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Spaces of urelements, II

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Spaces of Urelements, II.

NORBERT BRUNNER (*)

1. Introduction.

The present paper is concerned with the topological structure of the urelemente-line $L$. We shall give several characterisations of the continuous $T_2$-images of compact subsets of $L$ which can be considered as analogies to the Hahn-Mazurkiewicz characterisation of the Peano-continua [The exact analogue would be the theory of the continuous Hausdorff-images of the compact and locally connected subspaces of $\mathbb{R}$]. $L$ is obtained from the set $U$ of all urelements of the Mostowski-permutation-model as the Dedekind-completion of an ordered sum of countably many copies of $U$. In independence proofs concerning the axiom of choice $AC$ the line $L$ plays a role, similar to that of $\mathbb{R}$ in the Cohen-Halpern-Levy-model.

Our notation will follow [15], [19] in set theory (in particular, the definition of the Mostowski-model MM can be found in [19]) and [14] in topology. Unless stated otherwise, our proofs will be in $ZF^\omega$-set theory (i.e.: $AC$ and the axiom of foundation are not used). We recall from [8], that in MM $U$ carries a natural order $<$ which is dense, Dedekind-complete and without endpoints. Hence the order topology $U$ is $T_2$, locally compact and connected. Moreover, since $P(U)$ is Dedekind-finite, $U$ is countably compact (in the covering sense).

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1.1. DEFINITION. In the Mostowski-model we set $L = (U \times \mathbb{Z}) \cup \mathbb{Z}$. We order $L$ by $<$ defined through the clauses $(a, n) < (b, n)$, if $a < b$, and $(a, n) < (b, m)$ if $n < m$. $L$ is the order topology.

As is easily verified, $<$ is a dense and Dedekind-complete order on $L$. Therefore $A$ is connected and locally compact (no AC is needed for the proof of the Haar-König criterion which is applied here). From its construction it also follows, that $L$ is $\sigma$-compact and not countably compact.

There is some interest to the question, if $L$ is paracompact. Its subspace $U$ is not even metacompact (c.f. [7]), whence $L$ is not hereditarily paracompact. Also, it was shown by van Douwen [12], that one needs some form of $AC$ to prove, that every linearly ordered topological space (LOTS) is normal. In particular, there are non-normal LOTS in MM (but $U$ is normal). These LOTS cannot be paracompact, either, because one does not need $AC$ in the proof of «paracompact $+ T_2 \Rightarrow T_4$».

Another question about $L$ concerns the Lindelöf-property. For it is known (see [5]), that the assertion « $\sigma$-compact $\Rightarrow$ Lindelöf » is equivalent to the countable choice axiom $AC^\omega$ which is false in MM. The same is true (for locally compact $T_2$-spaces) concerning « $\sigma$-compact $\Rightarrow$ weakly Lindelöf » and « weakly Lindelöf $\Rightarrow$ Lindelöf », where a space is weakly Lindelöf, if each open cover has a countable refinement.

1.2. LEMMA. In the Mostowski-model, $L$ is a paracompact and weakly Lindelöf space which is not Lindelöf.

PROOF. We first prove paracompact and weakly Lindelöf. Let $0 \subset L$ be an open cover and let $e \in [U]^\omega = \{e \subseteq U : e \text{ finite}\}$ be a finite support of $0 : 0 \in \Delta(e) = \{x \in MM : \text{sym } x \ni \text{fix } e\}$. We will construct a countable and locally finite open refinement $V$. Let $f \in [U \setminus e]^\omega$ be a choice set for the connectedness components of $U \setminus e$ (in $U$). For each $x \in L \cap \Delta(e) (= \mathbb{Z} \cup (e \times \mathbb{Z}))$ we let $I_x$ be the least open interval around $x$ whose endpoints are in $f \times \mathbb{Z}$. Then for each $I_x$ there is a $Q \in 0$ such that $I_x \subseteq Q$ [To this end we take any $0 \in 0$ which contains $x$ and let $I$ be any open interval around $x$ such that $I \cap \Delta(e) = \{x\}$. Then for some $\pi \in \text{fix } e$ we have $\pi I = I_x$ and setting $Q = \pi 0 \in \pi 0 = 0$ we get $I_x \subseteq Q$]. Similarly, using a choice set $g$ of the components of $U \setminus (e \cup f)$, we can define intervals $I_x \in \Delta(g)$ where $x \in (f \times \mathbb{Z})$, such that $I_x \subseteq Q$, some $Q \in 0$, whence $V = \{I_x : x \in (L \cap \Delta(e)) \cup (f \times \mathbb{Z})\}$ is the desired locally finite countable refinement of $0$. 


We next show, that $L$ is not Lindelöf. We define

$$O_n = \{-n, n\}_{E: E \in \mathcal{F}} \setminus n, n < \omega \subseteq \mathcal{P}(L) \text{ and } |\text{supp } E| = n, \quad n \in \omega,$$

where $|\cdot|$ denotes the cardinality and $\text{supp}(E)$ is the least support (c.f. [19]); since $E \rightarrow \text{supp } E \in \Delta(\emptyset)$ we have that $O_n \in \Delta(\emptyset)$. Then $\emptyset = \bigcup \{O_n : n \in \omega\}$ is an open covering of $L$ in $\Delta(\emptyset)$ which does not have a countable subcovering. Assume on the contrary, that $\{V_n : n \in \omega\}$ is such a subcovering and let $e$ be the support of $n \rightarrow V_n$. Since $|e| \in \omega \subseteq L$, there is a $V_n \in \emptyset$ containing $|e|$; let $V_n \in O_m$. Then $|e| < m$, because $|e| \in V_n \in O_m$. On the other hand, $V_n = \{-m, m\}_{E \in \Delta(e)}$ [since $V_n \in \Delta(e)$ and $E$ is defined from $V_n$ via $E = \{-m, m\}_{E \in \Delta(e)}$, whence $\text{supp } E \subseteq e$ and $m = |\text{supp } E| < |e|$, a contradiction. Q.E.D.

Another useful property of $L$ is the fact, that each subspace satisfies the countable antichain condition [this follows from the remark in [7], that there is no subspace of $U$ with an infinite family of pairwise disjoint open sets]. Also, each nowhere dense set in a subspace of $L$ is countable [because each infinite subset of $U$ contains nonempty open intervals]. It follows, that the nowhere dense subsets of a compact subspace $X$ of $L$ are finite [because $X \subseteq [-n, n]$ for some $n \in \omega$, whence $X$ and $\mathcal{P}(X)$ are Dedekind-finite].

In general, the characterisation of a class of spaces which are built up from urelements may lead to rather obscure notions. This is to be expected, because the very existence of these spaces contradicts $AC$. Interestingly, the class of continuous $T_2$-images of compact subsets of $L$ can be described by very harmless looking properties. Our main result states, that these are just the Dedekind-finite Lindelöf $T_2$-spaces which are hereditarily locally Lindelöf and have at most finitely many isolated points. On the other hand, it is wellknown that in MM the hereditarily (weakly) Lindelöf $T_2$-spaces are well-orderable, while a Dedekind-set (a Dedekind-finite but infinite subset of $\mathbb{R}$) is hereditarily weakly Lindelöf but not Lindelöf (see [7]), when carrying the subspace topology of $\mathbb{R}$ [which is second countable]. We close this section with a proof of half of this theorem.

1.3. LEMMA. In the Mostowski-model, if $X$ is a $T_2$-space which is a continuous image of a compact subspace $Y$ of $L$, then $\mathcal{P}(X)$ is Dedekind-finite and $X$ is compact and hereditarily locally compact.
The set of isolated points of $X$ is finite and every subspace satisfies the countable antichain condition.

**Proof.** It follows from the previous remarks, saying $P(Y)$ is a Dedekind-finite space, every subspace satisfying the countable antichain condition, that the same is true for $X$, whence the set of isolated points is finite. If $A \subset X$, then $A \setminus A$ is finite [if $Y \rightarrow X$ denotes the continuous onto-mapping, $B = f^{-1}(A)$; then $\overline{B}$ is compact, whence $f(\overline{B}) = \overline{f(B)}$ and $A \setminus A = f(\overline{B}) \setminus f(B) \subseteq f(B \setminus B)$. As $\overline{B} \subseteq Y$ is nowhere dense, it is finite and so is $A \setminus A]$. So every subspace $A$ of $X$ is an open subspace of the compact space $A^{-}$, whence $A$ is locally compact and $X$ is hereditarily locally compact. Q.E.D.

If $AC$ holds, then a hereditarily locally compact $T_{2}$-space has a dense set of isolated points (a consequence of[17]).

2. **Main result.**

We collect the purely topological arguments needed in the proof of our theorem in the following two lemmas.

2.1. **Lemma.** If $R$ is wellorderable, then the following assertions on a Hausdorff-space $X$ with at most finitely many isolated points are equivalent:

(i) $X$ is Dedekind-finite, Lindelöf and hereditarily locally Lindelöf.

(ii) $X$ is compact and hereditarily locally compact.

(iii) $X$ is compact and every dense set is open.

Moreover, such a space is hereditarily normal and each nowhere dense set is finite.

**Proof.** (i) $\Rightarrow$ (ii) follows from [6]: A Dedekind-finite Lindelöf $+ T_{1}$-space is compact [c.f. the proof of 3.1(i)].

(ii) $\Rightarrow$ (iii): Let $D$ be dense and choose $x \in D$. Since $X$ is regular and $D$ is locally compact, it is regular and there are a regularly open set $0$ and a compact set $K \subseteq D$ such that $x \in 0 \cap D \subseteq K$. Since $K$ is closed [because it is compact], $x \in 0 = (0 \cap D)^{o} \subseteq K \subseteq D$, whence $x \in D^{o}$ and $D$ is open.
(iii) ⇒ « nowhere dense sets are finite »: We show: If \( S^o = \emptyset \), then \( S \) is finite. For \( S^o = \emptyset \) implies, that \( X \setminus S = S^c \) is dense, whence by (iii) \( O = \{ \{x\} \cup S^c : x \in S \} \) is an open cover of \( X \). Since \( X \) is compact, it has a finite subcover \( F \) and \( S = (\bigcup F) \cap S \) is finite, too.

(iii) ⇒ \( T_4 \): If \( Y \) is a subspace of \( X \), then by the preceding remark the boundaries of subsets of \( Y \) are finite. So \( Y \) is \( T_4 \), because more generally a \( T_4 \)-space \( Y \) in which all boundaries are compact is \( T_4 \). For let \( A, B \) be closed and disjoint. Comfort [11] has shown, that one does not need \( AC \) in order to separate the compact sets \( A \setminus A^o, B \setminus B^o \) by disjoint open sets \( P, Q \). Then \( A^o \cup (P \setminus B) \) and \( B^o \cup (Q \setminus A) \) are open sets separating \( A \) and \( B \).

(ii) ⇒ (i) is now trivial: If \( Y \subseteq X \), then \( Y \setminus Y \) is finite, whence \( Y \) is locally compact [since \( Y \) is open in the compact space \( Y^- \)].

(ii) ⇒ (i) is based on a lemma from [8]: If \( R \) is wellorderable and \( X \) is an antianticompact \( T_2 \)-space \( (i.e.: \) each infinite \( Y \subseteq X \) has an infinite compact subset), then \( X \) is Dedekind-finite. To show, that \( X \) is antianticompact, let \( Y \subseteq X \) be infinite. As was shown before, \( Y \setminus Y^o \) is finite and so \( Y^o \) is infinite, too. If every compact subset of \( Y^o \) is finite, then \( Y^o \) is discrete, since \( Y^o \) is a locally compact \( T_2 \)-space. It follows, that \( Y^o \) is an infinite set of isolated points of \( X \), a contradiction to our assumption. Q.E.D.

In 2.1(iii) one cannot weaken « compact » to « Lindelöf ». For as was shown by Sharma [23] (in response to a question in Bankston [2]), \( ZF^o + AC + \) the continuum hypothesis implies, that there is a Lindelöf \( T_2 \)-space \( X \) without isolated points (not Dedekind-finite of course) in which every dense set is open. The property, that dense sets are open was introduced by Hewitt in his doctoral thesis [18] who called it MI. It is motivated by some classical results in real analysis: A family \( S \) of sets is reducible, if there is a set \( D \) such that \( D \setminus 0 \neq \emptyset \) and \( O \setminus D \neq \emptyset \) for all \( 0 \neq \emptyset \) in \( S \) \( (i.e. \) \( D \) and its complement are dense). This notion can be traced back to Bernstein [3], who used it (without naming it so) in his construction of a Bernstein set. Hewitt’s paper was the first systematic study of reducible topologies which was later extended \( e.g. \) by Katetov [20], Ceder [10], Aull [1] and Elkin [13] (as for predecessors, the Lusin and Sierpinski paper [22] should be mentioned). MI is a strong negation of reducibility. [In 2.1 « irreducible » is too weak: a finite product of intervals of \( U \) is irreducible]. For example, every maximal \( T_4 \)-space \( (i.e.: \) the topology is maximal with respect to having no isolated points) is MI ([18]). In view of
the importance of Hewitt's [18] ideas and concepts for the characterisation of the continuous $T_2$-images $X$ of compact subsets of $L$ we decide to call the class of these spaces $X$ **Hewitt-spaces**. In contrast to the theory of the Peano-continua, a Hewitt-space $X$ satisfies $\text{ind}(X) < 1$ \cite{1.3 + 2.1}. Also, Hewitt spaces are not closed under finite products, but they are closed-hereditary.

2.2. **Lemma.** A Dedekind-finite Suslin-line which is not compact has only two $T_2$-compactifications, namely the one-point compactification and the order-compactification (which may coincide).

**Proof.** $X$ is a Souslin-line, if $X$ is a linearly ordered and connected topological space which satisfies the countable antichain condition but is not separable (automatically true, if $X$ is Dedekind-finite); let $<$ be the order of $X$ [it is dense and Dedekind-complete]. Since $X$ is locally compact, the one point compactification $a(X)$ exists and the order compactification $oX$ reduces to the addition of an upper and/or lower bound; we assume « and » \cite [i.e. $X = ]\leftarrow, \rightarrow[,$ the other cases being analogous to this one]. Let $k: X \to kX$ be any $T_2$-compactification, where $k$ is the embedding; we assume $kx = x$ for all $x \in X$. Since $kX$ is regular, the regularly closed sets form local bases.

We first prove the following remark: If $P$ is a regularly closed neighbourhood of $p \in kX \setminus X$, then $P \cap X$ contains an unbounded interval \cite{[14x-0]}; since there are only two types of unbounded intervals, $T_2$ implies: $|kX \setminus X| < 2$. Because $X$ is locally connected and satisfies the countable antichain condition, in $X (P \cap X)^o$ is the union of a countable family $P$ of pairwise disjoint open intervals, the components. As $X$ is Dedekind-finite, the countable set of their endpoints is finite, whence also $P$ is finite. One of the intervals in $P$ is unbounded, for otherwise $(P \cap X)^o$ is relatively compact in $X$, which is impossible \cite [because $P$ is regularly closed, $P = (P \cap X)^-$ and $P \cap X$ is regularly closed in $X$, whence in $X (P \cap X)^o = P \cap X$ is compact, yielding $P = P \cap X$, while $p \in P \setminus X$]. We now assume, that $kX \setminus X = \{p, q\}$, $p \neq q$, and show, that $k = o$. \cite {If $kX \setminus X = \{p\}$ we similarly will obtain $k = a$}. We set $P_x = \{p\} \cup x$, $Q_x = \{q\} \cup [x, \rightarrow[,$ $x \in X$, and fix two disjoint regularly closed neighbourhoods $P$ and $Q$ of $p$ and $q$. It follows from the above observation that—modulo an obvious change in notation—we have $P_x \subseteq P$ and $Q_y \subseteq Q$ for some $x, y$. If $x' < x$, then $kX \setminus P_x' = Q \cup [x', y]$ is closed in $kX$, whence $P_x'$ and $Q_y'$ are open neighbourhoods of $p$ and $q$, where $x > x'$ and $y < y'$. If $P' \subseteq P$ is any regularly closed neighbourhood
of \( p \), then since \( P' \cap Q_x = \emptyset \) the unbounded intervals in \( P' \cap X \) have to be unbounded to the left, whence \( P_x \subseteq P' \) for some \( x' \in X \). Hence \( \{ P_{x'}^x : x' < x \} \) and \( \{ Q_y^y : y < y' \} \) are neighbourhoods of \( p \) and \( q \), proving \( k = 0 \). Q.E.D.

2.3. **Theorem.** In the Mostowski-model the following assertions are equivalent for a Hausdorff-space \( X \).

(i) \( X \) is a continuous image of a compact subset of \( L \) (a Hewitt space).

(ii) \( X \) is obtained from a compact subset of \( L \) by a finite number of identification of finite sets.

(iii) \( X \) is Lindelöf, hereditarily locally Lindelöf, Dedekind-finite and the set of isolated points is finite.

Moreover, every Hewitt space is hereditarily locally connected and it is not hereditarily metacompact (unless it is finite).

**Proof.** (ii) means, that there is a compact \( Y \subseteq L \) and a mapping \( f : Y \to X \) which is a composite of homeomorphisms and quotient maps which identify finite sets with points. The «moreover» statements are consequences of (ii): Since every subset \( M \) of \( L \) is a locally finite union of possibly degenerate intervals, it is locally connected. As every subspace \( Z \) of \( X \) is the quotient of some \( M \subseteq L \), \( Z \) is locally connected in view of the ZF\(^\omega\)-result of Whyburn [25]. If \( X \) is infinite, then \( X \) is obtained from an infinite \( Y \subseteq L \) by the collapse of some finite sets \( E_i, i \in \mathbb{N} \). In \( L \left( Y \setminus \bigcup_{i \in \mathbb{N}} E_i \right)^\emptyset \neq \emptyset \), and so there is an open interval \( I \subseteq Y \) which is embedded homeomorphically into \( X \). Since a subset of \( U \) is metacompact iff it is compact ([7]) we obtain a non-metacompact subset of \( X \) in this way.

(ii) \( \Rightarrow \) (i) is trivial and (i) \( \Rightarrow \) (iii) is 1.3.

(iii) \( \Rightarrow \) (ii): We recall from the proof of 2.1, last part, that \( X \) is antianticompact. Let \( e \subseteq U \) be a support of \( X \) and \( X \). We first show, that \( |\text{supp}(x) \setminus e| < 1 \) for all \( x \in X \). Let \( x \not\in \Lambda(e) \). If we write \( \text{supp}(x) \setminus e = \text{im} (a) \) for some strictly increasing \( a \in (U \setminus e)^\ast \), \( n = |\text{supp}(x) \setminus e| \), then \( \text{orb}_e x = \{ px : x \in \text{fix } e \} \) can be identified with the set \( \text{orb}_e a \subseteq (U \setminus e)^\ast \), using a mapping \( f_0 : \text{orb}_e a \to \text{orb}_e x : = 0 \) in \( \Lambda(e) \). In [8] it was shown, that the antianticompactness of \( X \)
implies, that the relative topology $X|_{\text{orb}_{\alpha} x}$ is homeomorphic under $f_0$ with the topology on $\text{orb}_{\alpha} a$ which is generated by the product topology $\mathcal{U}$. Hence if $n > 2$, then there are some nontrivial intervals $I_i \subseteq U_i, i \in \mathbb{N}$, such that $\prod_{i \in \mathbb{N}} I_i \subseteq \text{orb}_{\alpha} a$. But this is impossible, since then by 2.1 every nowhere dense set—like $\{\alpha| n-1\} \times I_{n-1}$—should be finite. Therefore for $x \in X \setminus \Delta(e)$, $\text{orb}_{\alpha} x = 0$ is homeomorphic to the open interval $I_0 = \text{orb}_{\alpha} a \subseteq U$ whose endpoints are in $e \cup \{-, +\}$ and $f_0$ is the homeomorphism. We next show, that $O = \{\text{orb}_{\alpha} x : x \in \Delta(e)\}$ is finite. If on the contrary, $O$ is infinite, then by the pigeon-hole principle there is some component $I$ of $U \setminus e$ such that $Q = \{o \in O: \text{dom } f_o = I\}$ is infinite. Therefore, for any $u \in I$, the set $\{f_0(u): o \in Q\} \subseteq X$ is infinite and wellorderable, contradicting the Dedekind-finiteness of $X$. We now define a compact set $Y \subseteq L$ and a quotient map $f: Y \to X$. To this end we enumerate $O = \{o_i : i \in \mathbb{N}\}$ and $X \setminus \Delta(e) = \{x_i : i \in \mathbb{N}\}$; corresponding to $o_i$, we have a component $I_i$ of $U \setminus e$ and a homeomorphism $f_i: I_i \to o_i$. We set $Y = \{-i-1 : i \in \mathbb{N}\} \cup \{(I_i \times \{2i\})^- : i \in \mathbb{N}\}$. Since by 2.1 $f_i: I_i \to o_i$ is a $T_2$-compactification we conclude from 2.2, that $f_i = a$ or $f_i = 0$. Since $(I_i \times \{2i\})^- = aI_i$ and because $aI_i$ is obtained from $aI_i$ by collapsing the finite set of endpoints, we obtain onto mappings $g_i: (I_i \times \{2i\})^- \to o_i$ which collapse only finite sets and an onto mapping $g: \{-i-1: i \in \mathbb{N}\} \subseteq L \to X \setminus \Delta(e)$. Then $f = g \cup \bigcup \{g_i: i \in \mathbb{N}\}$ is the required mapping $Y \to X$. Q.E.D.

We finally observe, that in the Mostowski-model not every compact and connected linearly ordered space $X$ with $\mathbb{P}(X)$ Dedekind-finite is a Hewitt-space: If $X$ is $o(U.2)$, $U.2$ the lexicographic product, then $X$ is reducible.

3. Related results.

The notion of reducibility appears in several independence results. (i) Let us say, a family $\mathcal{S}$ of sets is $\sigma$-reducible, if it is a countable union of reducible sets. Then a nonprincipal ultrafilter on $\omega$ is $\sigma$-reducible, if it is not a Ramsey-point, the existence of Ramsey points being independent of ZFC (classical results due to Rudin, Booth and Kunen). (ii) If AC holds, then some nonempty $\mathcal{S} \subseteq [\omega]^\omega$ is not $\sigma$-reducible. But in the model of Blass [4], each nonempty set $\mathcal{S} \subseteq [\omega]^\omega$ is $\sigma$-reducible [which follows from a Feferman-type argument, c.f. [15]].
Also, combining Easton-forcing with the Blass-construction, we obtain a model such that every nonempty family $S$ of infinite sets is $\sigma$-reducible, provided that $\cup S$ is wellorderable. (iii) It follows from topological results due to Malhyn, that the existence of nonprincipal ultrafilters is equivalent to the existence of maximal $T_2$-spaces (in particular, the dense sets form an ultrafilter, if $X$ is maximal $T_2$).

One of the ingredients for the proof of 2.3 was the observation, that Dedekind-finite Lindelöf $T_1$-spaces are compact. There are even models, where several notions of compactness coincide; 3.1(ii) answers one of the problems listed in [21].

3.1. LEMMA. (i) In the Cohen-Halpern-Levy model, if $X$ is Lindelöf and $T_1$, then $X$ is compact.

(ii) In Gitik's model, if $X$ is countably compact, then $X$ is compact.

PROOF. In both cases, we shall actually prove a more general statement than asserted.

(i) If there is a Dedekind-set, then every Lindelöf $+$ $T_1$-space is compact.

As the first step we observe, that $\omega$ with the discrete topology is not Lindelöf. For, since there is an almost disjoint family $A$ of infinite subsets of $\omega$ such that $A$ is equipotent with $\mathbb{R}$ [a well-known ZF° result of Sierpinski for which Buddenhagen [9] has an elegant topological proof], our assumption implies, that there is an infinite, Dedekind-finite almost disjoint family $B \subseteq [\omega]^\omega$ such that $\cup B = \omega$.

If $\omega$ were Lindelöf, then there would exist a finite subcover $F$ of $B$, which is impossible, since for $B \in B \setminus F \cap \bigcup F \in [\omega]^{<\omega}$ while $B$ is infinite. We next show, that every Lindelöf $+$ $T_1$-space which is not compact contains a closed copy of $\omega$. Since it is not countably compact, there is a countable open cover $\langle O_n: n \in \omega \rangle$ such that $O_n \subseteq O_{n+1}$. Via $T_1$ we define an auxiliary cover $O: O = \{O_{n+1} \setminus \{x\}: x \in O_{n+1} \setminus O_n \text{ and } n \in \omega\}$. Then a countable subcover of $O$ defines an infinite sequence $x_k$, $x_k \in O_{n+1} \setminus O_n$ for some $n(k)$, and by a routine verification we see, that $\{x_k: k \in \omega\}$ is closed an discrete, hence Lindelöf which is impossible. As there exists a Dedekind-set in the Cohen-model, the result follows.

(ii) Specker's axiom, that every set has a grad implies, that every countably compact space is compact.
If we set $C_0 = \text{class of countable sets}$, $C'_\alpha = \text{class of countable unions of sets in}$ and then Specker's axiom [24] says, that $V = \bigcup \{C_\alpha : \alpha \in \Omega_n\}$. In [16] Gitik has constructed a model for this axiom under the assumption, that there are arbitrarily large strongly compact cardinals. We assume, that $(X, X)$ is countably compact but not compact. Let $\alpha$ be the least ordinal, such that there is an open cover $O \in C_\alpha$ without a finite subcover; $\alpha > 0$. $O$ is a countable union of families $O_n \in \bigcup \{C_\beta : \beta < \alpha\}$ and by countably compactness there is a $F \in [\omega]^{<\omega}$ such that $\{O_n : n \in F\}$ covers $X$. But $\operatorname{grad} \left( \bigcup \{O_n : n \in F\} \right) < \alpha$, whence by the minimality of $\alpha$ there is a finite subcover of $\bigcup \{O_n : n \in F\}$ and a fortiori of $O$, a contradiction. Q.E.D.

We finally note, that Specker's axiom is equivalent to the following assertion: $AT_2$-space is discrete, if countable intersections of open sets are open.

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