \sqrt{1 + \boldsymbol{p}^2} \right\}.$$

Now the details of the proof depend on the number of space dimensions: we examine, in the first time, the 1 dimensional case. Using the definitions

$$p_k = \frac{\operatorname{meas} \tilde{A}_k}{2L_k} > 0\,, \qquad \beta_k = \frac{1}{2}\|\chi(A_k) - \chi(\tilde{A}_k)\|_1 \tag{2.3a, b}$$

the following estimates hold:

$$\iint_{A_k \setminus \tilde{A}_k} d\boldsymbol{x}\, d\boldsymbol{p} \sqrt{1 + \boldsymbol{p}^2} \geqslant 2 \int_0^{L_x} d\boldsymbol{x} \int_{p_k}^{p_k + \beta_k/2L_x} d\boldsymbol{p} \sqrt{1 + \boldsymbol{p}^2} = 2L_x \int_{p_k}^{p_k + \beta_k/2L_x} \sqrt{1 + \boldsymbol{p}^2}\, dp$$

$$\iint_{\tilde{A}_k \setminus A_k} d\boldsymbol{x}\, d\boldsymbol{p} \sqrt{1 + \boldsymbol{p}^2} \leqslant 2L_x \int_{p_k - \beta_k/2L_x}^{p_k} \sqrt{1 + p^2}\, dp$$

and the equation (2.2) becomes

$$T(f^N) - T(\tilde{f}^N) \geqslant \frac{2\alpha L_x}{N} \sum_{k=1}^{N-1} \left\{ \int_{p_k}^{p_k + \beta_k/2L_x} \sqrt{1 + p^2}\, dp - \int_{p_k - \beta_k/2L_x}^{p_k} \sqrt{1 + p^2}\, dp \right\} = \tag{2.4}$$

$$= \frac{2\alpha L_x}{N} \sum_{k=1}^{N-1} \left\{ \int_{p_k}^{p_k + \beta_k/2L_x} \frac{\sqrt{1 + p^2} - 1}{p}\, p\, dp - \int_{p_k - \beta_k/2L_x}^{p_k} \frac{\sqrt{1 + p^2} - 1}{p}\, p\, dp \right\}.$$

Using the monotonicity of $(\sqrt{1 + p^2} - 1)/p$ in $[0, \infty)$, the following estimate holds

$$\int_{p_k}^{p_k+\beta_k/2L_x} \frac{\sqrt{1+p^2}-1}{p}\, p\, dp \geqslant \frac{\sqrt{1+p_k^2}-1}{p_k} \int_{p_k}^{p_k+\beta_k/2L_x} p\, dp =$$

$$= \frac{\sqrt{1+p_k^2}-1}{p_k}\, \frac{(\beta_k/L_x)p_k + \beta_k^2/4L_k^2}{2}$$

and the analogous one

$$\int_{p_k-\beta_k/2L_x}^{p_k} \frac{\sqrt{1+p^2}-1}{p}\, p\, dp \leqslant \frac{\sqrt{1+p_k^2}-1}{p_k} \cdot \frac{(\beta_k/L_k)p_k - \beta_k^2/4L_x^2}{2}$$

so that (2.4), taking (2.3*b*) into account, becomes

$$T(f^N) - T(\tilde{f}^N) \geqslant \frac{\alpha}{8NL_x} \sum_{k=1}^{N-1} \frac{\sqrt{1+p_k^2}-1}{p_k} \cdot \|\chi(A_k) - \chi(\tilde{A}_k)\|_1^2\,. \tag{2.5}$$

Now, using the Cauchy-Schwartz inequality in the following way:

$$\sum_{k=1}^{N-1} a_k \cdot 1 \leqslant \left(\sum_{k=1}^{N-1} a_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{N-1} 1\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{N-1} a_k^2\right)^{\frac{1}{2}} \sqrt{N-1} \tag{2.5}$$

that implies

$$\sum_{k=1}^{N-1} a_k^2 \geqslant \frac{1}{N-1}\left(\sum_{k=1}^{N-1} a_k\right)^2 > \frac{1}{N}\left(\sum_{k=1}^{N-1} a_k\right)^2$$

equation (2.5) becomes

$$T(f^N) - T(\tilde{f}^N) \geqslant \frac{1}{8\alpha L_x}\left(\sum_{k=1}^{N-1} \left(\frac{\sqrt{1+p_k^2}-1}{p_k}\right)^{\frac{1}{2}} \frac{\alpha}{N} \|\chi(A_k) - \chi(\tilde{A}_k)\|_1\right)^2. \tag{2.6}$$

The term $(\sqrt{1+p_k^2}-1)/p_k$ in (2.6) goes to 0 when $p_k \to 0$, and this gives some problems in underestimating the right hand side of (2.6) in terms of $\|f^N - \tilde{f}^N\|_1$.

We choose $\gamma \in (0, 1)$ and we determine an integer h such that

(2.7a) $$\left(\frac{\sqrt{1+p_k^2}-1}{p_k}\right)^{\frac{1}{2}} > \gamma \qquad \text{when } 1 \leqslant k < h\,.$$

(2.7b) $$\left(\frac{\sqrt{1+p_k^2}-1}{p_k}\right) \leqslant \gamma \qquad \text{when } h \leqslant k \leqslant N-1$$

so that the equation (2.6) becomes

(2.8) $$T(f^N) - T(\tilde{f}^N) \geqslant \frac{1}{8\alpha L_x}\left[\gamma \sum_{k=1}^{N-1} \frac{\alpha}{N} \|\chi(A_k) - \chi(\tilde{A}_k)\|_1 - \right.$$
$$\left. - \sum_{k=h}^{N-1}\left(\gamma - \left(\frac{\sqrt{1+p_k^2}-1}{p_k}\right)^{\frac{1}{2}} \cdot \frac{\alpha}{N} \|\chi(A_k) - \chi(\tilde{A}_k)\|_1\right)\right]^2 \stackrel{\text{def}}{=}$$
$$\stackrel{\text{def}}{=} \frac{1}{8\alpha L_x}\left[\gamma \|f^N - \tilde{f}^N\|_1 - Q\right].$$

Using (2.7a), and taking into account that

$$\operatorname{meas} A_k = \operatorname{meas} \tilde{A}_k \;\Rightarrow\; \|\chi(A_k) - \chi(\tilde{A}_k)\|_1 \leqslant 2 \operatorname{meas} \tilde{A}_k = 4L_x p_k\,,$$

we obtain

(2.9) $$Q \leqslant \sum_{k=h}^{N-1} \gamma \frac{\alpha}{N} 4L_x p_k\,.$$

Solving (2.7b) with respect to p_k and substituting the result into (2.9), we obtain

$$p_k \leqslant \frac{2\gamma^2}{1-\gamma^4} \qquad \text{when } h \leqslant k \leqslant N-1 \;\Rightarrow\; Q \leqslant \frac{8L_x\gamma^3\alpha}{1-\gamma^4}$$

and (2.8) becomes

$$T(f^N) - T(\tilde{f}^N) \geqslant \frac{1}{8\alpha L_x}\left[\gamma \|f^N - \tilde{f}^N\|_1 - \frac{8L_x\gamma^3\alpha}{1-\gamma^4}\right]^2$$

and, ending N to infinity, using the dominated convergence theorem, we have:

$$T(f) - T(\tilde{f}) \geqslant \frac{1}{8\alpha L_x}\left[\gamma\|f - \tilde{f}\|_1 - \frac{8L_x\gamma^3\alpha}{1-\gamma^4}\right]^2$$

and this proves Lemma 2 in one space dimension.

In two space dimensions, the proof must be modified as follows. We define $p_k, \beta_k, \delta_k, \delta_k'$ through the conditions:

$$p_k = \left(\frac{\text{meas}\,\tilde{A}_k}{\pi L_x L_y}\right), \qquad \beta_k = \frac{1}{2}\|\chi(A_k) - \chi(\tilde{A}_k)\|_1 \tag{2.10a}$$

$$\pi(p_k + \delta_k)^2 - \pi p_k^2 = \pi p_k^2 - \pi(p_k - \delta_k')^2 = \frac{\beta_k}{L_x L_y} \tag{2.10b}$$

and, by monotonicity of $\left((\sqrt{1+p^2}-1)/p\right)$ in $[0, \infty)$, we have

$$T(f^N) - T(\tilde{f}^N) \geqslant \frac{2\pi L_x L_y \alpha}{N}\sum_{k=1}^{N-1}\frac{\sqrt{1+p_k^2}-1}{p_k}\left[\int_{p_k}^{p_k+\delta_k} p^2\,dp - \int_{p_k-\delta_k'}^{p_k} p^2\,dp\right] \tag{2.11}$$

and, performing an obvious change of variable

$$T(f^N) - T(\tilde{f}^N) \geqslant \frac{\alpha L_x L_y}{N}\sum_{k=1}^{N-1}\frac{\sqrt{1+p_k^2}-1}{p_k}\cdot \tag{2.12}$$

$$\cdot\left[\int_{A=0}^{\beta_k/L_xL_y}\left(\sqrt{p_k^2 + \frac{A}{\pi}} - \sqrt{p_k^2 - \frac{\beta_k}{\pi L_x L_y} + \frac{A}{\pi}}\right)dA\right] \geqslant$$

$$\geqslant \frac{\alpha L_x L_y}{N}\sum_{k=1}^{N-1}\frac{\sqrt{1+p_k^2}-1}{p_k}\cdot\frac{1/\pi\cdot(\beta_k/L_xL_y)^2}{\sqrt{p_k^2 + \beta_k/\pi L_x L_y} + p_k}\,.$$

We bound the denominator in (2.12) by observing that, by definition of β_k

$$\beta_k \leqslant \tfrac{1}{2}\cdot 2\,\text{meas}\,A_k = L_x\cdot L_y\pi\cdot p_k^2$$

so that

$$T(f^N) - T(\tilde{f}^N) \geqslant \frac{\alpha}{4\pi L_x L_y(1+\sqrt{2})}\sum_{k=1}^{N-1}\frac{\sqrt{1+p_k^2}-1}{p_k^2}\|\chi(A_k) - \chi(\tilde{A}_k)\|_1^2$$

and using the Cauchy-Schwartz inequality

$$T(f^N)-(\tilde f^N)\geqslant \frac{1}{4\pi\alpha(1+\sqrt{2})L_xL_y}\cdot \cdot\left[\sum_{k=1}^{N-1}\frac{(\sqrt{1+p_k^2}-1)^{\frac12}}{p_k}\cdot\frac{\alpha}{N}\|\chi(A_k)-\chi(\tilde A_k)\|_1\right]^2 .$$

We observe that when $p_k\to 0$, the coefficient of $(\alpha/N)\|\chi(A_k)-\chi(A_k)\|_1$ approaches a non null value, so here we have not the difficulty of the previous case. Despite of this, this coefficient goes to zero when $p_k\to\infty$, so that, in order to underestimate $T(f^N)-T(\tilde f^N)$ in terms of $\|f^N-\tilde f^N\|_1$, we must assume the compactness of the supports of f and $\tilde f$. In this hypothesis, when $N\to\infty$,

$$T(f)-T(\tilde f)\geqslant C_3\|f-\tilde f\|_1^2 .$$

In 3 space dimensions, we define p_k, β_k, δ_k, δ_k' in the following way:

$$\text{(2.13}a) \qquad p_k=\left(\frac{\beta\ \mathrm{meas}\,\tilde A_k}{4\pi L_xL_yL_z}\right)^{\frac13},\qquad \beta_k=\frac12\|\chi(A_k)-\chi(\tilde A_k\|_1$$

$$\text{(2.13}b) \qquad \frac43\pi(p_k+\delta_k)^3-\frac43\pi p_k^3=\frac43\pi p_k^3-\frac43\pi(p_k-\delta_k')^3=\frac{\beta_k}{L_xL_yL_z}$$

so that the equations (2.11) and (2.12) become

$$T(f^N)-T(\tilde f^N)\geqslant\frac{4\pi L_xL_yL_z\alpha}{N}\sum_{k=1}^{N-1}\frac{\sqrt{1+p_k^2}-1}{p_k}\left[\int_{p_k}^{p_k+\delta_k}p^3\,dp-\int_{p_k-\delta_k'}^{p_k}p^3\,dp\right]\geqslant$$

$$\geqslant\frac{\alpha L_xL_yL_z}{N}\sum_{k=1}^{N-1}\frac{\sqrt{1+p_k^2}-1}{p_k}\cdot \cdot\frac{(3/4\pi)(\beta_k/L_xL_yL_z)^2}{(p_k^3+3\beta_k/4\pi L_xL_yL_z)^{\frac23}+(p_k^3+3\beta_k/4\pi L_xL_yL_z)^{\frac13}p_k+p_k^2} .$$

By definition of β_k, we obtain

$$\beta_k\leqslant\frac12\cdot 2\cdot\mathrm{meas}\,\tilde A_k=\frac{4\pi}{3}L_x\cdot L_y\cdot L_z\cdot p_k^3$$

so that

$$T(f^N) - T(\tilde{f}^N) \geq \frac{\alpha}{NL_xL_yL_z} \cdot \frac{3}{16\pi(\sqrt[3]{4} + \sqrt[3]{2} + 1)} \sum_{k=1}^{N-1} \cdot \frac{\sqrt{1+p_k^2} - 1}{p_k^3} \|\chi(A_k) - \chi(\tilde{A}_k)\|_1^2 .$$

and using the Cauchy-Schwartz inequality

$$T(f^N) - T(\tilde{f}^N) \geq \frac{3}{\alpha L_xL_yL_z \cdot 16\pi(\sqrt[3]{4} + \sqrt[3]{2} + 1)} \cdot \left[\sum_{k=1}^{N-1} \left(\frac{\sqrt{1+p_k^2} - 1}{p_k^3}\right)^{\frac{1}{2}} \frac{\alpha}{N} \|\chi(A_k) - \chi(\tilde{A}_k)\|_1\right]^2 .$$

As well as in 2 space dimensions, the coefficient of $\alpha/N\|\chi(A_k) - \chi(\tilde{A}_k)\|_1$ approaches a non null value when $p_k \to 0$. Moreover, this coefficient goes to zero when $p_k \to \infty$, and so, as in 2 space dimensions, we must require the compactness of the supports of f and $\tilde{f}$. In this hypothesis, when $N \to \infty$

$$T(f) - T(\tilde{f}) \geq C_3\|f - \tilde{f}\|_1^2$$

and this completes the proof of Lemma 2.

We note that in the analogous problem of stability for the non relativistic Poisson-Vlasov model [17], in 2 space dimensions, the hypothesis of compactness of the support of the perturbation was not necessary. □

Proof of Lemma 3. Let f_t be the time evolution of f_0 by (1.3). Using (1.4) we have

$$T(f_t) + U(f_t) = T(f_0) + U(f_0)$$

and, by positivity of the potential energy

$$T(f_t) \leq T(f_0) + U(f_0)$$

Subtracting $T(\tilde{f})$ we have:

$$T(f_t) - T(\tilde{f}) \leq [T(f_0) - T(\tilde{f})] + U(f_0) \tag{2.14}$$

Now

$$T(f_0)-T(\tilde f)=\int_{x\in T^\nu} dx\int_{p\in\mathbb{R}^\nu} dp(f_0-\tilde f)\sqrt{1+p^2}\leqslant\int_x dx\int_p dp|f_0-\tilde f|\sqrt{1+p^2}\,.$$

By hypothesis $f_0\in\mathfrak{J}(\tilde f, M)$, so that $f_0\in\mathfrak{J}(\tilde f, M')$, i.e., the integral in the last equation is convergent, and we have the decomposition:

$$(2.15)\qquad T(f_0)-T(\tilde f)\leqslant\int dx\int_{\|p\|<P} dp|f_0-\tilde f|\sqrt{1+p^2}+Q(P)\leqslant$$
$$\leqslant\sqrt{1+P^2}\|f_0-\tilde f\|_1+Q(P)$$

and $Q(P)\to 0$ when $P\to\infty$. (A possible choiche for P is $(\|f_0-\tilde f\|_1)^{-\frac{1}{2}}$).

Using the periodicity of the boundary conditions,

$$U(f_0)=\frac{1}{2}\int dx\|\nabla_x\varphi_0\|^2=\frac{1}{2}\int dx(-\varphi_0)\Delta\varphi_0=\frac{1}{2}\int dx\,\varphi_0(x)\varrho_0(x)=$$
$$=\frac{1}{2}\int_{x\in T^\nu} dx\,\varphi_0(x)\left[1-\int_p f_0(x,p)\,dp\right]=\frac{1}{2}\int_{x\in T^\nu} dx\,\varphi_0(x)\int_{p\in\mathbb{R}^\nu} dp\,[\tilde f(x,p)-f_0(x,p)]=$$
$$=\frac{1}{2}\int_{x\in T^\nu} dx\int_{x'\in T^\nu} dx'\,G(x,x')\varrho_0(x')\int_{p\in\mathbb{R}^\nu} dp\,[\tilde f(x,p)-f_0(x,p)]$$

where G is the Green function for Δ with periodic boundary conditions.

By the boundedness of $\|\varrho_0(\cdot)\|_\infty$, (following by $f_0\in\mathfrak{J}(\tilde f, M)\cap\cap L_\infty(T^\nu\times\mathbb{R}^\nu)$ that implies $f_0\in L_1(T^\nu\times\mathbb{R})^\nu\cap L_\infty(T^\nu\times\mathbb{R}^\nu)$), we have

$$(2.16)\qquad U(f_0)\leqslant\frac{1}{2}\|\varrho_0(\cdot)\|_\infty\cdot\sup_{x\in T^\nu}\int_{x'\in T^\nu} dx'|G(x,x')|\int_{x\in T^\nu} dx\cdot$$
$$\cdot\int_{p\in\mathbb{R}^\nu} dp|\tilde f(x,p)-f_0(x,p)|=\frac{1}{2}\|\varrho_0(\cdot)\|_\infty\cdot\sup_{x\in T^\nu}\int_{x'\in T^\nu} dx'|G(x,x')|\cdot\|\tilde f-f_0\|_1=$$
$$=c\|\tilde f-f_0\|_1\,.$$

In concluding, taking (2.15) and (2.16) into account, (2.14) becomes

$$T(f_t)-T(\tilde f)\leqslant g(\|\tilde f-f_0\|_1)$$

where $g(x)\to 0$ when $x\to 0$. □

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