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Extension of CR-Forms and Related Problems.

ALESSANDRO PEROTTI (*)

Introduction.

Let D be a bounded domain of \mathbf{C}^n , $n \geq 3$, whose boundary contains a real hypersurface S of class C^1 , connected, with boundary ∂S , and such that $A = \partial D \setminus S$ is a non empty piecewise C^1 real hypersurface. Assume that there exists a family $\{V_j\}_{j \in \mathbb{N}}$ of $(n-3)$ -complete open sets such that $\bar{V}_{j+1} \subset V_j$, $(\bigcap_{j \in \mathbb{N}} V_j) \cap \bar{D} = A$.

In particular these conditions are satisfied when A is contained in the zero-set of a pluriharmonic function, a situation which is considered in [11].

We show (Theorem 1) that every locally Lipschitz CR-form of type $(p, 0)$ on S extends, in a unique way, by a $(p, 0)$ -form holomorphic on D and continuous on $D \cup S$.

This result is obtained employing the techniques used in [11], and is based on the existence of primitives of Martinelli-Bochner-Kopelman integral kernel adapted to the sets V_j .

The result shown here sharpens what obtained in [11], [13], [14], where the extension problem is posed only for CR-functions, and on more particular domains.

Furthermore, we consider CR-forms of type (p, q) on S , with $q > 0$. In this case extendibility depends on the Levi convexity of S , as shown by Andreotti and Hill [4] and Kohn and Rossi [8] when S is the boundary of a compact region. However, under the following assumptions we can obtain a jump theorem.

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Let D be a bounded domain of \mathbf{C}^n , $n \geq 2$, of the type considered above, with A of class C^1 . Let $2 < s \leq n - 2$ be a fixed integer. Assume that there exists a family $\{V_j\}_{j \in \mathbb{N}}$ of $(n - s - 2)$ -complete open sets such that $\bar{V}_{j+1} \subset V_j$, $(\bigcap_{j \in \mathbb{N}} V_j) \cap \bar{D} = \bar{A}$.

Then we can show (Theorem 2) that if $0 < p \leq n$, $1 \leq q \leq s - 1$, every regular CR-form of type (p, q) on S is the jump across S between two $\bar{\partial}$ -closed forms defined on D and on $\mathbf{C}^n \setminus (\bar{D} \cup \bigcap_{j \in \mathbb{N}} \bar{V}_j)$.

Finally, we give some applications of the jump theorem under pseudoconvexity assumptions on S .

We obtain some results about the $\bar{\partial}_b$ -problem and the Cauchy problem for $\bar{\partial}$ -operator. In particular, extension theorems for CR-forms of type (p, q) are obtained (Theorems 3, 4). We prove these results for forms of class C^m , $m < +\infty$. In the case of C^∞ forms, these problems have been considered by Andreotti and Hill in [4], under weaker conditions for S .

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1. Preliminaries.

1) We recall the Martinelli-Bochner-Koppelman formula (see [6] Ch. 1 and [1] Ch. 1).

Let Δ be the diagonal of $\mathbf{C}^n \times \mathbf{C}^n$. For $(z, \zeta) \in \mathbf{C}^n \times \mathbf{C}^n \setminus \Delta$, we consider the differential form

$$U(z, \zeta) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^n (-1)^{j-1} \frac{\bar{z}_j - \bar{\zeta}_j}{|z - \zeta|^{2n}} (\bar{d}z_1 - \bar{d}\zeta_1) \wedge \dots \wedge (\widehat{\bar{d}z_j - \bar{d}\zeta_j} \wedge \dots \wedge (\bar{d}z_n - \bar{d}\zeta_n) \wedge dz_1 \wedge \dots \wedge dz_n .$$

$U(z, \zeta) \in C_{(n, n-1)}^\infty(\mathbf{C}^n \times \mathbf{C}^n \setminus \Delta)$, and we have the decomposition

$$U(z, \zeta) = \sum_{q=0}^{n-1} U_{0,q}(z, \zeta)$$

in forms $U_{0,q}(z, \zeta)$ of type $(n, n - q - 1)$ with respect to z and type $(0, q)$ with respect to ζ .

Let $\mu_q(z, \zeta)$ be the form such that $U_{0,q} = \mu_q \wedge dz_1 \wedge \dots \wedge dz_n$. For

$0 \leq p \leq n$ and $0 \leq q \leq n - 1$, we consider the forms

$$U_{p,q}(z, \zeta) = (-1)^{p(n-1)} \mu_q(z, \zeta) \wedge \sum'_{|I|=p} \sigma(I) dz[I] \wedge d\zeta_I,$$

where $dz[I] = dz_1 \wedge \dots \wedge \widehat{dz}_{i_1} \wedge \dots \wedge \widehat{dz}_{i_p} \wedge \dots \wedge dz_n$, $\sigma(I)$ is the sign determined by $dz_I \wedge dz[I] = \sigma(I) dz_1 \wedge \dots \wedge dz_n$, and the sum is taken on increasing multiindices. $U_{p,q}$ is C^∞ on $\mathbf{C}^n \times \mathbf{C}^n \setminus \Delta$, of type $(n-p, n-q-1)$ in z and (p, q) in ζ . We set $U_{p,-1} \equiv U_{p,n} \equiv 0$.

REMARK. The forms $U_{p,q}$ introduced above differ in sign from the corresponding forms defined in [1]. This is due to the fact that they are considered as forms on the product manifold $\mathbf{C}^n \times \mathbf{C}^n$, and not as double forms.

Let D be a bounded domain of \mathbf{C}^n with piecewise C^1 boundary. The orientation of D is defined by the form $dx_1 \wedge \dots \wedge dx_n \wedge dy_1 \wedge \dots \wedge dy_n$, where $z_\alpha = x_\alpha + iy_\alpha$ ($\alpha = 1, \dots, n$), and ∂D has the orientation induced from D .

For $0 \leq p, q \leq n$, let f be a continuous (p, q) -form on \bar{D} such that $\bar{\delta}f$ (defined in the weak sense) is also continuous on \bar{D} . Then the Martinelli-Bockner-Koppelman formula holds:

$$\begin{aligned} \int_{\partial D} f(z) \wedge U_{p,q}(z, \zeta) - \int_{\bar{D}} \bar{\delta}f(z) \wedge U_{p,q}(z, \zeta) + \bar{\delta} \int_{\bar{D}} f(z) \wedge U_{p,q-1}(z, \zeta) &= \\ &= \begin{cases} (-1)^q f(\zeta) & \text{if } \zeta \in D, \\ 0 & \text{if } \zeta \notin \bar{D}. \end{cases} \end{aligned}$$

The form $U(z, \zeta)$ is $\bar{\delta}$ -closed on $\mathbf{C}^n \times \mathbf{C}^n \setminus \Delta$ (see [6] 1.7). Since the component of $\bar{\delta}U$ of type $(n, n-q)$ in z and $(0, q)$ in ζ is $\bar{\delta}_z U_{0,q} + \bar{\delta}_\zeta U_{0,q-1}$, the condition $\bar{\delta}U = 0$ is equivalent to the property

$$\bar{\delta}_z U_{0,q} = -\bar{\delta}_\zeta U_{0,q-1} \quad \text{for } 0 \leq q \leq n$$

(indices z, ζ mean differentiation with respect to z and ζ respectively).

This property obviously holds for μ_q , and so for $U_{p,q}$:

$$(1) \quad \bar{\delta}_z U_{p,q} = -\bar{\delta}_\zeta U_{p,q-1} \quad \text{for } 0 \leq p, q \leq n.$$

In particular, $\bar{\delta}_z U_{p,0} = \bar{\delta}_\zeta U_{p,n-1} = 0$.

2) In order to apply the integral representation formula which we have just mentioned, we need some primitives of the kernel $U(z, \zeta)$.

We recall that an open set D of \mathbb{C}^n is called *q-complete* if there exists an exhaustion function for D which is strongly *q-plurisubharmonic* (i.e. its Levi form has at least $n - q$ positive eigenvalues at every point of D). Such an open set is *q-cohomologically complete*, i.e. $H^p(D, \mathcal{F}) = 0$ for every $p > q$ and every coherent analytic sheaf \mathcal{F} on D ([3]).

PROPOSITION 1. *For fixed $0 \leq s \leq n - 2$, let $V \subseteq \mathbb{C}^n$, $n > 1$, be a $(n - s - 2)$ -complete open set. Then we can find forms $\eta_{p,q}(z, \zeta)$ ($0 \leq p \leq n$, $0 \leq q \leq s$), of class C^∞ on $V \times (\mathbb{C}^n \setminus \bar{V})$, of type $(n - p, n - q - 2)$ in z and (p, q) in ζ , such that*

$$U_{p,0}(z, \zeta) = \bar{\partial}_z \eta_{p,0}(z, \zeta)$$

$$U_{p,q}(z, \zeta) = \bar{\partial}_z \eta_{p,q}(z, \zeta) + \bar{\partial}_\zeta \eta_{p,q-1}(z, \zeta) \quad \text{for } 1 \leq q \leq s.$$

PROOF. Set $U_p(z, \zeta) := \sum_{q=0}^{n-1} U_{p,q}(z, \zeta) \in C^\infty_{(n,n-1)}(\mathbb{C}^n \times \mathbb{C}^n \setminus \Delta)$. From (1)

we have that U_p is $\bar{\partial}$ -closed on $\mathbb{C}^n \times \mathbb{C}^n \setminus \Delta$. Let $\{B_\alpha\}_{\alpha \in J}$ be a locally finite family of Stein open subsets of $\mathbb{C}^n \setminus \bar{V}$ which covers $\mathbb{C}^n \setminus \bar{V}$. For fixed $\alpha \in J$, $V \times B_\alpha$ is $(n - s - 2)$ -complete, therefore $(n - s - 2)$ -cohomologically complete. Then we can find for any $0 \leq p \leq n$ a form $\eta_p^\alpha \in C^\infty_{(n,n-2)}(V \times B_\alpha)$ such that $\bar{\partial}\eta_p^\alpha = U_p$ on $V \times B_\alpha$.

Let $\{\varphi_\alpha\}_{\alpha \in J}$ be a C^∞ partition of unity subordinate to the covering $\{B_\alpha\}_{\alpha \in J}$.

$$\text{Set } \eta_p(z, \zeta) := \sum_{\alpha \in J} \varphi_\alpha(\zeta) \eta_p^\alpha(z, \zeta) \in C^\infty_{(n,n-2)}(V \times (\mathbb{C}^n \setminus \bar{V})).$$

Then we have $\bar{\partial}\eta_p = U_p + \sum_{\alpha \in J} \bar{\partial}_\zeta \varphi_\alpha \wedge \eta_p^\alpha$.

Let $\eta_p = \sum_{q=0}^{n-2} \eta_{p,q}$ be the decomposition of η_p in forms $\eta_{p,q}$ of type $(n - p, n - q - 2)$ in z and (p, q) in ζ . By comparison of types, we obtain in particular $\bar{\partial}_z \eta_{p,0} = U_{p,0}$ and the proposition is proved for $s = 0$.

Now take $s \geq 1$. We have $\bar{\partial}(\eta_p^\alpha - \eta_p^\beta) = 0$ on $V \times (B_\alpha \cap B_\beta)$, and then we can find $\gamma_p^{\alpha\beta} \in C^\infty_{(n,n-3)}(V \times (B_\alpha \cap B_\beta))$ such that $\bar{\partial}\gamma_p^{\alpha\beta} = \eta_p^\alpha - \eta_p^\beta$.

Set $\gamma_p(z, \zeta) := \sum_{\alpha, \beta \in J} \bar{\partial}_z \varphi_\alpha(\zeta) \wedge \varphi_\beta(z) \gamma_p^{\alpha\beta}(z, \zeta) \in C_{(n, n-2)}^\infty(V \times (\mathbb{C}^n \setminus \bar{V}))$. Then

$$\begin{aligned} \bar{\partial}(\eta_p + \gamma_p) &= U_p + \sum_{\alpha \in J} \bar{\partial}_z \varphi_\alpha \wedge \eta_p^\alpha - \sum_{\alpha, \beta \in J} \bar{\partial}_z \varphi_\alpha \wedge \bar{\partial}_z \varphi_\beta \wedge \gamma_p^{\alpha\beta} - \\ &\quad - \sum_{\alpha, \beta \in J} \bar{\partial}_z \varphi_\alpha \wedge \varphi_\beta(\eta_p^\alpha - \eta_p^\beta) = U_p - \sum_{\alpha, \beta \in J} \bar{\partial}_z \varphi_\alpha \wedge \bar{\partial}_z \varphi_\beta \wedge \gamma_p^{\alpha\beta}. \end{aligned}$$

If $\gamma_p = \sum_{q=1}^{n-2} \gamma_{p,q}$ is the decomposition in forms $\gamma_{p,q}$ of type $(n-p, n-q-2)$ in z and (p, q) in ζ , by comparison of types we obtain

$$\bar{\partial}_z \eta_{p,0} = U_{p,0}; \quad \bar{\partial}_z(\eta_{p,1} + \gamma_{p,1}) + \bar{\partial}_z \eta_{p,0} = U_{p,1}$$

and the proposition is proved for $s = 1$.

If $s > 1$, it is sufficient to solve the equation $\bar{\partial} \tau_p^{\alpha\beta\delta} = \gamma_p^{\alpha\beta} + \gamma_p^{\beta\delta} + \gamma_p^{\delta\alpha}$ on $V \times (B_\alpha \cap B_\beta \cap B_\delta)$ and then consider the form

$$\tau_p(z, \zeta) := \sum_{\alpha, \beta, \delta \in J} \bar{\partial}_z \varphi_\alpha(\zeta) \wedge \bar{\partial}_z \varphi_\beta(\zeta) \wedge \varphi_\delta(z) \tau_p^{\alpha\beta\delta}(z, \zeta).$$

Then we have

$$\bar{\partial}(\eta_p + \gamma_p + \tau_p) = U_p + \sum_{\alpha, \beta, \delta} \bar{\partial}_z \varphi_\alpha \wedge \bar{\partial}_z \varphi_\beta \wedge \bar{\partial}_z \varphi_\delta \wedge \tau_p^{\alpha\beta\delta},$$

and therefore

$$\bar{\partial}_z \eta_{p,0} = U_{p,0}; \quad \bar{\partial}_z(\eta_{p,1} + \gamma_{p,1}) + \bar{\partial}_z \eta_{p,0} = U_{p,1};$$

$$\bar{\partial}_z(\eta_{p,2} + \gamma_{p,2} + \tau_{p,2}) + \bar{\partial}_z(\eta_{p,1} + \gamma_{p,1}) = U_{p,2}.$$

By induction we can prove the proposition for any $0 < s < n-2$. ■

REMARK. Given a sequence $\{V_j\}_{j \in \mathbb{N}}$ of open sets satisfying the hypothesis of Proposition 1, with $\bar{V}_{j+1} \subset V_j$, the forms $\eta_{p,q}^j$ defined on $V_j \times (\mathbb{C}^n \setminus \bar{V}_j)$ can be constructed in such a way that $\eta_{p,q}^j(z, \zeta) = \eta_{p,q}^{j+1}(z, \zeta)$ if $z \in V_{j+1}$ and $\text{dist}(\zeta, V_j) > 1/j$. In fact, we can construct the covering \mathfrak{U}^{j+1} of $\mathbb{C}^n \setminus \bar{V}_{j+1}$ recursively by taking a locally finite covering \mathfrak{U}_1^{j+1} of $\bar{V}_j \setminus \bar{V}_{j+1}$ with balls of radius less than $1/4(j+1)$, and then setting $\mathfrak{U}^{j+1} := \mathfrak{U}^j \cup \mathfrak{U}_1^{j+1}$, where \mathfrak{U}^j is the covering of $\mathbb{C}^n \setminus \bar{V}_j$ already constructed. Then on the set $\{\zeta \in \mathbb{C}^n : \text{dist}(\zeta, V_j) > 1/j\}$ the coverings \mathfrak{U}^j and \mathfrak{U}^{j+1} coincide, and if $B \in \mathfrak{U}^j$ is such that

$B \cap \{\zeta \in \mathbf{C}^n : \text{dist}(\zeta, V_j) > 1/j\} \neq \emptyset$, then we have $B \cap B_1 = \emptyset$ for every $B_1 \in \mathfrak{U}_1^{j+1}$. Therefore, the partition of unity subordinate to \mathfrak{U}^{j+1} can be taken equal to that subordinate to \mathfrak{U}^j on the set

$$\{\zeta \in \mathbf{C}^n : \text{dist}(\zeta, V_j) > 1/j\}.$$

2. Extension of CR-forms of type $(p, 0)$.

1) Let D be a bounded domain of \mathbf{C}^n , $n \geq 3$, with the following properties:

- I) ∂D contains a real hypersurface S of class C^1 , connected with boundary ∂S ;
- II) $A := \partial D \setminus S \neq \emptyset$ is a piecewise C^1 real hypersurface;
- III) there exists a family $\{V_j\}_{j \in \mathbb{N}}$ of $(n-3)$ -complete open sets such that $\overline{V}_{j+1} \subset V_j$, $(\bigcap_{j \in \mathbb{N}} V_j) \cap \overline{D} = A$.

Among the open sets of this type there are those considered in [11] where A is contained in the zero-set of a pluriharmonic function.

REMARK. It can be shown (the proof is not trivial) that the complement of a $(n-2)$ -complete open set cannot have compact components. This implies that if D verifies properties I), II), III), and j is so large that $S \setminus V_j$ is connected, then the component of $\mathbf{C}^n \setminus (\overline{D} \cup \overline{V}_j)$ whose boundary contains $S \setminus V_j$ is unbounded.

In order to obtain an extension theorem for CR-forms of type $(p, 0)$ (the weak solutions of the tangential Cauchy-Riemann equation), we shall need the following result, that in the case of functions is proved in [11].

PROPOSITION 2. *Let Σ be an oriented C^1 real hypersurface of \mathbf{C}^n . Let f be a locally Lipschitz CR-form of type $(p, 0)$ on Σ . For any C^1 $(n+r)$ -chain C_{n+r} of Σ and any $(n-p, r-1)$ -form θ , of class C^∞ on a neighbourhood of C_{n+r} , the following formula holds:*

$$\int_{C_{n+r}} f \wedge \bar{\partial} \theta = (-1)^p \int_{\partial C_{n+r}} f \wedge \theta.$$

PROOF. We can repeat the proof given in [11], using the kernel $U_{p,0}$ in place of the Martinelli-Bochner kernel $U_{0,0}$. ■

2) Now we are able to prove the extension theorem for CR-forms of type $(p, 0)$.

THEOREM 1. *Let $D \subseteq \mathbb{C}^n$, $n \geq 3$, be a bounded domain that verifies conditions I), II), III), and $0 < p \leq n$. Then every locally Lipschitz CR-form f on \mathring{S} of type $(p, 0)$ extends, in a unique way, by a $(p, 0)$ -form F , holomorphic on D and continuous on $D \cup \mathring{S}$.*

PROOF. Let $j \in \mathbb{N}$ be a fixed integer.

Let D' be an open set with C^1 boundary such that $D \setminus V_j \subset D' \subset D$ and $\bar{D}' \cap \bar{A} = \emptyset$. We set $S' := S \cap \bar{D}'$ and $A' := \partial D' \setminus S'$.

Suppose we have found the extension F . Then the Martinelli-Bochner-Koppelman formula applied on D' gives

$$F(\zeta) = \int_{S'} f(z) \wedge U_{p,0}(z, \zeta) + \int_{A'} F(z) \wedge U_{p,0}(z, \zeta) \quad \text{if } \zeta \in D' .$$

For $z \in D \cap V_j$ and $\zeta \in \mathbb{C}^n \setminus \bar{V}_j$, we have

$$F(z) \wedge U_{p,0}(z, \zeta) = (-1)^p d_z(F(z) \wedge \eta_{p,0}^j(z, \zeta)) ,$$

where $\eta_{p,0}^j$ is the C^∞ form on $V_j \times (\mathbb{C}^n \setminus \bar{V}_j)$ given by Proposition 1. Therefore

$$(*) \quad F(\zeta) = \int_{S'} f(z) \wedge U_{p,0}(z, \zeta) - (-1)^p \int_{\partial S'} f(z) \wedge \eta_{p,0}^j(z, \zeta) \quad \text{if } \zeta \in D \setminus \bar{V}_j .$$

Since $(*)$ holds for every $j \in \mathbb{N}$, the uniqueness of the extension follows.

Now we prove existence. Let $F(\zeta)$ be the $C^\infty(p, 0)$ -form defined on $\mathbb{C}^n \setminus (S \cup \bar{V}_j)$ by $(*)$. First we show that F is holomorphic on $\mathbb{C}^n \setminus (S \cup \bar{V}_j)$:

$$\begin{aligned} \bar{\partial} F(\zeta) &= (-1)^{p-1} \int_{S'} f(z) \wedge \bar{\partial}_z U_{p,0}(z, \zeta) - \int_{\partial S'} f(z) \wedge \bar{\partial}_z \eta_{p,0}^j(z, \zeta) = \\ &= (-1)^p \int_{S'} f(z) \wedge \bar{\partial}_z U_{p,1}(z, \zeta) - \int_{\partial S'} f(z) \wedge \bar{\partial}_z \eta_{p,0}^j(z, \zeta) . \end{aligned}$$

REMARK. By definition of integration with respect to z (see [6] Ch. 1), if $f(z)$ and $\alpha(z, \zeta)$ are differential forms and C is a chain of dimension $\dim C = \deg f + \deg \alpha(\cdot, \zeta)$, we have

$$\bar{\partial}_\zeta \left(\int_C f(z) \wedge \alpha(z, \zeta) \right) = (-1)^{\deg \alpha(\cdot, \zeta)} \int_C f(z) \wedge \bar{\partial}_\zeta \alpha(z, \zeta).$$

We go back to our proof.

From Proposition 2 and Proposition 1 we obtain

$$\begin{aligned} \bar{\partial} F(\zeta) &= \int_{\partial S'} f(z) \wedge [U_{p,1}(z, \zeta) - \bar{\partial}_\zeta \eta_{p,0}^j(z, \zeta)] = \\ &= \int_{\partial S'} f(z) \wedge \bar{\partial}_z \eta_{p,1}^j(z, \zeta) = (-1)^p \int_{\partial(\partial S')} f(z) \wedge \eta_{p,1}^j(z, \zeta) = 0. \end{aligned}$$

We set

$$\begin{aligned} F_1(\zeta) &:= \int_{S'} f(z) \wedge U_{p,0}(z, \zeta) \quad \text{for } \zeta \in \mathbf{C}^n \setminus S, \\ F_2(\zeta) &:= (-1)^p \int_{\partial S'} f(z) \wedge \eta_{p,0}^j(z, \zeta) \quad \text{for } \zeta \in \mathbf{C}^n \setminus \bar{V}_j, \end{aligned}$$

and denote by F_i^+ , F_i^- ($i = 1, 2$) their restrictions to $D \setminus \bar{V}_j$ and $\mathbf{C}^n \setminus (\bar{D} \cup \bar{V}_j)$ respectively. Since f is locally Lipschitz, F_i^\pm extend continuously to $S \setminus \bar{V}_j \subset S'$, and we have $F_1^+ - F_1^- = f$ on $S \setminus \bar{V}_j$.

Moreover, $F_2^+ = F_2^-$ on $S \setminus \bar{V}_j$ and therefore F extends continuously to $(D \setminus \bar{V}_j) \cup (S \setminus \bar{V}_j)$ and $F = f + F_1^- - F_2^-$ on $S \setminus \bar{V}_j$.

Now take the integer j as in the remark in section 2.1. Let W be a bounded Stein neighbourhood of \bar{D} and $\zeta \in \mathbf{C}^n \setminus (\bar{V}_j \cup \bar{W})$ fixed. On W we can find a primitive ψ of $U_{p,0}(\cdot, \zeta)$. Then from Proposition 2 we have

$$\int_{S'} f(z) \wedge U_{p,0}(z, \zeta) = (-1)^p \int_{\partial S'} f(z) \wedge \psi(z)$$

and therefore

$$F(\zeta) = (-1)^p \int_{\partial S'} f(z) \wedge [\psi(z) - \eta_{p,0}^j(z, \zeta)].$$

Since $\bar{\partial}_z[\psi - \eta_{p,0}^j(\cdot, \zeta)] = 0$ on $V_j \cap W$, there exists ψ' such that $\psi - \eta_{p,0}^j(\cdot, \zeta) = \bar{\partial}_z \psi'$, and then

$$\int_{\partial S'} f \wedge [\psi - \eta_{p,0}^j(\cdot, \zeta)] = \int_{\partial S'} f \wedge \bar{\partial}_z \psi' = (-1)^p \int_{\partial(\partial S')} f \wedge \psi' = 0.$$

Therefore $F \equiv 0$ on $\mathbf{C}^n \setminus (\bar{V}_j \cup \bar{W})$ and the remark in 2.1 implies that $F_1^- - F_2^- = 0$ on $S \setminus \bar{V}_j$, and $F = f$ on $S \setminus \bar{V}_j$.

Thus we have found an extension F_j of f on $D \setminus \bar{V}_j$, for any $j \in \mathbb{N}$ sufficiently large. If $j' > j$, the $(p, 0)$ -form $F_j - F_{j'}$ has components which are holomorphic on $D \setminus \bar{V}_j$ and vanish on $S \setminus \bar{V}_j$. Therefore $F_{j'} - F_j = 0$ on $D \setminus \bar{V}_j$. In fact, if g is such a component, the function obtained extending g by zero on a connected neighbourhood of a point of $S \setminus \bar{V}_j$ is holomorphic in the weak sense, and therefore zero by uniqueness of analytic continuation.

By the same reasoning we can obtain again the uniqueness of the extension. ■

3. Applications of the extension theorem.

1) Let $\varphi_1, \dots, \varphi_m$ be pluriharmonic C^2 functions on \mathbf{C}^n . Let D be a domain verifying I) and II) and such that $A \subset \bigcup_{i=1}^m \{\varphi_i = 0\}$ and $D \subset \bigcap_{i=1}^m \{\varphi_i > 0\}$. This situation was considered in [11] for $m = 1$ and [13] for $m = 2$.

Set $\psi_j := \prod_{i=1}^m \varphi_i - 1/j$ and $V_j := \{z \in \mathbf{C}^n : \varphi_j(z) < 0\}$. For $z_0 \in \partial V_j$, the holomorphic tangent space to ∂V_j at z_0 is

$$T_{z_0}(\partial V_j) = \left\{ w \in \mathbf{C}^n : \partial \psi_j(z_0)(w) = \sum_{i=1}^m \partial \varphi_i(z_0)(w) \prod_{h \neq i} \varphi_h(z_0) = 0 \right\},$$

and the Levi form of ψ_j is given by

$$\mathcal{L}_{\psi_j, z_0}(w) = \sum_{i=1}^m \sum_{k \neq i} \overline{\partial \varphi_i(z_0)(w)} \partial \varphi_k(z_0)(w) \prod_{h \neq i, k} \varphi_h(z_0).$$

Consider the sets

$$V_{z_0, i} := \{z \in \mathbf{C}^n : \varphi_i(z) < \varphi_i(z_0)\} \quad (i = 1, \dots, m).$$

The subspace of $T_{z_0}(\partial V_j)$

$$E_{z_0} := \{w \in \mathbb{C}^n : \partial\varphi_i(z_0)(w) = 0 \text{ for } i = 1, \dots, m\} = \bigcap_{i=1}^m T_{z_0}(\partial V_{z_0, i})$$

has dimension not less than $n - m$, and $\mathcal{L}_{\psi_j, z_0} E_{z_0} = 0$. Therefore the Levi form of ψ_j restricted to $T_{z_0}(\partial V_j)$ has at most $n - 1 - (n - m) = m - 1$ negative eigenvalues at each point of ∂V_j . The set V_j is then (weakly) $(m - 1)$ -pseudoconvex, and so it is $(m - 1)$ -complete (see [16]).

If $m < n - 2$ the family $\{V_j\}_{j \in \mathbb{N}}$ satisfies condition III) of the extension theorem, which can then be applied on D .

2) Theorem 1 can also be applied to deduce the well known theorem on global extension of CR-forms of type $(p, 0)$ defined on the boundary of a bounded domain of \mathbb{C}^n (see [7] Th. 2.3.2' and [1] Th. 3.2).

COROLLARY 1. *Let U be a bounded domain of \mathbb{C}^n , $n \geq 3$, with ∂U of class C^1 and connected. Then every locally Lipschitz CR-form of type $(p, 0)$ on ∂U extends, in a unique way, by a $(p, 0)$ -form holomorphic on U and continuous on \bar{U} .*

PROOF. Let $z_0 \in \partial U$ and $r_0 > 0$ such that $D := U \setminus B(z_0, r_0) \neq \emptyset$ and $S := \partial U \setminus B(z_0, r_0)$ is connected. We set $A := \partial B(z_0, r_0) \cap U$ and $V_j := B(z_0, r_0 + 1/j)$, $j \in \mathbb{N}$. Since the set D verifies conditions I), II), III) of Theorem 1, we obtain the extension on D .

Since z_0 and r_0 are arbitrary, we have the extension on the whole U . ■

REMARK. Corollary 1 holds also for $n = 2$ and for CR-forms of type $(p, 0)$ only continuous on ∂U (see [1]).

3. The extension theorem can also be applied to the following situation, that generalizes the result contained in [14] Th. 1.

Let D be a domain of \mathbb{C}^n , $n \geq 3$, verifying I) and II). Suppose A relatively open in a hypersurface M defined by a C^∞ function ϱ on a Stein open set U of \mathbb{C}^n . Suppose also that there exists $r > 0$ such that ϱ is strongly $(n - 3)$ -plurisubharmonic on the set $\{z \in U : 0 < \varrho(z) < r\}$.

Then the extension theorem holds on D . In fact, the open sets $V_j := \{z \in U : \varrho(z) < 1/j\}$ ($j > 1/r$) verify condition III) of Theorem 1. To see this, consider the functions $\varphi_j := \varphi - \varepsilon_j \log(-\varrho + 1/j)$, where

φ is a strongly plurisubharmonic exhaustion function for U . If $\varepsilon_j > 0$ is small enough, we obtain, after restricting U if necessary, that φ_j is a strongly $(n-3)$ -plurisubharmonic exhaustion function for V_j .

4. « Jump » theorem for CR-forms of type (p, q) .

1) Now we consider CR-forms of type (p, q) , with $q > 0$. As we shall see later, in this case it is not possible to obtain an extension theorem as for $(p, 0)$ -forms without imposing a pseudoconvexity condition on S .

However, we can prove a « jump » theorem, i.e. a CR-form can be written as the difference between two $\bar{\partial}$ -closed forms, defined on the two sides of the hypersurface (additive Riemann-Hilbert problem).

In the case when the CR-forms are defined on the boundary of a compact set, this result is proved in [1] Th. 2.10-2.11.

In the following we deal with bounded domains of \mathbf{C}^n , $n \geq 2$, satisfying conditions I) and II) of 2.1, where A is of class C^1 and has the following property:

III') for a fixed $2 \leq s \leq n-2$, there exists a family $\{V_j\}_{j \in \mathbb{N}}$ of $(n-s-2)$ -complete open sets such that $\bar{V}_{j+1} \subset V_j$, $(\bigcap_{j \in \mathbb{N}} V_j) \cap \bar{D} = \bar{A}$.

THEOREM 2. Let $D \subseteq \mathbf{C}^n$, $n \geq 2$, be a bounded domain satisfying properties I), II), III'). Let $0 < p < n$, $1 \leq q \leq s-1$ or $q = n-1$. Consider a (p, q) -form f of class C^1 on a neighbourhood of S , and suppose f is CR on \dot{S} for $q \neq n-1$. Then there exist two C^∞ forms of type (p, q) F^+ on D and F^- on $\mathbf{C}^n \setminus (\bar{D} \cup (\bigcap_{j \in \mathbb{N}} \bar{V}_j))$, continuous up to \dot{S} , with $\bar{\partial}F^+ = \bar{\partial}F^- = 0$ and $F^+ - F^- = f$ on \dot{S} .

PROOF. Let $j \in \mathbb{N}$ be a fixed integer.

Let \tilde{f} be a C^1 extension of f on a neighbourhood of \bar{D} . From Martinelli-Bochner-Koppelman formula we get

$$(1) \quad \int_S f(z) \wedge U_{p,q}(z, \zeta) + \int_A \tilde{f}(z) \wedge U_{p,q}(z, \zeta) - \int_D \bar{\partial} \tilde{f}(z) \wedge U_{p,q}(z, \zeta) + \\ + \bar{\partial} \int_D \tilde{f}(z) \wedge U_{p,q-1}(z, \zeta) = \begin{cases} (-1)^q \tilde{f}(\zeta) & \text{if } \zeta \in D \\ 0 & \text{if } \zeta \notin \bar{D} \end{cases}$$

and by Proposition 1, for $\zeta \notin \bar{V}_j$ we have

$$(2) \quad \int_A \tilde{f} \wedge U_{p,q} = \int_A \tilde{f} \wedge [\delta_z \eta_{p,q}^j + \delta_\zeta \eta_{p,q-1}^j] = \\ = -(-1)^{p+q} \int_{\partial S} f \wedge \eta_{p,q}^j - (-1)^{p+q} \int_A \delta_z \tilde{f} \wedge \eta_{p,q}^j + \int_A \tilde{f} \wedge \delta_\zeta \eta_{p,q-1}^j.$$

Then, if $1 < q < s-1$ we set

$$F^\pm := (-1)^q \int_S f \wedge U_{p,q} - (-1)^p \int_{\partial S} f \wedge \eta_{p,q}^j + (-1)^q \int_A \tilde{f} \wedge \delta_\zeta \eta_{p,q-1}^j + \\ + (-1)^q \delta_\zeta \int_D \tilde{f} \wedge U_{p,q-1}$$

on $D \setminus \bar{V}_j$ and on $\mathbf{C}^n \setminus (\bar{D} \cup \bar{V}_j)$ respectively.

Therefore we have

$$\delta F^\pm = (-1)^{p-1} \int_S f \wedge \delta_\zeta U_{p,q} - (-1)^q \int_{\partial S} f \wedge \delta_\zeta \eta_{p,q}^j = \\ = (-1)^p \int_S f \wedge \delta_z U_{p,q+1} - (-1)^q \int_{\partial S} f \wedge \delta_\zeta \eta_{p,q}^j = \\ = (-1)^q \int_{\partial S} f \wedge [U_{p,q+1} - \delta_\zeta \eta_{p,q}^j] = (-1)^q \int_{\partial S} f \wedge \delta_z \eta_{p,q+1}^j = 0,$$

since f is CR.

If $q = n-1$, we consider the forms

$$F^\pm := (-1)^{n-1} \int_S f \wedge U_{p,n-1} + (-1)^{n-1} \int_A \tilde{f} \wedge U_{p,n-1} + (-1)^{n-1} \delta_\zeta \int_D \tilde{f} \wedge U_{p,n-2},$$

which are $\tilde{\delta}$ -closed since $\tilde{\delta}_\zeta U_{p,n-1} = 0$.

From (1) and (2) we now obtain

$$\begin{cases} F^+ = \tilde{f} + (-1)^p \int_A \delta_z \tilde{f} \wedge \eta_{p,q}^j + (-1)^q \int_D \delta_z \tilde{f} \wedge U_{p,q} \\ F^- = (-1)^p \int_A \delta_z \tilde{f} \wedge \eta_{p,q}^j + (-1)^q \int_D \delta_z \tilde{f} \wedge U_{p,q} \end{cases}$$

(the first integral is missing for $q = n-1$).

Since the integral $\int_D \bar{\partial}_z \tilde{f} \wedge U_{p,q}$ is absolutely convergent for every $\xi \in \mathbf{C}^n$, F^+ and F^- extend continuously up to $S \setminus \bar{V}_i$, and we have

$$F^+|_{S \setminus \bar{V}_j} - F^-|_{S \setminus \bar{V}_j} = \tilde{f}|_{S \setminus \bar{V}_j} = f|_{S \setminus \bar{V}_j}.$$

By the remark following Proposition 1, F^\pm define, as $j \in \mathbf{N}$ varies, two C^∞ forms, $\bar{\partial}$ -closed on D and on $\mathbf{C}^n \setminus (\bar{D} \cup \bigcap_{j \in \mathbf{N}} \bar{V}_j)$ respectively, continuous up to \mathring{S} , and such that $F^+ - F^- = f$ on \mathring{S} . ■

REMARK. If $S \in C^\infty$ and $f \in C_{(p,q)}^m(S)$ ($m \geq 2$), then F^\pm extend up to \mathring{S} as forms of class $C^{(m,\lambda)}$, with $\lambda \in (0, 1)$ (i.e. the coefficients of F^\pm are C^m and their derivatives of order m are λ -Hölder).

This follows from Proposition 0.10 of [2] applied to the integral $\int_D \bar{\partial}_z \tilde{f} \wedge U_{p,q}$, where $\{D_\epsilon\}_{\epsilon > 0}$ is an increasing family of open sets with C^∞ boundary such that $\bigcup_{\epsilon > 0} D_\epsilon = D$ and $\bigcup_{\epsilon > 0} (S \cap \bar{D}_\epsilon) = \mathring{S}$.

5. Applications.

1) A first application of Theorem 2 allows to obtain an extension theorem for CR-forms of type (p, q) .

Let D be a domain which satisfies I) and II), with S contained in a smooth and strictly pseudoconvex hypersurface Σ and A of class C^1 . Assume that \bar{A} has a fundamental system of Stein neighbourhoods $\{V_j\}_{j \in \mathbf{N}}$ with boundaries ∂V_j transversal to Σ .

THEOREM 3. Let $0 < p < n$ and $1 < q < n - 3$. Let f be a (p, q) -form of class C^m on S ($2 < m < +\infty$) and W a neighbourhood of \bar{A} . If f is CR on \mathring{S} , then there exists a (p, q) -form F of class C^{m-2} on $D \cup \mathring{S}$, $\bar{\partial}$ -closed on D , and such that $F|_{S \setminus W} = f|_{S \setminus W}$.

In the proof of this theorem we need the following approximation lemma (for a proof see [15] p. 244 or [4] p. 785):

LEMMA. Let $V \subseteq \mathbf{C}^n$ be an open set and $G := \{z \in V : g(z) < 0 \text{ and } h(z) < 0\}$ where g, h are C^∞ on V and $dg(z) \neq 0$ if $g(z) = 0$, $dh(z) \neq 0$ if $h(z) = 0$, $dg \wedge dh(z) \neq 0$ if $g(z) = h(z) = 0$. We suppose that \bar{G} is a compact connected region of \mathbf{C}^n . Let W be a neighbourhood of the set

$\{z \in V : g(z) = h(z) = 0\}$. Then there exists a domain $G' \subset G$ defined on V by a C^∞ function F such that $\partial G' \setminus \partial G \subset W$ and

$$\mathcal{L}_{F,z} \geq \alpha(z) \mathcal{L}_{g,z} + \beta(z) \mathcal{L}_{h,z}$$

for every $z \in \partial G'$, where $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ and

$$\text{supp } \alpha \cap \partial G' \subset W \cup \{g = 0\}, \quad \text{supp } \beta \cap \partial G' \subset W \cup \{h = 0\}.$$

PROOF OF THEOREM 3. Let $\Sigma = \{\varrho = 0\}$, where ϱ is a strongly plurisubharmonic function on a neighbourhood U of \bar{D} . Let $\varrho' : U \rightarrow \mathbb{R}$ be a C^∞ function such that $\varrho' \leq \varrho$ on U , $\varrho' = \varrho = 0$ on $S \setminus W$ and $\varrho' < \varrho = 0$ on ∂S , and with the same convexity properties as ϱ .

Take $V_j \subseteq W$. We may suppose that V_j is defined by a strongly plurisubharmonic function ψ on a neighbourhood of V_j .

Let ψ' be a C^∞ function on \mathbb{C}^n such that $\psi' = \psi$ on a small neighbourhood V of $(\partial V_j \setminus D) \cap \{\varrho' > 0\}$ and $\psi' < 0$ on the component of $\{\varrho' < 0\} \setminus V$ which contains D .

The open set $D' := \{\varrho' < 0\} \cap \{\psi' < 0\}$ contains $\bar{D} \setminus \dot{S}$, and $\partial D' \supset S \setminus W$. Applying the lemma to D' we can obtain a C^∞ domain $D'' \subset D'$, strictly pseudoconvex, such that $D'' \supset \bar{D} \setminus \dot{S}$ and $\partial D'' \supset S \setminus W$.

Since condition III') of 4.1 is verified for $s = n - 2$, we can apply Theorem 2 on D and obtain two $\bar{\partial}$ -closed forms F^+ on D and F^- on $\mathbb{C}^n \setminus \bar{D}$, of class $C^{(m,\lambda)}$ up to \dot{S} ($0 < \lambda < 1$), such that $F^+ - F^- = f$ on \dot{S} .

The form $F^-|_{\mathbb{C}^n \setminus \bar{D}}$ is C^m up to the boundary $\partial D''$. Let \tilde{F}^- be a C^m extension to \mathbb{C}^n , and let $\beta := \bar{\partial} \tilde{F}^- \in C^{m-1}_{(\varrho, \varrho+1)}(\mathbb{C}^n)$. Then we have $\bar{\partial} \beta = 0$ and $\text{supp } \beta \subseteq \overline{D''}$. According to Theorem 4.3 of [1] (see also [9]), we can find a $u \in C^{m-2}_{(\varrho, 0)}(\mathbb{C}^n)$ with $\text{supp } u \subseteq \overline{D''}$ and $\bar{\partial} u = \beta$.

Let $F := F^+ - \tilde{F}^- + u \in C^{m-2}_{(\varrho, 0)}(D \cup \dot{S})$. Then $\bar{\partial} F = -\bar{\partial} \tilde{F}^- + \bar{\partial} u = 0$, and we have $F|_{S \setminus W} = (F^+ - F^-)|_{S \setminus W} = f|_{S \setminus W}$. ■

REMARKS. (1) In particular, Theorem 3 can be applied when S is strictly pseudoconvex and A is contained in the zero-set of a plurisubharmonic function.

(2) If S is q -pseudoconvex, a theorem analogous to Theorem 4.3 of [1] holds for forms of certain types depending on q (see [12]). This can be applied as before to obtain the extension.

(3) If CR-forms are C^∞ , Theorem 3 is a particular case of a more general theorem which can be deduced from results of Andreotti and

Hill [4] and which holds under weaker convexity assumptions on S . These results are based on a difficult cohomology vanishing theorem, while in the preceding theorem only integral representation formulas are used.

Let D be a domain verifying conditions I) and II) of 2.1, with $S \subset \Sigma := \{z \in U : \varrho(z) = 0\}$, U open set of \mathbf{C}^n . Suppose that $\mathcal{L}_{\varrho,z}$ has at least $r + 1$ positive eigenvalues for z in a neighbourhood W of S .

Let $\psi : U \rightarrow \mathbb{R}$ be strongly $(n - r - 1)$ -plurisubharmonic on a neighbourhood of the set $\{\psi = 0\}$, such that $d\psi \neq 0$ on $\{\psi = 0\}$, $\psi < 0$ on \bar{D} and $\{\psi = 0\} \cap \Sigma \subset W$. Let $D' := \{\varrho < 0\} \cap \{\psi < 0\}$ be such that $D' \subset\subset U$ and $d\varrho \wedge d\psi \neq 0$ on $\Sigma \cap \{\psi = 0\}$.

From the results of [4] we can obtain the following theorem.

THEOREM. *If $0 \leq p \leq n$ and $0 \leq q \leq r - 1$, every CR-form of type (p, q) of class C^∞ on S extends on $D \cup S$ by a $\bar{\partial}$ -closed C^∞ form.*

PROOF. We can apply the lemma to D' and obtain a domain $D'' = \{F = 0\}$ contained in D' such that $D'' \supset \bar{D} \setminus S$, $\partial D'' \supset S$ and $\mathcal{L}_{F,z|T_z(\partial D'')}$ has at least r positive eigenvalues at each point $z \in \partial D''$. In fact, $\mathcal{L}_{\varrho,z}$ is positive definite on a r -dimensional subspace of $T_z(\partial D'')$, for every $z \in \partial D'' \cap W$. The same holds for $\mathcal{L}_{\psi,z}$, for z in a neighbourhood of $\{\psi = 0\}$.

Let $V \subset U$ be an open neighbourhood of D'' such that $D \cup S \subset V$, $\partial D'' \setminus S \subset \partial V$. Now the theorem follows from Theorem 6 of [4] part I and Theorem 6 of [4] part II, since $S = \{z \in V : F(z) = 0\}$ and we have $D'' = V^- := \{z \in V : F(z) < 0\} \supset D$. ■

Now suppose that S is a C^∞ real hypersurface defined by $\varrho = 0$ and let x_0 be a point of strict pseudoconvexity. Then there exists an open neighbourhood U of x_0 such that $\bar{U} \cap \{\varrho < 0\}$ is biholomorphic to the intersection of a strictly convex set with a halfspace, and so it verifies the conditions considered in Remark (1).

Thus we obtain a local extension theorem for CR-forms, which generalizes the local extension theorem for CR-functions of H. Lewy [10]:

COROLLARY 2. *There exists a neighbourhood S' of x_0 , relative to S , such that every CR (p, q) -form of class C^m ($2 \leq m \leq +\infty$) on a neighbourhood of S' in S ($0 \leq p \leq n$, $0 \leq q \leq n - 3$), extends by a $\bar{\partial}$ -closed form on an open set D contained in the convex side, such that $\partial D \supset S'$.* ■

REMARK. For C^∞ forms, also this result is a particular case of a theorem of Andreotti and Hill (see Theorem 6 of [4] part I and Theorem 2

of [4] part II), which assures local extendibility of C^∞ CR-forms of type (p, q) near a point $x_0 \in S$ where the Levi form has at least $r + 1$ positive eigenvalues, for $0 < q < r - 1$.

2) Now we consider the problem of extension of CR-forms outside D .

Suppose that S is strictly pseudoconvex and A is contained in the zero-set of a pluriharmonic function φ .

THEOREM 4. *For $0 < p < n$, $1 < q < n - 3$, let f be a CR-form of type (p, q) of class C^m on S ($2 \leq m \leq +\infty$). Then*

(a) *if $m < +\infty$ and W is an open neighbourhood of \bar{A} , there exists a (p, q) -form F of class C^{m-1} on $(\mathbf{C}^n \setminus \bar{D}) \cup \mathring{S}$ which is $\bar{\delta}$ -closed on $\mathbf{C}^n \setminus \bar{D}$ and such that $F|_{S \setminus W} = f|_{S \setminus W}$;*

(b) *if $m = +\infty$, there exists a (p, q) -form F , C^∞ on $(\mathbf{C}^n \setminus \bar{D}) \cup \mathring{S}$, such that $\bar{\delta}F = 0$ on $\mathbf{C}^n \setminus \bar{D}$ and $F|_{\mathring{S}} = f|_{\mathring{S}}$.*

PROOF. Since condition III') is verified for $s = n - 2$, we can apply Theorem 2 on D and obtain two $\bar{\delta}$ -closed forms F^+ on D and F^- on $\mathbf{C}^n \setminus \bar{D}$, of class C^m up to \mathring{S} , such that $F^+ - F^- = f$ on \mathring{S} .

Let $m < +\infty$. Let $\lambda > 0$ so small that the set $D' := \{z \in D : -\varphi(z) + \lambda|z|^2 > 0\}$ contains $D \setminus W$. Applying the lemma to D' we get a strictly pseudoconvex domain D'' with C^∞ boundary which contains $D \setminus W$.

Let $u \in C_{(p,q-1)}^m(\bar{D}'')$ be such that $\bar{\delta}u = F^+$ (see [15] Th. 3). Let \tilde{u} be a C^m extension of u to \mathbf{C}^n . Set $F := \bar{\delta}\tilde{u} - F^-$. Then $F \in C_{(p,q)}^{m-1}((\mathbf{C}^n \setminus \bar{D}) \cup \mathring{S})$ and $\bar{\delta}F = 0$, $F|_{S \setminus W} = (F^+ - F^-)|_{S \setminus W} = f|_{S \setminus W}$.

Now take $m = +\infty$. According to Theorem 2 of [13], we can find $u \in C_{(p,q-1)}^\infty(D \cup \mathring{S})$ such that $\bar{\delta}u = F^+$. Let \tilde{u} be a C^∞ extension of u to \mathbf{C}^n .

Then $F := \bar{\delta}\tilde{u} - F^-$ is the desired extension of f . ■

3) Under the same assumptions, we consider the inhomogeneous $\bar{\delta}_b$ -problem on \mathring{S} :

$$(1) \quad \bar{\delta}_b u = f$$

where f is a $(p, q + 1)$ -form on \mathring{S} and u is a (p, q) -form on \mathring{S} .

COROLLARY 3. *Let $0 < q < n - 4$, and let f be a CR-form of type $(p, q + 1)$ of class C^m on \mathring{S} ($3 \leq m \leq +\infty$). Then*

(a) if $q > 0$ and $m = +\infty$, there exists a solution $u \in C_{(p,q)}^\infty(\mathring{S})$ of (1);

(b) if $q = 0$ or $m < +\infty$ and W is an open neighbourhood of \bar{A} , there exists a (p, q) -form $u \in C_{(p,q)}^{m-2}(S \setminus \bar{W})$ such that $\bar{\delta}_b u = f|_{S \setminus W}$.

PROOF. For $j \in \mathbb{N}$ sufficiently large, consider the set $D_j := \{z \in D : \varphi(z) > 1/j\}$. Let $S_j := S \cap \bar{D}_j$.

According to Theorem 3, we can find a form $\tilde{f}_j \in C_{(p,q+1)}^{m-2}(D_j \cup \mathring{S}_j)$ such that $\bar{\delta}\tilde{f}_j = 0$ on D_j and $\tilde{f}_j|_{\mathring{S}_{j-1}} = f|_{\mathring{S}_{j-1}}$.

As in the proof of Theorem 4, we can construct a smooth strictly pseudoconvex domain D'_j such that $\bar{D}_{j-1} \subset \bar{D}'_j \subset D_j \cup \mathring{S}_j$. Let $u_j \in C_{(p,q)}^{m-2}(\bar{D}'_j)$ be such that $\bar{\delta}u_j = \tilde{f}_j$ on D'_j . Then $\bar{\delta}_b(u_j|_{\mathring{S}_{j-1}}) = f|_{\mathring{S}_{j-1}}$.

Since $\bar{\delta}(u_j - u_{j+1|D'_j}) = 0$, if $q > 0$ and $m = +\infty$ we can find a $v_j \in C_{(p,q-1)}^\infty(\bar{D}'_j)$ such that $u_j - u_{j+1|D'_j} = \bar{\delta}v_j$. Let \tilde{v}_j be a C^∞ extension of v_j to \mathbb{C}^n . Replace u_{j+1} by $u'_{j+1} := u_{j+1} + \bar{\delta}\tilde{v}_j$. Then $u'_{j+1} \in C_{(p,q)}^\infty(\bar{D}'_{j+1})$ and $u'_{j+1|D'_j} = u_j$, $\bar{\delta}_b(u'_{j+1|S_j}) = f|_{S_j}$, and therefore we can glue the forms together and obtain the desired form u .

Now take $q = 0$ or $m < +\infty$. For j sufficiently large, we have $\mathring{S}_{j-1} \supset S \setminus \bar{W}$. Then $u_j|_{S \setminus \bar{W}} \in C_{(p,q)}^{m-2}(S \setminus \bar{W})$ and $\bar{\delta}_b(u_j|_{S \setminus \bar{W}}) = f|_{S \setminus \bar{W}}$. ■

REMARK. Results similar to this have been obtained by Boggess [5] using an explicit integral formula for the solutions.

4) Finally, under the same hypotheses we consider the general Cauchy problem for $\bar{\delta}$:

$$(2) \quad \bar{\delta}u = f, \quad u|_{\mathring{S}} = g$$

where f is a $(p, q+1)$ -form on $D \cup \mathring{S}$, g is a (p, q) -form on \mathring{S} and u is a (p, q) -form on $D \cup \mathring{S}$.

COROLLARY 4. Let $0 \leq q \leq n-3$, $f \in C_{(p,q+1)}^m(D \cup \mathring{S})$ and $g \in C_{(p,q)}^m(\mathring{S})$ ($2 \leq m \leq +\infty$) such that $\bar{\delta}f = 0$ on D and $\bar{\delta}_b b = f|_{\mathring{S}}$. Then

(a) if $q = 0$ or $q > 1$ and $m = +\infty$, there exists a solution $u \in C_{(p,q)}^\infty(D \cup \mathring{S})$ of (2);

(b) if $q = 1$ or $q > 0$ and $m < +\infty$, and W is an open neighbourhood of \bar{A} , there exists a form $u \in C_{(p,q)}^{m-2}((D \cup S) \setminus \bar{W})$ such that $\bar{\delta}u = f$ on $D \setminus \bar{W}$, $u|_{S \setminus \bar{W}} = g|_{S \setminus \bar{W}}$.

For $q = 0$ the solution is unique.

PROOF. Let $m = +\infty$. According to Theorem 2 of [13], we can find $w \in C_{(p,q)}^\infty(D \cup \mathring{S})$ such that $\bar{\partial}w = f$ on D . Then $\bar{\partial}_b(g - w|_{\mathring{S}}) = 0$.

If $q = 0$, from Theorem 1 we can obtain an holomorphic extension $h \in C_{(p,0)}^\infty(D \cup \mathring{S})$ of $g - w|_{\mathring{S}}$ (see Proposition 0.10 of [2] and the remark following Theorem 2).

We set $u := w + h$. Then we have $\bar{\partial}u = \bar{\partial}w = f$ on D and $u|_{\mathring{S}} = w|_{\mathring{S}} + h|_{\mathring{S}} = g$.

If $q > 1$, from Corollary 3 we get a form $v \in C_{(p,q-1)}^\infty(\mathring{S})$ such that $\bar{\partial}_b v = g - w|_{\mathring{S}}$. Let \tilde{v} be a C^∞ extension of v on $D \cup \mathring{S}$, and $u := w + \bar{\partial}\tilde{v}$. Then $\bar{\partial}u = \bar{\partial}w = f$ and $u|_{\mathring{S}} = g$.

Now take $m < +\infty$ or $q = 1$. For $j \in \mathbb{N}$, let D_j , S_j and D'_j be as in the proof of Corollary 3.

If $q > 0$, take j such that $D_j \supset D \setminus W$, and let D' be a smooth strictly pseudoconvex domain such that $D_j \subset D' \subset D$. Let $W \in C_{(p,q)}^m(\overline{D'})$ be such that $\bar{\partial}w = f|_{D'}$. Then $\bar{\partial}_b(g|_{S_j} - w|_{S_j}) = 0$. According to Theorem 3, we can find a $\bar{\partial}$ -closed extension $v \in C_{(p,q)}^{m-2}((D \cup \mathring{S}) \setminus \overline{W})$ of $(g - w)|_{S_j \setminus \overline{W}}$.

Then $u := w + v \in C_{(p,q)}^{m-2}((D \cup \mathring{S}) \setminus \overline{W})$, and $\bar{\partial}u = f$ on $D \setminus \overline{W}$, $u|_{S \setminus \overline{W}} = g|_{S \setminus \overline{W}}$.

Now suppose $q = 0$. For any j , let $w_j \in C_{(p,0)}^m(\overline{D}'_j)$ be such that $\bar{\partial}w_j = f$ on D'_j .

Then $\bar{\partial}_b((g - w_j)|_{S_{j-1}}) = 0$, and from Theorem 1 we get an holomorphic extension $h_j \in C_{(p,0)}^m(D_{j-1} \cup \mathring{S}_{j-1})$ of $(g - w_j)|_{S_{j-1}}$.

Let $u_j := w_j + h_j \in C_{(p,0)}^m(D_{j-1} \cup \mathring{S}_{j-1})$. Then $\bar{\partial}u_j = f$ on D_{j-1} and $u_j|_{S_{j-1}} = g|_{S_{j-1}}$.

We have $\bar{\partial}(u_j - u_{j+1}|_{D_{j-1}}) = 0$ and $(u_j - u_{j+1})|_{S_{j-1}} = 0$. Then $u_{j+1}|_{D_{j-1} \cup \mathring{S}_{j-1}} = u_j$, and setting $u|_{D_{j-1} \cup \mathring{S}_{j-1}} := u_j$ we obtain the solution $u \in C_{(p,0)}^m(D \cup \mathring{S})$.

If $u_1, u_2 \in C_{(p,0)}^m(D \cup \mathring{S})$ are two solutions of (2), then $u_1 \equiv u_2$, since $\bar{\partial}(u_1 - u_2) = 0$ and $(u_1 - u_2)|_{\mathring{S}} = 0$. ■

REMARK. For C^∞ forms, these results are contained in those of Andreotti and Hill (Proposition 4.1 of [4] part I).

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