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On the Existence of Solutions of the Darboux Problem for the Hyperbolic Partial Differential Equations in Banach Spaces.

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SUMMARY - We are interested in the existence of solutions of the Darboux problem for the hyperbolic equation $z_{xy} = f(x, y, z, z_{xy})$ on the quarter-plane $x \geqslant 0$, $y \geqslant 0$. Here f is a function with values in a Banach space satisfying some regularity Ambrosetti type condition expressed in terms of the measure of noncompactness α and a Lipschitz condition in the last variable.

1. Let $J = [0, \infty)$ and $Q = J \times J$. Let $(E, \|\cdot\|)$ be a Banach space and let f be an E-valued function defined on $\Omega = Q \times E \times E$. We are interested in the existence of solutions of the Darboux problem for the hyperbolic partial differential equation with implicit derivative

$$z_{xy} = f(x, y, z, z_{xy}) \tag{+}$$

via a fixed point theorem of Sadovskii [12].

Let σ, τ be functions from J to E such that $\sigma(0) = \tau(0)$. By (PD) we shall denote the problem of finding a solution (in the classical sense) of equation (+) satisfying the initial conditions

$$z(x, 0) = \sigma(x), \qquad z(0, y) = \tau(y) \qquad \text{for } x, y \geqslant 0 .$$

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We deal with (PD) using a method developed by Ambrosetti [2] and Goebel and Rzymowski [7] concerning Cauchy problem for ordinary differential equations with the independent variable in a compact interval of J.

2. Denote by S_∞ the set of all nonnegative real sequences and $\emptyset$ the zero sequence. For $\xi = (\xi_n)$, $\eta = (\eta_n) \in S_\infty$ we write $\xi < \eta$ if $\xi \neq \eta$ and $\xi_n \leqslant \eta_n$ for $n = 1, 2, \ldots$.

Let $\mathfrak{X}_0$ be a closed convex subset of a Hausdorff locally convex topological vector space. Let Φ be a function which maps each nonempty subset Z of $\mathfrak{X}_0$ to a sequence $\Phi(Z) \in S_\infty$ such that (1) $\Phi(\{z\} \cup \cup Z) = \Phi(Z)$ for $z \in \mathfrak{X}_0$, (2) $\Phi(\overline{\mathrm{co}}\, Z) = \Phi(Z)$ (here $\overline{\mathrm{co}}\, Z$ is the closed convex hull of Z), and (3) if $\Phi(Z) = \emptyset$ then $\bar{Z}$ is compact.

For such Φ we have the following theorem of Sadovskii (cf. [12], Theorem 3.4.3):

If T is a continuous mapping of $\mathfrak{X}_0$ into itself and $\Phi(T[Z]) < \Phi(Z)$ for arbitrary nonempty subset Z of $\mathfrak{X}_0$ with $\Phi(Z) > \emptyset$, then T has a fixed point in $\mathfrak{X}_0$:

3. Let α denote the Kuratowski's measure of noncompactness in E (see e.g. [6], [8]). Moreover if Z is a set of functions on Q

$$Z(x, y) = \{z(x, y) : z \in Z\}$$

and

$$\int_0^x \int_0^y Z(t, s)\, dt\, ds = \left\{ \int_0^x \int_0^y z(t, s)\, dt\, ds : z \in Z \right\}$$

for $x, y \in J$.

The Lemma below is an adaptation of the corresponding result of Goebel and Rzymowski ([3], [7]).

Lemma. If W is a bounded equicontinuous subset of usual Banach space of continuous E-valued functions defined on a compact subset $P = [0, a] \times [0, a]$ of Q, then

$$\alpha\left(\int_0^x \int_0^y W(t, s)\, dt\, ds \right) \leqslant \int_0^x \int_0^y \alpha\big(W(t, s) \big)\, dt\, dt$$

for (x, y) in P.

Our result reads as follows.

THEOREM. Let σ, σ', τ and τ' be continuous on J. Let f be uniformly continuous on bounded subsets of Ω and

$$\|f(x, y, u, v)\| \leqslant G(x, y, \|u\|, \|v\|) \quad \text{for } (x, y, u, v) \in \Omega .$$

Suppose that for each bounded subset P of Q there exist nonnegative constants $k(P)$ and $L(P) < \frac{1}{2}$ such that

$$\alpha(f[x, y, U, v]) \leqslant k(P)\alpha(U)$$

and

$$\|f(x, y, u, v_1) - f(x, y, u, v_2)\| \leqslant L(P)\|v_1 - v_2\|$$

for all $(x, y) \in P$, u, v, v_1, v_2 in E and for any nonempty bounded subset U of E. Assume in addition that the function $(x, y, r, s) \mapsto \mapsto G(x, y, r, s)$ is monotonic nondecrasing for each $(x, y) \in Q$ (m.e. $0 \leqslant r_1 \leqslant r_2$ and $0 \leqslant s_1 \leqslant s_2$ implies $G(x, y, r_1, s_1) \leqslant G(x, y, r_2, s_2)$) and the scalar inequality

$$G\left(x, y, \int_0^x \int_0^y g(t, s)\,dt\,ds,\ g(x, y)\right) \leqslant g(x, y)$$

has a locally bounded solution g_0 existing on Q.

Under the hypotheses, (PD) has at least one solution on Q.

PROOF. Without loss of generality we may assume that $\sigma \equiv 0$ and $\tau \equiv 0$. Therefore, (PD) is equivalent to the functional-integral equation

$$w(x, y) = f\left(x, y, \int_0^x \int_0^y w(t, s)\,dt\,ds,\ w(x, y)\right).$$

Denote by $C(Q, E)$ the space of all continuous functions from Q to E ($C(Q, E)$ is a Fréchet space whose topology is introduced by seminorms of uniform convergence on compact subsets of Q), and by $\mathfrak{X}$ the set of all $z \in C(Q, E)$ with

$$\|z(x, y)\| \leqslant g_0(x, y) \quad \text{on } Q .$$

Let P be a bounded subset of Q. From the uniform continuity of f on bounded subsets of Ω follows the existence of a function $\delta_P\colon(0,\infty)\to(0,\infty)$ such that

$$\Big\|f\Big(x',y',\int_0^{x'}\int_0^{y'}z(t,s)\,dt\,ds,\,z(x,y)\Big)-\\ -f\Big(x'',y'',\int_0^{x''}\int_0^{y''}z(t,s)\,dt\,ds,\,z(x,y)\Big)\Big\|<\varepsilon$$

for any $z\in\mathfrak{X}$, $(x,y)\in P$ and (x',y'), $(x'',y'')\in P$ with $|x'-x''|<\delta_P(\varepsilon)$ and $|y'-y''|<\delta_P(\varepsilon)$.

Consider the set $\mathfrak{X}_0$ of $z\in\mathfrak{X}$ possessing the following property: for each bounded subset P of Q, $\varepsilon>0$ and $|x'-x''|<\delta_P(\varepsilon)$, $|y'-y''|<\delta_P(\varepsilon)$ (here (x',y'), $(x'',x'')\in P$) there holds $\|z(x',y')-z(x'',y'')\|<\big(1-L(P)\big)^{-1}\varepsilon$. The set $\mathfrak{X}_0$ is a closed convex and almost equicontinuous subset of $C(Q,E)$. To apply our fixed point theorem we define a continuous mapping T of $C(Q,E)$ into itself by the formula

$$(Tw)(x,y)=f\Big(x,y,\int_0^{x}\int_0^{y}w(t,s)\,dt\,ds,\,w(x,y)\Big).$$

Let $z\in\mathfrak{X}_0$. Then

$$\|(Tz)(x,y)\|\leqslant G\Big(x,y,\int_0^{x}\int_0^{y}\|z(t,s)\|\,dt\,ds,\,\|z(x,y)\|\Big)\leqslant g_0(x,y)$$

for $(x,y)\in Q$. Further, for $\varepsilon>0$ and (x',y'), $(x'',y'')\in P$ such that $|x'-x''|<\delta_P(\varepsilon)$, $|y'-y''|<\delta_P(\varepsilon)$ we have

$$\|(Tz)(x',y')-(Tz)(x'',y'')\|\leqslant\Big\|f\Big(x',y',\int_0^{x'}\int_0^{y'}z(t,s)\,dt\,ds,\,z(x',y')\Big)-\\ -f\Big(x',y',\int_0^{x'}\int_0^{y'}z(t,s)\,dt\,ds,\,z(x'',y'')\Big)\Big\|+\Big\|f\Big(x',y',\int_0^{x'}\int_0^{y'}z(t,s)\,dt\,ds,\,z(x'',y'')\Big)-$$

$$-f\Big(x'', y'', \int_0^{x''}\int_0^{y''} z(t,s)\,dt\,ds,\ z(x'',y'')\Big)\| \leqslant$$

$$\leqslant L(P)\|z(x',y') - z(x'',y'')\| + \varepsilon \leqslant \big(1 - L(P)\big)^{-1}\varepsilon ,$$

i.e. $Tz \in \mathfrak{X}_0$. Consequently, $T[\mathfrak{X}_0] \subset \mathfrak{X}_0$.

Let n be a positive integer and let Z be a nonempty subset of $\mathfrak{X}_0$. Put $P_n = [0, n]\times[0, n]$, $k_n = k(P_n)$ and $L_n = L(P_n)$. Now we shall show the basic inequality:

$$(*)\qquad \sup_{(x,y)\in P_n} \exp\big(-p_n(x+y)\big)\,\alpha\big(T[Z](x,y)\big) \leqslant (p_n^{-2}k_n + 2L_n)\cdot \sup_{(x,y)\in P_n} \exp\big(-p_n(x+y)\big)\,\alpha\big(Z(x,y)\big),$$

where $p_n \geqslant 0$.

To this end, fix (x, y) in P_n. By Lemma, we obtain

$$\alpha\Big(\int_0^x\int_0^y Z(t,s)\,dt\,ds\Big) \leqslant \int_0^x\int_0^y \alpha\big(Z(t,s)\big)\,dt\,ds \leqslant$$

$$\leqslant \sup_{(t,s)\in P_n} \exp\big(-p_n(t+s)\big)\,\alpha\big(Z(t,s)\big)\cdot \int_0^x\int_0^y \exp\big(p_n(t+s)\big)\,dt\,ds <$$

$$< p_n^{-2}\cdot\exp\big(p_n(x+y)\big)\cdot \sup_{(t,s)\in P_n} \exp\big(-p_n(t+s)\big)\,\alpha\big(Z(t,s)\big).$$

It is easy to verify (see [11], p. 476) that

$$\alpha\big(T[Z](x,y)\big) \leqslant k_n\cdot\alpha\Big(\int_0^x\int_0^y Z(t,s)\,dt\,ds\Big) + 2L_n\cdot\alpha\big(Z(x,y)\big).$$

Therefore

$$\exp\big(-p_n(x+y)\big)\,\alpha\big(T[Z](x,y)\big) <$$

$$< (p_n^{-2}k_n + 2L_n)\cdot \sup_{(t,s)\in P_n} \exp\big(-p_n(t+s)\big)\,\alpha\big(Z(t,s)\big)$$

and our inequality is proved.

Let $p_n^2 > (1-2L_n)^{-1}k_n$ $(n = 1, 2, \ldots)$. Define:

$$\Phi(Z) = \Big(\sup_{(x,y)\in P_1} \exp\big(-p_1(x+y)\big)\alpha\big(Z(x,y)\big)\,, \qquad \sup_{(x,y)\in P_2} \exp\big(-p_2(x+y)\big)\alpha\big(Z(x,y)\big), \ldots\Big)$$

for any nonempty subset Z of $\mathfrak{X}_0$. Evidently, $\Phi(Z) \in S_\infty$. By Ascoli theorem and properties of α our function Φ satisfy conditions (1)-(3) listed in Section 2. From inequality $(*)$ it follows that $\Phi(T[Z]) < \Phi(Z)$ whenever $\Phi(Z) > \emptyset$, and all assumptions of Sadovskii's fixed point theorem are satisfied. Consequently, T has a fixed point in $\mathfrak{X}_0$ which completes the proof.

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