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Arbitrarily Large Indecomposable Divisible Torsion Modules Over Certain Valuation Domains.

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Since Shelah [S] has succeeded in establishing the existence of arbitrarily large indecomposable torsion-free abelian groups, a number of improved versions and generalizations have been published. The generalizations are primarily concerned with torsion-free modules over domains. However, the question as to the existence of large indecomposable torsion modules has not been given due attention. Here we wish to remedy this situation by dealing with this question in one of the special cases in which arbitrarily large torsion indecomposables are least likely to exist: the domain is a valuation domain (i.e. its ideals form a chain under inclusion) and the torsion modules are divisible.

We will show that there exist valuation domains R such that there are arbitrarily large torsion divisible R-modules. (Actually, we prove a stronger version of this claim.) As we rely heavily on results in our recent volume [FS], and as one of the results we require here has been proved by using Jensen's $\diamondsuit_{\aleph_1}$ (which holds in the constructible universe), we will work in ZFC $+ \diamondsuit_{\aleph_1}$. An important role is played in our construction by a recent result due to Corner [C]; the author is indebted to Prof. A. Orsatti for mentioning to him Corner's theorem. (It should be mentioned that Franzen and Göbel [FG] proved independently and simultaneously a result almost equivalent to Corner's.)

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1. Let R be a valuation domain and M an R-module. M is called *divisible* if $rM = M$ for every $0 \neq r \in R$.

LEMMA 1. If the global dimension m of the valuation domain R is finite, then, for every divisible R-module M,

$$\text{i.d. } M \leqq m - 1 .$$

PROOF. In [F1] it is shown that $\operatorname{Ext}_R^m(X, M) = 0$ for any R-module X of proj. dim. $\leqq m$ whenever M is divisible. Hence the claim is evident. □

An R-module M is called *uniserial* (or chained) if its submodules form a chain under inclusion. The field of quotients, Q, of R is a uniserial R-module, and so are I/J for submodules $0 \leqq J < I \leqq Q$. A uniserial R-module which is of this form for suitable I, J is said to be *standard*. A divisible standard uniserial R-module is isomorphic to Q/J for some $0 \leqq J \leqq Q$.

Let Γ be the direct sum of $\aleph_1$ copies of $\mathbb{Z}$, lexicographically ordered over an index set which carries the inverse order of ω_1. There exists a valuation domain R with value group Γ such that there is a continuous well-ordered ascending chain of submodules of Q,

$$R < J_0 < \ldots < J_\nu < \ldots < J_{\omega_1} = Q \qquad (\nu < \omega_1)$$

satisfying:

(i) for each $\nu < \omega_1$, J_ν/R is countable;

(ii) for every limit ordinal $\nu < \omega_1$, R/J_ν^{-1} is not complete in its R-topology.

Such an R has been constructed in [FS, p. 151]. It was shown there that

LEMMA 2. (ZFC $+ \diamondsuit_{\aleph_1}$) Over such a ring R, there exist non-standard uniserial divisible torsion modules.

The proof shows that there are $2^{\aleph_1}$ non-isomorphic ones. Observe that if U, V are non-isomorphic uniserial divisible torsion modules the annihilators of whose elements are principal ideals, then $\operatorname{Hom}_R(U, V) = 0$.

We can establish:

LEMMA 3. (ZFC + $\diamondsuit_{\aleph_1}$) Let R be as indicated and V any uniserial divisible torsion R-module. Then

$$\mathrm{Ext}^1_R(Q, V) \neq 0 .$$

For the proof we refer to [F2]. □

Let R be a commutative ring and S an R-algebra. Corner [C] defines a *fully rigid system* for S as a family $\{G_X\ (X \subseteq I)\}$ of faithful right S-modules, indexed by the subsets X of I such that

$$(1) \qquad G_X \leqslant G_Y \quad \text{for } X \subseteq Y \subseteq I,$$

$$(2) \qquad \mathrm{Hom}_R(G_X, G_Y) = \begin{cases} S & \text{if } X \subseteq Y, \\ 0 & \text{otherwise.} \end{cases}$$

He proves, by making use of an extended version of a lemma by Shelah [S]:

LEMMA 4. Suppose there exists a fully rigid system $\{H_X\ (X \subseteq I)\}$ for S where $|I| \geqq 6$. Then for every infinite cardinal $\lambda \geqq |H_I|$ there exists a fully rigid system $\{G_X\ (X \subseteq \lambda)\}$ for S such that $|G_X| = \lambda\ (X \subseteq \lambda)$.

Corner's proof shows that if the H_X's are torsion divisible, then so are the G_X. This observation is relevant in the proof of Theorem 8.

2. From now on, R will denote the valuation domain featured in Lemma 2. It is easy to see that its global dimension is 2.

Recall that a module C over a domain R is said to be *cotorsion* if $\mathrm{Ext}^1_R(F, C) = 0$ for all torsion-free R-modules F; see [FS, p. 243]. Matlis [M] defines a weaker version by restricting F to the field Q of quotients. It is readily seen that the two definitions are equivalent for C of injective dimension 1.

LEMMA 5. (ZFC + $\diamondsuit_{\aleph_1}$) Let U, V be uniserial divisible torsion R-modules such that $\mathrm{Hom}(U, V) = 0$ and the annihilator of a is a principal ideal for each $a \in U$ and $a \in V$. Then $\mathrm{Ext}^1_R(U, V)$ satisfies:

(*a*) it is cotorsion;

(*b*) it is divisible;

(*c*) its torsion submodule is uniserial and divisible;

(*d*) it is a mixed module.

Proof. By hypothesis, in the exact sequence $0 \to U[r] \to U \xrightarrow{r} U \to 0$ $(r \in R)$ the first module is cyclically presented; here $U[r] = \{x \in U | rx = 0\}$. Hence we obtain the exact sequence (the last Ext is 0, because V is divisible, see [FS, p. 39])

$$0 = \mathrm{Hom}\,(U, V) \to \mathrm{Hom}\,(U[r], V) \cong V[r] \to \to \mathrm{Ext}^1\,(U, V) \xrightarrow{r} \mathrm{Ext}^1\,(U, V) \to \mathrm{Ext}^1\,(U[r], V) = 0\,.$$

This implies (*b*) and that the submodule of $\mathrm{Ext}^1\,(U, V)$ annihilated by r is $\cong V[r]$. Therefore, the torsion submodule of $\mathrm{Ext}^1\,(U, V)$ is the union of unbounded cyclically presented submodules whence (*c*) is evident.

Consider an injective resolution of V, $0 \to V \to E(V) \to I \to 0$ where $E(V)$ denotes the injective hull of V. As V is divisible, by Lemma 1, its injective dimension is at most the global dimension of R minus 1, so i.d. $V \leqq 1$. Hence I is injective and we get an exact sequence

$$0 = \mathrm{Hom}\,(U, V) \to \mathrm{Hom}\,(U, E(V)) \to \mathrm{Hom}\,(U, I) \to \mathrm{Ext}^1\,(U, V) \to 0\,.$$

As U is divisible, the Hom modules are torsion-free, and as $E(V)$, I are injective, they are pure-injective as well. If the middle module is cotorsion and the submodule has injective dimension $\leqq 1$ (see [FS, XII, 3.1]), then the quotient is cotorsion (see [FS, XII. 3.4]), proving (*a*).

By Lemma 3, a torsion uniserial divisible R-module W over the indicated R satisfies $\mathrm{Ext}^1\,(Q, W) \neq 0$. Hence the torsion submodule W of $\mathrm{Ext}^1\,(U, V)$ cannot be cotorsion and (*d*) follows at once from (*a*) and (*c*). This completes the proof. □

Observe that, for every divisible uniserial torsion module U, $\mathrm{End}\, U \cong \tilde{R} =$ the completion of R in the R-topology. This follows in the same way as $\mathrm{End}\, Q/R \cong \tilde{R}$ (see [M]).

Lemma 6. *Hypotheses as in the preceding Lemma. If A is an extension of V by U, representing an element of $\mathrm{Ext}^1_R\,(U, V)$ with annihilator 0, then*

(*a*) $\mathrm{Hom}_R\,(U, A) = 0$,

(*b*) $\mathrm{End}_R\, A \cong \tilde{R}$.

PROOF. (*a*) By contradiction, assume there is a non-zero homomorphism $\eta: U \to A$. As $\mathrm{Hom}(U, V) = 0$ implies $\mathrm{Im}\,\eta \nleq V$, it is clear that η induces a non-zero map $\bar{\eta}: U \to A/V \simeq U$. In view of the structure of U, $\bar{\eta}$ has to be onto. Consequently, $V + \mathrm{Im}\,\bar{\eta} = A$. Furthermore, $U \cong A/V \cong \mathrm{Im}\,\bar{\eta}/(\mathrm{Im}\,\bar{\eta} \cap V)$ implies that $\mathrm{Im}\,\bar{\eta} \cap V$ is cyclic, say $= V[r]$ for some $r \in R$. The canonical map $V \to V/V[r]$ induces the extension $0 \to V/V[r] \to A/V[r] \to U \to 0$ which has to be splitting because of $A/V[r] = V/V[r] \oplus \mathrm{Im}\,\bar{\eta}/V[r]$. But the exact sequence

$$\mathrm{Ext}^1\big(U, V[r]\big) \to \mathrm{Ext}^1(U, V) \to \mathrm{Ext}^1\big(U, V/V[r]\big)$$

shows that an induced extension of $V/V[r]$ by U can split only if it comes from the bounded R-module $\mathrm{Ext}^1\big(U, V[r]\big)$. As A was chosen to have 0 annihilator, $A/V[r]$ cannot split. This contradiction verifies (*a*).

(*b*) First observe that $\mathrm{Hom}(U, V) = 0$ implies $\mathrm{Hom}(V, U) = 0$. Now if $\eta \in \mathrm{End}\,A$ is followed by the natural projection $A \to A/V$, then $\mathrm{Hom}(V, U) = 0$ shows the full invariance of V in A. If $\eta \in \mathrm{End}\,A$ maps V to 0, then it induces a homomorphism $U \to A$ which must vanish in view of (*a*). We infer that 0 is the only endomorphism of A that carries V to 0.

Multiplications by $\tilde{r} \in \tilde{R}$ are evidently endomorphisms of A and different elements of $\tilde{R}$ induced different endomorphisms. Thus $\tilde{R}$ may be viewed as a subring of $\mathrm{End}\,A$. Any $\eta \in \mathrm{End}\,A$ induces an endomorphism of V which must act as a multiplication by some $\tilde{r} \in \tilde{R}$. Now $\eta - \tilde{r} \in \mathrm{End}\,A$ vanishes on V, so it must be 0 as was shown in the preceding paragraph. Therefore $\eta = \tilde{r}$, and we arrive at the desired conclusion $\mathrm{End}\,A \simeq \tilde{R}$. □

3. We have come to the construction of fully rigid systems for finite sets $\{1, 2, \ldots, n\} = X$. The endomorphism rings will be isomorphic to $\tilde{R}$, so the modules will be faithful $\tilde{R}$-modules.

Consider a rigid system $\{U_1, \ldots, U_n, V\}$ of divisible uniserial torsion R-modules, i.e. the trivial homomorphism is the only homomorphism between different members of the system. We keep assuming that the annihilators of the elements are principal ideals. Define M to be an extension of V by $U_1 \oplus \ldots \oplus U_n$ which represents an element of

$$\mathrm{Ext}^1_R(U_1 \oplus \ldots \oplus U_n, V) = \mathrm{Ext}^1_R(U_1, V) \oplus \ldots \oplus \mathrm{Ext}^1_R(U_n, V)$$

all of whose coordinates in $\mathrm{Ext}^1_R(U_i, V)$ have 0 annihilator. For a subset $Y \subseteq X$, define M_Y as the extension of V by $\bigoplus_{i \in Y} U_i$ by dropping the coordinates of M in $\mathrm{Ext}^1_R(U_i, V)$ with $i \in X \setminus Y$; or, alternately, as the pull-back

$$\begin{array}{ccccccccc} 0 & \to & V & \to & M_Y & \to & \bigoplus_{i \in Y} U_i & \to & 0 \\ & & \| & & \downarrow & & \downarrow \delta & & \\ 0 & \to & V & \to & M & \to & \bigoplus_{i \in X} U_i & \to & 0 \end{array}$$

with the canonical injection δ. In particular, $M_\emptyset = V$.

From these definitions it is evident that $Y \subset Z$ implies $M_Y \subset M_Z$.

If $i \neq j$, then $\mathrm{Hom}(M_i, M_j) = 0$. Indeed, this is a simple consequence of the exact sequence $0 \to V \to M_j \to U_j \to 0$ and of $\mathrm{Hom}(M_i, V) = 0 = \mathrm{Hom}(M_i, U_j)$. Also, $\mathrm{Hom}(M_i, M_Z) = 0$ if $i \notin Z$.

Now suppose $Y, Z \subseteq X$ and $\varphi: M_Y \to M_Z$ is an R-homomorphism. If there is an $i \in Y \setminus Z$, then from the exact sequence

$$0 \to M_i \to M_Y \to \bigoplus_{Y \setminus i} U_j \to 0$$

we deduce $\mathrm{Hom}_R(M_Y, M_Z) = 0$. On the other hand, if $Y \subseteq Z$ and if $i \in Y$, then M_i is fully invariant in M_Y and in M_Z, thus φ induces an endomorphism of M_i which is, by Lemma 6, a multiplication by some $\tilde{r} \in \tilde{R}$. As an endomorphism of M_i is completely determined by its action on V and as M_Y is generated by the M_i with $i \in Y$, it follows that φ is a multiplication by $\tilde{r}$.

This completes the proof of the following theorem. Notice that an R-module is necessarily indecomposable if its endomorphism ring is a domain.

THEOREM 7. (ZFC + $\diamondsuit_{\aleph_1}$) *There are valuation domains R of global dimension 2 such that for every integer $n \geqslant 1$, there is a fully rigid system of indecomposable divisible torsion R-modules of cardinality $|\tilde{R}|$ (indexed by a set of n elements) whose endomorphism rings are isomorphic to $\tilde{R}$.* □

An application of Corner's theorem (Lemma 4) yields our main result. As noted earlier, if the initial system consists of divisible torsion modules, then all fully rigid systems obtained from it will contain only divisible torsion modules.

THEOREM 8. (ZFC + $\diamondsuit_{\aleph_1}$) There exist valuation domains R such that for every infinite cardinal $\varkappa \geqq |\tilde{R}|$ there is a fully rigid system of indecomposable divisible torsion R-modules M_X (indexed by subsets of a set of cardinality $\varkappa$) such that $|M_X| = \varkappa$ and End $M_X \simeq \tilde{R}$ for each $X \subseteq \varkappa$. □

In view of Lemma 1, gl.d. $R = 2$ implies that i.d. $M_X = 1$ for each divisible M_X in the fully rigid system. (Injective modules of large cardinalities cannot be indecomposable.)

REMARKS. 1) If gl.d. $R = 1$, then the only indecomposable divisible R-modules are the field Q of quotients of R and Q/R.

2) If M is a divisible module over a valuation domain R and if p.d. $M = 1$, then by [F1, Thm 18] M is a summand of a direct sum of copies of the module ∂ constructed there. If follows from C. Walker's extension of a Kaplansky lemma [W] that M cannot be indecomposable if its cardinality exceeds the cardinality of ∂.

As the modules M_X in our Theorem 8 have projective dimension 2, it is clear that neither the global dimension of R nor the projective dimensions of M_X can be decreased in Theorem 8. In other words, this result is the best possible one as far as projective and injective dimensions are concerned.

4. As an application of the main result we prove an analogous statement on cotorsion modules.

We start with a simple lemma where R can be any domain.

LEMMA 9. Let D be an indecomposable, h-reduced, divisible torsion R-module of injective dimension 1. Then its cotorsion hull

$$D^\bullet \simeq \operatorname{Ext}^1_R(Q/R, D)$$

is an indecomposable divisible cotorsion R-module. Moreover, End $D^\bullet \simeq$ End D (naturally).

PROOF. We have the exact sequence $0 \to D \to D^\bullet \to \oplus Q \to 0$ which implies rightaway that $D^\bullet$ is divisible. Any direct decomposition of $D^\bullet$ induces a direct decomposition of D, unless one of the summands is torsionfree divisible which is not the case. Hence $D^\bullet$ is an indecomposable divisible cotorsion module. It is straightforward to verify the isomorphism of the endomorphism rings of D and $D^\bullet$. □

It is now easy to derive:

THEOREM 10. (ZFC $+ \diamondsuit_{\aleph_1}$) There exist valuation domains R such that for each infinite cardinal $\varkappa \geqq |\tilde{R}|$ there is a fully rigid system of indecomposable divisible cotorsion R-modules C_X (indexed by subsets of a set of cardinality $\varkappa$) such that $|C_X| \leqq \varkappa^{\aleph_1}$ and End $C_X \cong \tilde{R}$ for each $X \subseteq \varkappa$.

PROOF. Using Theorem 9 and putting $C_X = M_X^\bullet$ (note that M_X will be h-reduced whenever the module V used in their construction is not isomorphic to Q/R), we have almost everything from Lemma 9 except for the estimate of the cardinality of C_X. As $|Q|$ is now $\aleph_1$, we can write $0 \to H \to F \to Q \to 0$ with F free of rank $\aleph_1$, and H a pure submodule of F. As such $|H| \leqq \aleph_1$, so $|\mathrm{Hom}_R(H, M_X)| \leqq$ $\leqq |M_X|^{|H|} = \varkappa^{\aleph_1}$. But $\mathrm{Ext}^1(Q, M_X)$ is an epic image of $\mathrm{Hom}(H, M_X)$, so by the exact sequence $0 \to M_X \to M_X^\bullet \to \mathrm{Ext}^1(Q, M_X) \to 0$, the estimate is clear. □

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