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Direct Products of Linearly Compact Primary Rings.

P. N. ANH (*)

SUMMARY - Numakura [4], [5] gave criterions for compact rings to be direct products of primary rings. In this note we extend these results to linearly compact rings. As a consequence we get a characterization of those rings which are direct products of division rings, of uni-serial rings (i.e. such artinian rings in which the $P^k$ ($k = 1, 2, ...$) are all the one-sided ideals where $P$ denotes the unique maximal ideal) and of rings of complete discrete valuations on division rings.

1. All rings considered will be associative rings with identity. In what follows $R$ denotes always a ring and $J$ is its Jacobson radical (briefly: radical). A ring $R$ is said to be topologically artinian or topologically noetherian if it is the inverse limit of artinian or noetherian left $R$-modules endowed with the inverse limit topology. A topological ring $R$ is called linearly compact if open left ideals form a base for neighbourhoods of zero and every finitely solvable system of congruences $x \equiv x_k (\text{mod} \ L_k)$ where the $L_k$ are closed left ideals, is solvable. Remark that every topologically artinian ring is linearly compact (see [2] satz 4) and every compact ring is topologically artinian (see [4] Lemma 5).

By a classical result of Artin every artinian primary ring is a matrix ring over a local ring. In [3] Leptin proved that every topologically artinian primary ring is a matrix ring (not necessarily of finite size) over a local ring. These results can be considered as a generalization

of Artin-Wedderburn Theorem on simple artinian rings. Throughout
this note a primary ring is understood in a more general sense than
what is usual. We call a ring primary if the factor by its radical is
the endomorphism ring of a vector space over a division ring. Now
our purpose is to characterize direct products of linearly compact
rings of some kind.

Before turning to the main results we need some preparations.

For every subset \( A \) of a topological space we denote by \( \overline{A} \) its
topological closure. Define for every ideal \( I \) of a topological ring the
following ideals

\[
\begin{align*}
\overset{\lambda}{I} &= \overline{I}, \\
\overset{\mu+1}{I} &= \overline{\mu I}, \\
\overset{\lambda}{\bigcap}_{\mu < \lambda} I &= \bigcap_{\mu < \lambda} \overset{\lambda}{I} \\
\overset{\lambda}{\bigcap}_{\mu < \lambda} I &= \bigcap_{\mu < \lambda} I \\
\overset{\lambda}{I} &= \bigcap_{\mu < \lambda} I \\
\overset{\lambda}{I} &= \bigcap_{\mu < \lambda} I \\
\overset{\lambda}{I} &= \bigcap_{\mu < \lambda} I
\end{align*}
\]

Then there is an ordinal \( \eta \) for which \( \xi I = \eta I \), \( I = I \), \( I = I \) for all
\( \xi > \eta \). These \( \xi I \), \( I \), \( I \) will be denoted by \( \overset{\xi}{I} \), \( I \), \( I \) and \( \overset{\xi}{\bigcap} I = I \),
\( \overset{\xi}{\bigcap} I = I \), \( \overset{\xi}{\bigcap} I = I \) hold clearly. If \( \overset{\xi}{I} = 0 \), then \( I \) is called
\( l(r^-) \) transfinitely nilpotent. If \( I = 0 \), then \( I \) is said to be transfinitely
nilpotent.

**Proposition 1 ([2] (3)).** If the set \( \{L_\alpha\} \) of closed left ideals in a linearly
compact ring has the finite intersection property, then

\[ A + \bigcap L_\alpha = \bigcap (A + L_\alpha) \]

holds for every closed left ideal \( A \).

**Proposition 2 ([2] Lemma).** For any two (one-sided) ideals \( A \)
and \( B \) of a topological ring we have

\[ \overline{AB} = \overline{AB} = \overline{AB} = \overline{AB} \]

**Proposition 3 ([2] Satz 9).** The radical of a topologically artinian
ring is \( r \)-transfinitely nilpotent.

**Proposition 4 ([2] (4)).** If \( A \) and \( B \) are closed (left) ideals in a lin-
early compact ring, then \( A + B \) is closed.
PROPOSITION 5 ([2] Satz 4). Every continuous isomorphism between topologically artinian rings is a topological isomorphism.

PROPOSITION 6 ([3] Satz 1). Every linearly compact module over a semisimple linearly compact ring is a direct product of simple modules.

PROPOSITION 7. Let $A$ and $B$ be closed (left) ideals in a linearly compact $R$. If $A + B = R$ holds, then we have $A_\xi + B_\xi = R$ for every ordinal $\xi > 1$.

PROOF. We prove the assertion by transfinite induction. By the assumption we have $A_1 + B_1 = A + B = R$. Suppose that $A_\xi + B_\xi = R$, then there are elements $a \in A_\xi$, $b \in B_\xi$ such that $a + b = 1$ and so $a^2 = 1 - 2b + b^2$. By $2b - b^2 \in B_\xi$ we have $A_{\xi+1} + B_\xi = R$. This implies the existence of elements $a^* \in A_{\xi+1}$ and $b^* \in B_\xi$ such that $a^* + b^* = 1$. This shows that $(b^*)^2 = 1 - 2a^* + (a^*)^2 = 1 - c$ with $c = 2a^* - (a^*)^2 \in A_{\xi+1}$ from which we obtain $A_{\xi+1} + B_{\xi+1} = R$.

If we have now the equality $A_\xi + B_\xi = R$ for every ordinal $\xi < \lambda$ where $\lambda$ is a limit ordinal, then for each $\mu < \lambda$ it is true that

$$A_\xi + B_\mu = A_\xi + B_\lambda + B_\mu = R \quad \text{if} \quad \lambda > \xi > \mu$$

and

$$A_\xi + B_\mu = A_\lambda + A_\mu + B_\mu = R \quad \text{if} \quad \lambda > \mu > \xi$$

that is, $A_\xi + B_\mu = R$ holds for every $\xi$, $\mu < \lambda$.

Now we have by Proposition 1 for every fixed $\mu < \lambda$

$$R = \bigcap_{\xi < \lambda} (A_\xi + B_\mu) = \bigcap_{\xi < \lambda} A_\xi + B_\mu = A_\lambda + B_\mu$$

which implies

$$R = \bigcap_{\mu < \lambda} (A_\lambda + B_\mu) = A_\lambda + \bigcap_{\mu < \lambda} B_\mu = A_\lambda + B_\lambda.$$

This completes the proof.

PROPOSITION 8. For any closed (left) ideals $A$ and $B$ satisfying $A + B = R$ in a linearly compact ring $R$, $AB = BA$ implies $AB = A \cap B = AB$ and $AB = BA$ implies $AB = A \cap B$. 

2. A topological ring $R$ is called of finite type if the set of all isomorphism classes of simple submodules in all factors of any finitely generated discrete $R$-module is finite. Consider now a topologically artinian ring $R$. Let $\{P_i | i \in I\}$ be the set of all maximal closed ideals of $R$ where $I$ is the index set and $P_i \neq P_j$ for $i \neq j$, $i, j \in I$.

In case $i \neq j$ by Proposition 4 the ideal $P_i + P_j$ is closed in $R$ and hence we get $P_i + P_j = R$. Therefore we obtain $(P_i)_\xi + (P_j)_\xi = R$ for every ordinal $\xi$ by Proposition 7. Thus $(P_i)_\star + (P_j)_\star = R$ holds. Henceforth for any finite set $\{(P_1)_\star, \ldots, (P_n)_\star\}$ we have by the Chinese Remainder Theorem

$$(P_i)_\star + \bigcap_{k \neq i} (P_k)_\star = R, \quad k = 1, \ldots, n.$$ 

**Proposition 9.** Let $R$ be a topologically artinian ring of finite type. If $P_iP_j = P_jP_i$, $P_i(P_j)_\lambda = (P_j)_\lambda P_i$, $(P_j)_\lambda (P_i)_\xi = (P_i)_\xi (P_j)_\lambda$ hold for any $i \neq j$, $i, j \in I$ and any limit ordinals $\xi, \lambda$ then $\bigcap_i (P_i)_\star = 0$.

**Proof.** First we obtain $(P_i)_\xi (P_j)_\eta = (P_j)_\eta (P_i)_\xi$ for any $i \neq j$, $i, j \in I$ and any ordinals $\xi$ and $\eta$. We proceed by double transfinite induction. Suppose that for some $\xi$ and some fixed $\eta$, we have $(P_\chi P_i)_\eta = (P_i)_\eta (P_\chi)_x$ for every $\chi \leq \xi$. Then we have by Proposition 2 and the associativity of the multiplication of ideals

$$(P_i)_\xi (P_j)_\eta = (P_i)_\xi (P_j)_\eta (P_i)_\eta = (P_i)_\xi ((P_j)_\eta P_i) = (P_i)_\xi (P_j)_\eta P_i = ((P_i)_\xi (P_j)_\eta) P_i = (P_i)_\xi (P_j)_\eta P_i = (P_i)_\xi (P_j)_\eta (P_i)_\xi+1$$
if now $(P_i)_\xi (P_i)_\eta = (P_i)_\eta (P_i)_\xi$ holds for some fixed $\eta$ and every $\xi < \lambda$ where $\lambda$ is a limit ordinal, then there exists a limit ordinal $\tau$ such that $\eta = \tau + n$ for some natural number $n$, and hence we have again by Proposition 2 and the associativity of the multiplication of ideals:

\[
(P_i)_\lambda (P_i)_\eta = ((P_i)_\lambda (P_i)_\eta) P_i^n = ( (P_i)_\eta (P_i)_\lambda) P_i^n = (P_i)_\tau ((P_i)_\lambda P_i^n) = \]

\[= (P_i)_\tau (P_i^n (P_i)_\lambda) = ((P_i)_\tau P_i^n) (P_i)_\lambda = (P_i)_\eta (P_i)_\lambda .
\]

Finally we let $\eta$ vary and complete the proof in the same way.

Assume now indirectly that $\bigcap_i (P_i)_\ast \neq 0$. Then there is an element $c \in R$ and an open left ideal $L$ with $c \in \bigcap_i (P_i)_\ast$ and $c \notin L$. Since $R$ is of finite type, there are finitely many maximal closed ideals $P_i$, say $P_1, \ldots, P_n$ such that every simple submodule of each factor of $R/L$ is annihilated by some one of them. Consider now the submodule $M$ of $R/L$ consisting of those elements which are annihilated by $(P_1)_\ast \ldots (P_n)_\ast$. We claim $M = R/L$. In fact, if $M \neq R/L$, then there exists an element $m \in R/L$ such that $(Rm + M)/M$ is simple, because $R/L$ is artinian. Therefore $(Rm + M)/M$ is annihilated by some $P_i$, say $P_1$. We obtain now

\[
[(P_1)_\ast \ldots (P_n)_\ast]m = [(P_1)_\ast P_1 \ldots (P_n)_\ast]m =
\]

\[= [(P_1)_\ast \ldots (P_n)_\ast] P_1 m \subseteq [(P_1)_\ast \ldots (P_m)_\ast] M = 0
\]

and consequently $m \in M$, which is a contradiction. Thus $M = R/L$ holds. This implies by $c \in \bigcap_i (P_i)_\ast \subseteq (P_1)_\ast \ldots (P_n)_\ast$ that $c \cdot R/L = 0$, i.e. $c \in L$, a contradiction. Therefore $\bigcap_i (P_i)_\ast = 0$ and the proof is complete.

As $(P_i)_\ast$ is a closed ideal of the topologically artinian ring $R$, the factor $R_i = R/(P_i)_\ast$ is a topologically artinian ring, too. We denote by $\bar{R}$ the direct product of the $R_i$, $i \in I$. Then $\bar{R}$ endowed with the product topology is a topologically artinian ring. $R$ is topologically isomorphic to $\bar{R}$ by the following proposition.

**Proposition 10.** If $A_i$ ($i \in I$) are closed ideals in a topologically artinian ring $R$ such that $\bigcap A_i = 0$ and $A_i + A_j = 0$ hold for all $i \neq j$; $i, j \in I$, then $R$ is topologically isomorphic to the direct product $\prod_i R/A_i$. 


PROOF. Taking the mapping \( \varphi : R \to \prod_{i \in I} R/A_i \) define by \( \varphi(x) = (\ldots, x_i, \ldots) \) with \( x_i = x + (A_i) \in R/A_i \) for every \( i \in I \), we have a homomorphism from \( R \) into \( \prod_{i \in I} R/A_i \). Further \( \varphi(x) = 0 \) implies \( x_i = 0 \) for each \( i \in I \), i.e. \( x \in \bigcap_{i \in I} A_i = 0 \), hence \( x = 0 \). Thus \( \varphi \) is an injective homomorphism, and it is clear that \( \varphi \) is continuous.

Finally, let \( (\ldots, \tilde{x}_i, \ldots) \) be any element of \( \prod_{i \in I} R/A_i \), then \( M_i = \varphi_i^{-1}(\tilde{x}_i) \) is a coset of the ideal \( A_i \) in \( R \) where \( \varphi_i : R \to R/A_i \) is the natural projection of \( R \) onto \( R/A_i \). Hence we can express \( M_i \) as \( x_i + A_i \), \( x_i \in M_i \). Taking any finite number of \( M_i \), say \( M_1, \ldots, M_n \), we have \( \bigcap_{k=1}^n M_k \neq \emptyset \) by the Chinese Remainder Theorem, hence \( \bigcap_{i \in I} M_i \neq \emptyset \), since \( R \) is topologically artinian. Choosing an element \( x \in \bigcap_{i \in I} M_i \), it is obvious that \( \varphi(x) = (\ldots, \tilde{x}_i, \ldots) \). This implies that \( \varphi \) is a continuous isomorphism. Since \( R \) is topologically artinian, by Proposition 5 \( \varphi \) is a homeomorphism. This completes the proof of the assertion that \( R \) is topologically isomorphic to \( \prod_{i \in I} R/A_i \).

**Proposition 11.** The rings \( R_i, i \in I \) are primary rings.

**Proof.** Let \( \tilde{P}_i = P_i/(P_i)_+ \). Then \( R_i/\tilde{P}_i \cong R/P_i \) which is the endomorphism ring of a vector space over a division ring. Since the radical of \( R_i \) is the intersection of all maximal closed ideals which are exactly the images of maximal closed ideals of \( R \) by the natural homomorphism \( R \to R_i, \tilde{P}_i \) is clearly the radical of \( R_i \). This means that the rings \( R_i (i \in I) \) are primary rings.

The above considerations yield the following

**Theorem 12.** Let \( R \) be a topologically artinian ring. The following conditions are equivalent

1) \( R \) is a direct product of topologically artinian primary rings \( R_i, i \in I \).

2) Let \( \{P_i|i \in I\} \) be the set of all maximal closed ideals in \( R \) then the equalities \( P_i P_j = P_j P_i \), \( P_i(P_j)_\lambda = (P_j)_\lambda P_i \), \( (P_i)_\mu((P_j)_\lambda = (P_j)_\lambda(P_i)_\mu \) hold for all \( i \neq j, i, j \in I \) and limit ordinals \( \mu, \lambda \) and \( R \) is of finite type.

3) The equalities \( \tilde{P}_i \tilde{P}_j = \tilde{P}_j \tilde{P}_i \), \( \tilde{P}_i(P_j)_\lambda = (P_j)_\lambda \tilde{P}_i \), \( (P_j)_\mu \tilde{P}_i = (P_i)_\lambda \) hold for all \( i \neq j, i, j \in I \) and limit ordinals \( \mu, \lambda \), and \( R \) is of finite type.

4) Any ideal \( K \) of \( R \) such that \( K = \overline{K^2} \), is a unital ring.
PROOF. 1 \(\Rightarrow\) 2. This implication is trivial, since maximal closed ideals in \(R\) are the products of a maximal closed ideal in one component with the other components. For any discrete, finitely generated module, let \(\{x_1, \ldots, x_n\}\) be a generator set of \(M\). The annihilator \(\text{ann}_R x_i\) of \(x_i\) is an open left ideal of \(R_i\) and hence it contains almost all \(R_i\). Therefore \(\text{ann}_R (x_1, \ldots, x_n)\) contains almost all \(R_i\). This shows that almost all \(R_i\) is contained in the annihilator \(\text{ann}_R M\) of \(M\) and henceforth \(M\) can be considered as a discrete module over a finite direct sum of primary rings \(R_i\), say \(R_1 \times \ldots \times R_n\): Since the set of isomorphism classes of discrete simple factors of \(R_1 \times \ldots \times R_n\) is clearly finite, \(M\) is obviously of finite type, i.e. \(R\) is of finite type.

2 \(\Rightarrow\) 3. This implication is also trivial, since \(A = B\) implies \(\overline{A} = \overline{B}\).

3 \(\Rightarrow\) 1. This implication is the consequence of the fact that \(R\) and \(\overline{R}\) are topologically isomorphic.

1 \(\Rightarrow\) 4. Let \(\varphi_i\) denote the natural projection of \(R\) onto \(R_i\) for each \(i \in I\). For \(K_i = \varphi_i(K)\) it follows by [6] Lemma 6.1 that \(K\) is the direct product of the \(K_i, i \in I\). Since \(K = \overline{K}^2\), we have \(K_i = \overline{K}^2\), and hence either \(K_i = 0\) or \(K_i = R_i\), because \(R_i\) is a primary ring and its radical is \(r\)-transfinitely nilpotent. This shows that \(K\) is a unital ring.

4 \(\Rightarrow\) 1. From the assumption it follows by \(P_i^* = P_i\) that \(P_i\) has an identity \(e_i\). It is easy to see that \(R_i = \overline{P_i}/P_i\) is a primary ring. To come to the end of the proof of the implication 4 \(\Rightarrow\) 1 we show \(\bigcap_{i \in I} P_i = 0\). For this aim let \(x\) be an arbitrary element in \(\bigcap_{i \in I} P_i\). Then we have \(x = xe_i\) for each \(i \in I\) and \(x \in \bigcap_{i \in I} P_i = J\). Assume that \(x\) is an element of \(J\). By \((J_{\xi}/J_{\xi+1}) J = 0\) we can consider \(J_{\xi}/J_{\xi+1}\) as a right \(\overline{R}/J\)-module. Since \(x = xe_j\), we have \(\overline{a}(1 - \overline{e}_i) = 0\) and hence \(\overline{a} \cdot 1 = 0\) where \(\overline{a}\) denotes the image of \(a \in J_{\xi}\) in \(J_{\xi}/J_{\xi+1}\). This implies that \(x\) belongs to \(J_{\xi+1}\). On the other hand the radical \(J\) is \(r\)-transfinitely nilpotent so we have \(x = 0\). Similarly, to proposition 7 it is routine to verify that \(P_i + P_j = \overline{R}\) for every ordinal \(\xi\) and \(i \neq j\). Therefore we have \(P_i + P_j = \overline{R}\) for all \(i \neq j\) and thus by Proposition 10 \(R\) is topologically isomorphic to the product of the primary rings \(R_i\) and this completes the proof of Theorem 12.

If \(\bigcap J^n = 0\) holds, we have the following.
COROLLARY 13. Let $R$ be a topologically artinian ring satisfying $\bigcap J^n = 0$. The following assertions are equivalent

1) $R$ is a direct product of topologically artinian primary rings.

2) $P_i P_j = P_i P_j$ where $\{P_i : i \in I\}$ is the set of maximal closed ideals of $R$.

3) $Q_i Q_j = Q_i Q_j$ where $Q_i = \bigcap P_i^n$.

4) Every ideal $K$ of $R$ with $K = K^2$ is a unital ring.

5) $\bigcap Q_i = 0$.

First we prove the following.

PROPOSITION 14. Let $R$ be a topologically artinian ring satisfying $\bigcap J^n = 0$. For any open left ideal $L$ there are (not necessarily different) maximal closed ideals $P_1, \ldots, P_n$ with $P_n \ldots P_1 \subseteq L$.

PROOF. Since $L$ is open, the left $R$-module $R/L$ is artinian. Because $L = L + \left( \bigcap J^n \right) = \bigcap (L + J^n)$ holds, there is a natural number $t$ with $J^{t+k} + L = J^t + L$ for every non-negative integer $k$. Hence $L$ contains $J^t$. The artinian module $R/(J + L)$ can be considered as a left $R/J$-module, and then by Proposition 6 it is a finite direct sum of simple modules, i.e. it is noetherian. Consider the artinian $R$-module $(J + L)/(J^2 + L)$. By $J[(J + L)/(J^2 + L)] = 0$ it can be considered as an $R/J$-module and hence by Proposition 6 it is a finite direct sum of simple modules. Thus it is noetherian. Iterating this process in $t$ steps we get that $(J^{t-1} + L)/(J^t + L) = (J^{t-1} + L)/L$ is noetherian. Thus $R/L$ is noetherian, i.e., it has a composition series $R/L = M_1 \supset M_2 \supset \ldots \supset M_n \supset 0$ where $M_i/M_{i+1}$ is simple. Let $P_k$ be the annihilator ideal of $M_k/M_{k+1}$ ($k = 1, \ldots, n$), then the $P_k$ are (not necessarily different) maximal closed ideals and $(P_n \ldots P_1)R/L = 0$, thus $P_n \ldots P_1 \subseteq L$. This completes the proof of Proposition 14.

PROOF OF COROLLARY 13. By Theorem 12 we have $1 \iff 4$ and $1 \implies 1 \implies 3$. Next we show $2 \implies 5$. Suppose that $P_i P_j = P_j P_i$ holds for any $i, j \in I$. If there were an element $0 \neq c \in \bigcap Q_i$, then we should have an open left ideal $L$ with $c \notin L$. By Proposition 14 there are maximal closed ideals $P_1, \ldots, P_n$ with $P_1 \ldots P_n \subseteq L$. This
shows by Propositions 7 and 8 that

\[ c \in \bigcap_{i \in I} Q_i \subseteq \bigcap_{i=1}^{n} (P_i') \subseteq P_1 \ldots P_n \subseteq L \]

which contradicts \( c \notin L \).

Similarly we can see that \( 3 \Rightarrow 5 \).

Finally, suppose \( \bigcap_{i \in I} Q_i = 0 \). Let \( R_i = R/Q_i \). From Proposition 7 we get immediately \( Q_i + P_j = R \) for all \( i \neq j \). This shows clearly that \( P_i/Q_i \) is the radical of \( R_i \). By \( R/P_i \cong R_i/(P_i/Q_i) \) the rings \( R_i \) \((i \in I)\) are primary rings. Consider the direct product \( \bar{R} = \prod \bar{R}_i \). By Proposition 10 we have \( R \cong \bar{R} \). This completes the proof of Corollary 13.

**Proposition 15.** If in a topologically artinian ring \( R \) the products of any two maximal open left ideals commute, then so do those of any two maximal closed ideals, and \( R/P \) is a division ring for each maximal closed ideal \( P \).

**Proof.** Let \( P \) be a maximal closed ideal. Since \( P \) contains the radical, \( R/P \) is a primitive ring, consequently it is the endomorphism ring of a vector space over a division ring. To show that \( R/P \) is a division ring, we assume indirectly that \( R/P \) is the endomorphism ring of a right vector space \( V \) over a division ring and the dimension of \( V \) is at least 2. Hence there is a basis \( \{v_i : i \in I\} \) in \( V \) such that the cardinal number of \( I \) is greater than 1. Let \( i, j \) be any two distinct index in \( I \) and we define the endomorphisms \( e_i \) and \( e_j \) by setting

\[
e_i(v_k) = \begin{cases} v_k, & k \neq i \\ 0, & k = i \end{cases}, \quad e_j(v_k) = \begin{cases} v_k, & k \neq j \\ 0, & k = j \end{cases}.
\]

It is obvious that \((R/P)e_i\) and \((R/P)e_j\) are two maximal open left ideals in \( R/P \). For the endomorphism \( e_{ij} \) define by

\[
e_{ij}(v_k) = \begin{cases} v_k, & k \neq j, \; k \neq i \\ v_i, & k = j \\ v_j, & k = i \end{cases},
\]

we obtain \( e_i e_{ij} = e_j \) from which it follows that \((R/P)e_i \cdot (R/P)e_j = (R/P)e_j \). Similarly we can see that \((R/P)e_i \cdot (R/P)e_i = (R/P)e_i \).
Since \((R/P)e_i\) and \((R/P)e_j\) are two distinct maximal open left ideals, we have that their products do not commute. This contradicts the assumption. This implies that \(P\) is a maximal open left ideal in \(R\) and hence the validity of the proposition is verified.

As an immediate consequence of Proposition 15 and Corollary 13 we have

**Corollary 16.** A topologically artinian ring satisfying \(\bigcap \mathcal{J}^n = 0\) is a direct product of local rings if and only if the products of any two maximal open left ideals commute.

In what follows, let \(R\) be a topologically artinian local ring satisfying \(\bigcap \mathcal{P}^n = 0\) where \(P\) denotes its maximal open ideal. We assume that \(R\) satisfies the additional condition:

There exists no one-sided open ideal between \(P\) and \(P^2\).

**Proposition 17.** If \(P^2\) is not open in \(P\), then \(P = 0\).

**Proof.** For any open left ideal \(L\) in \(P\) the left ideal \(P^2 + L\) is open and it holds \(P \supset P^2 + L \supset P^2\). Hence \(P^2 + L = P\). This shows that \(P^2\) is everywhere dense in \(P\), so \(P^2 = P\) is true. By induction we get \(P = \bigcap \mathcal{P}^n = 0\).

**Proposition 18.** \(\{\mathcal{P}^n, n = 1, 2, \ldots\}\) forms a fundamental system of neighbourhoods of zero.

**Proof.** Let \(L\) be any open left ideal in \(R\). Since \(L = L + (\bigcap \mathcal{P}^n) = \bigcap (L + \mathcal{P}^n)\) and \(R/L\) is an artinian module, there is an integer \(n\) with \(\mathcal{P}^n \subseteq L\). This means by [4] Lemma 6 that \(L\) is a power of \(P\). (Note that in this case \(P^k\) is a union of translates of \(P^t\) for every \(t > k\)). Suppose now \(\mathcal{P}^n = 0\) for some \(n\), and let \(k\) be the least integer with this property, i.e. \(P^k = 0, P^{k-1} \neq 0\). We prove that \(R\) is artinian in this case, or in other words, 0 is open. In fact, since all non-zero open left ideals are powers of \(P\) containing \(P^{k-1}\), they cannot form a fundamental system of neighbourhoods of zero, hence 0 must also be open.

**Proposition 19.** Every non-zero one-sided ideal (closed or not) in \(R\) coincides with some \(\mathcal{P}^n(P^n = R)\), \(n = 0, 1, 2, \ldots\)

**Proof.** Let \(0 \neq L \neq R\) be any left ideal in \(R\) and \(0 \neq e \in L\). The \(Re\) is a closed left ideal in \(R\). This implies \(Re = Re + (\bigcap \mathcal{P}^n) = \)
= \bigcap (Re + P^n). Since Re + P^n is open, Re + P^n = P^{*n} for each n. By \bigcap P^n = 0, Re \neq 0 there exists an integer t with Re = P^t. Therefore L is open and hence by the proof of Proposition 18 L equals a power of P. By the assumption it is easy to see that R is topologically artinian from the right, too. By symmetry the assertion of Proposition 19 is true.

**Theorem 20.** Let R be a topologically artinian local ring satisfying \bigcap P^n = 0 and suppose that there exists no one-sided ideal between P and P^2 where P denotes its maximal non-zero open ideal. Then R has no zero-divisors if and only if P^n \neq 0 for each n.

**Proof.** The necessity is obvious.

Conversely, by Proposition 19 there is an element a contained in P but not in P^2, and then P = Ra = aR and by induction P^n = a^n R = Ra^n. If x and y are non-zero elements in R, then there are integers k, l such that x \in P^k - P^{k+1}, y \in P^l - P^{l+1}. Thus it follows that we can express x and y as x = ua^k, y = va^l where u, v must be units in R. Therefore xy = ua^k va^l = uv_1 a^{k+l} where v_1 is a unit with the property v_1 a^k = a^k v. This implies xy \neq 0.

If a and b are any two non-zero elements of R, from the proof of Theorem 20 and by Proposition 19 we have Ra = P^k, Rb = P^l for some integers k, l. Hence a, b have a common left multiple c = b, a = a, b = 0. By Ore's well-known result one can now construct a quotient division ring A of R whose elements are the quotients a^{-1} b. We define the following valuations \nu on R: \nu(0) = + \infty; if a \neq 0 is any element of R, then there is an integer n with a \in P^n - P^{n+1}, and let \nu(a) = n. It is routine to check that \nu defines a discrete valuation on R in the sense of [1] Chap. VI. In the classical way we can extend \nu to A and it is easy to see that \nu is a discrete valuation on A. We show that R is the ring of the valuation \nu on A. In fact, let x = b^{-1} a be any element of A which is not contained in R where a, b \in R. Then we can write a = c^k u, b = c^l t for c \in P - P^2, and u, t units in R. Since x \notin R and x = b^{-1} a = (c^l t)^{-1} c^k u = t^{-1} c^{k-l} u, v^{-1}, u \in R, we have k < l. Hence x^{-1} = a^{-1} b = u^{-1} c^{k-l} t \in R. Since R is complete, A is complete in the topology induced by \nu.

From the above we have

**Theorem 21.** Let R be a linearly compact local ring satisfying \bigcap P^n = 0 where P denotes its maximal open ideal. The following conditions are equivalent.
1) $P^n \neq 0$ for each $n$ and there exists no one-sided ideal between $P$ and $P^2$.

2) $R$ is the ring of a complete, discrete valuation on a division ring.

**Theorem 22.** Let $R$ be a linearly compact ring with $\bigcap \overline{J} = 0$. $R$ is a direct product of rings of complete, discrete valuations on division rings, of local uni-serial rings, and of division rings if and only if the products of any two maximal open left ideals commute and there exists no one-sided open ideal between $P$ and $P^2$ for each maximal open ideal $P$ of $R$.

In what follows we shall investigate topologically noetherian linearly compact rings. First we prove

**Theorem 23.** A linearly compact ring $R$ is topologically noetherian in the equivalent Leptin-topology if and only if $*J = 0$, i.e. its radical is $\mu$-transfinitely nilpotent.

**Proof.** Assume that $R$ is topologically noetherian. For any open left ideal $L$ of $R$ we have $J[(*J + L)/L] = (*J + L)/L$ and hence by Nakayama's Lemma we have $(*J + L)/L = 0$, consequently $*J \subseteq L$ and therefore $*J = 0$. Conversely if $*J = 0$, then we prove by induction that $R/\mu J$ is topologically noetherian for every ordinal $\mu$ from which the statement follows clearly. For $\mu = 1$ the assertion is obvious. If $R/\mu J$ is topologically noetherian, then $\mu J/\mu + 1 J$ can be considered as $R/\mu J$-module and hence it is obviously topologically noetherian in the Leptin-topology. Therefore $R/\mu + 1 J$ is such, too. If $\lambda$ is a limit ordinal and $R/\mu J$ is topologically noetherian for all $\mu < \lambda$, there $R/J$ is the inverse limit of $R/\mu J$ is trivially topologically noetherian, too. This completes the proof.

A topological ring $R$ is called of cofinite type if the set of all isomorphism classes of simple factors of all submodules in any finitely generated discrete $R$-module is finite. Similarly to Proposition 9 we can prove the next statement.

**Proposition 24.** Let $R$ be a topologically noetherian linearly compact ring of cofinite type. If the set $\{P_i : i \in I\}$ of all maximal closed ideals in $R$ satisfies $P_i P_j = P_j P_i$, $P_i(P_j)_\lambda = (P_j)_\lambda P_i$ and $(P_j)_\lambda(P_i)_\xi = (P_i)_\xi(P_j)_\lambda$ for any $i \neq j$, $i, j \in I$ and any limit ordinals $\xi, \lambda$ then $\bigcap_{i \in I} (*)(P_i) = 0$.  

PROOF. Similarly to Proposition 9 we obtain $(P_i)_{\xi}(P_j)_{\eta} = (P_j)_{\eta}(P_i)_{\xi}$ for any $i \neq j$, $i, j \in I$ and any ordinals $\xi$ and $\eta$. Assume now indirectly that $\bigcap_{i \in I} \star(P_i) \neq 0$. Then there is an element $c \in R$ and an open left ideal $L$ with $c \in \bigcap_{i \in I} \star(P_i)$ and $c \notin L$. Since $R$ is of cofinite type, there are finitely many maximal closed ideals $P_i$, say $P_1, \ldots, P_n$ such that every simple factor of each submodule in $R/L$ is annihilated by some one of them. By Zorn’s Lemma there exists a minimal submodule $M$ of $R/L$ such that $(R/L)/M$ is annihilated by $\star(P_1) \ldots \star(P_n)$. We claim $M = 0$. In fact, if $M \neq 0$, then $M$ as a submodule of the noetherian module $R/L$ is finitely generated. Therefore $JM \neq M$. Since $M/JM$ can be considered as a module over $R/J$, it is a finite direct sum of simple $R$-modules by Proposition 6. Thus there is a submodule $N$ of $M$ such that $M/N$ is annihilated by some $P_i$, say $P_1$. Henceforth we obtain now

$$\star(P_1) \ldots \star(P_n) \cdot [(R/L)/N]] =$$

$$= P_1\{\star(P_1) \ldots \star(P_n)\} \cdot [(R/L)/N]] \subseteq P_1 \cdot M/N = 0,$$

which contradicts to the minimality of $M$. Thus $M = 0$ holds. This implies by $c \in \bigcap_{i \in I} \star(P_i) \subseteq \star(P_1) \ldots \star(P_n)$ that $c \cdot R/L = 0$, i.e. $c \in L$, which is impossible. Therefore $\bigcap_{i \in I} \star(P_i) = 0$ and we are done.

Consider now a topologically noetherian linearly compact ring $R$ satisfying the condition of Proposition 24. In the case $i \neq j$ by Proposition 4 the ideal $P_i + P_j$ is closed in $R$ and hence we get $P_i + P_j = R$. Similarly to Proposition 7 we obtain $\xi(P_i) + \xi(P_j) = R$ for every ordinal $\xi$. Thus $\star(P_i) + \star(P_j) = R$ holds. Henceforth for any finite set $\{\star(P_1), \ldots, \star(P_n)\}$ we have by the Chinese Remainder Theorem

$$\star(P_i) + \bigcap_{k \neq i} \star(P_k) = R, \quad i = 1, \ldots, n.$$ 

As $\star(P_i)$ is a closed ideal of $R$, the factor ring $R_i = R/\star(P_i)$ is a topologically noetherian linearly compact ring, too. Similarly to Proposition 11, one can see that the rings $R_i$ are primary rings. As it was done in Proposition 10, we can prove that $R$ is isomorphic to the direct product $\prod R_i$ and this isomorphism is continuous, but in general, it is not a topological isomorphism. Furthermore, the topology in $\prod R_i$
induced by this continuous isomorphism is equivalent to the product topology. Therefore this isomorphism is topological if we endow \( R \) with the equivalent Leptin-topology. Thus the proof of the following theorem is similar to that of Theorem 12 and hence we omit it.

**Theorem 24.** Let \( R \) be a topologically noetherian linearly compact ring endowed with the equivalent Leptin-topology. The following conditions are equivalent.

1) \( R \) is a direct product of topologically noetherian linearly compact primary ring \( R_i \).

2) \( R \) is of cofinite type and if \( \{ P_i : i \in I \} \) is the set of all maximal closed ideals in \( R \), then the equalities \( P_i P_j = P_j P_i, P_i(P_j)_\lambda = (P_j)_\lambda P_i, \)
\( (P_i)_\mu (P_j)_\lambda = (P_j)_\lambda (P_i)_\mu \) hold for all \( i \neq j, i, j \in I \) and limit ordinals \( \mu, \lambda \).

3) \( R \) is of cofinite type and the equalities \( \overline{P_i} P_j = \overline{P_j} P_i, P_i(P_j)_\lambda = \overline{P_i(P_j)}_\lambda = \overline{(P_i)_\lambda P_j} = (P_j)_\lambda (P_i)_\mu \) hold for all \( i \neq j, i, j \in I \) and limit ordinals \( \mu, \lambda \).

4) Any ideal \( K \) of \( R \) such that \( K = \overline{K} \) is a unital ring.

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**References**


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