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On the Existence of Solutions of the Darboux Problem
for the Hyperbolic Partial Differential Equations
in Banach Spaces.

Bogdan Rzepecki (*)

Summary - We are interested in the existence of solutions of the Darboux
problem for the hyperbolic equation \( z_{xy} = f(x, y, z, z_{xy}) \) on the quarter-
plane \( x > 0, y > 0 \). Here \( f \) is a function with values in a Banach space
satisfying some regularity Ambrosetti type condition expressed in terms
of the measure of noncompactness \( \alpha \) and a Lipschitz condition in the
last variable.

1. Let \( J = [0, \infty) \) and \( Q = J \times J \). Let \((E, \| \cdot \|)\) be a Banach
space and let \( f \) be an \( E \)-valued function defined on \( \Omega = Q \times E \times E \).
We are interested in the existence of solutions of the Darboux problem
for the hyperbolic partial differential equation with implicit derivative

\[
(+) \quad z_{xy} = f(x, y, z, z_{xy})
\]

via a fixed point theorem of Sadovskii [12].

Let \( \sigma, \tau \) be functions from \( J \) to \( E \) such that \( \sigma(0) = \tau(0) \). By (PD)
we shall denote the problem of finding a solution (in the classical sense)
of equation (\( + \)) satisfying the initial conditions

\[
z(x, 0) = \sigma(x), \quad z(0, y) = \tau(y) \quad \text{for} \ x, y > 0.
\]

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We deal with (PD) using a method developed by Ambrosetti [2] and Goebel and Rzymowski [7] concerning Cauchy problem for ordinary differential equations with the independent variable in a compact interval of $J$.

2. Denote by $S_\infty$ the set of all nonnegative real sequences and $\emptyset$ the zero sequence. For $\xi = (\xi_n), \eta = (\eta_n) \in S_\infty$ we write $\xi < \eta$ if $\xi \neq \eta$ and $\xi_n < \eta_n$ for $n = 1, 2, \ldots$.

Let $X_0$ be a closed convex subset of a Hausdorff locally convex topological vector space. Let $\Phi$ be a function which maps each non-empty subset $Z$ of $X_0$ to a sequence $\Phi(Z) \in S_\infty$ such that (1) $\Phi(\{z\} \cup Z) = \Phi(Z)$ for $z \in X_0$, (2) $\Phi(\overline{co} Z) = \Phi(Z)$ (here $\overline{co} Z$ is the closed convex hull of $Z$), and (3) if $\Phi(Z) = \emptyset$ then $\overline{Z}$ is compact.

For such $\Phi$ we have the following theorem of Sadovskii (cf. [12], Theorem 3.4.3):

If $T$ is a continuous mapping of $X_0$ into itself and $\Phi(T[Z]) < \Phi(Z)$ for arbitrary nonempty subset $Z$ of $X_0$ with $\Phi(Z) > \emptyset$, then $T$ has a fixed point in $X_0$.

3. Let $\alpha$ denote the Kuratowski's measure of noncompactness in $E$ (see e.g. [6], [8]). Moreover if $Z$ is a set of functions on $Q$ $Z(x, y) = \{z(x, y) : z \in Z\}$

and

$$\int_0^x \int_0^y Z(t, s) \, dt \, ds = \left\{ \int_0^x \int_0^y z(t, s) \, dt \, ds : z \in Z \right\}$$

for $x, y \in J$.

The Lemma below is an adaptation of the corresponding result of Goebel and Rzymowski ([3], [7]).

**Lemma.** If $W$ is a bounded equicontinuous subset of usual Banach space of continuous $E$-valued functions defined on a compact subset $P = [0, a] \times [0, a]$ of $Q$, then

$$\alpha \left( \int_0^x \int_0^y W(t, s) \, dt \, ds \right) < \int_0^x \int_0^y \alpha(W(t, s)) \, dt \, ds$$

for $(x, y)$ in $P$.

Our result reads as follows.
THEOREM. Let \( \sigma, \sigma', \tau \) and \( \tau' \) be continuous on \( J \). Let \( f \) be uniformly continuous on bounded subsets of \( \Omega \) and

\[
\| f(x, y, u, v) \| < G(x, y, \| u \|, \| v \|) \quad \text{for} \ (x, y, u, v) \in \Omega.
\]

Suppose that for each bounded subset \( P \) of \( \Omega \) there exist nonnegative constants \( k(P) \) and \( L(P) \leq \frac{1}{2} \) such that

\[
\alpha(f[x, y, U, v]) < k(P)\alpha(U)
\]

and

\[
\| f(x, y, u, v_1) - f(x, y, u, v_2) \| < L(P)\| v_1 - v_2 \|
\]

for all \( (x, y) \in P, u, v, v_1, v_2 \) in \( E \) and for any nonempty bounded subset \( U \) of \( E \). Assume in addition that the function \((x, y, r, s) \mapsto G(x, y, r, s)\) is monotonic nondecreasing for each \((x, y) \in Q\) (m.e. \( 0 < r_1 < r_2 \) and \( 0 < s_1 < s_2 \) implies \( G(x, y, r_1, s_1) < G(x, y, r_2, s_2) \)) and the scalar inequality

\[
G\left(x, y, \int_0^x \int_0^y g(t, s) \, dt \, ds, g(x, y)\right) < g(x, y)
\]

has a locally bounded solution \( g_0 \) existing on \( Q \).

Under the hypotheses, \((PD)\) has at least one solution on \( Q \).

PROOF. Without loss of generality we may assume that \( \sigma \equiv 0 \) and \( \tau \equiv 0 \). Therefore, \((PD)\) is equivalent to the functional-integral equation

\[
w(x, y) = f\left(x, y, \int_0^x \int_0^y w(t, s) \, dt \, ds, w(x, y)\right).
\]

Denote by \( C(Q, E) \) the space of all continuous functions from \( Q \) to \( E \) \((C(Q, E)\) is a Frechet space whose topology is introduced by seminorms of uniform convergence on compact subsets of \( Q)\), and by \( \mathcal{F} \) the set of all \( z \in C(Q, E) \) with

\[
\| z(x, y) \| < g_0(x, y) \quad \text{on} \ Q.
\]

Let \( P \) be a bounded subset of \( Q \). From the uniform continuity
of $f$ on bounded subsets of $\Omega$ follows the existence of a function $\delta_p : (0, \infty) \to (0, \infty)$ such that

$$\|f(x', y', \int_0^x \int_0^y z(t, s) dt ds, z(x, y)) - f(x'', y'', \int_0^x \int_0^y z(t, s) dt ds, z(x, y))\| < \varepsilon$$

for any $z \in \mathcal{X}$, $(x, y) \in P$ and $(x', y'), (x'', y'') \in P$ with $|x' - x''| < \delta_p(\varepsilon)$ and $|y' - y''| < \delta_p(\varepsilon)$.

Consider the set $\mathcal{X}_0$ of $z \in \mathcal{X}$ possessing the following property: for each bounded subset $P$ of $Q$, $\varepsilon > 0$ and $|x' - x''| < \delta_p(\varepsilon)$, $|y' - y''| < \delta_p(\varepsilon)$ (here $(x', y')$, $(x'', y'') \in P$) there holds $\|z(x', y') - z(x'', y'')\| < (1 - L(P))^{-1} \varepsilon$. The set $\mathcal{X}_0$ is a closed convex and almost equi-continuous subset of $C(Q, E)$. To apply our fixed point theorem we define a continuous mapping $T$ of $C(Q, E)$ into itself by the formula

$$(Tw)(x, y) = f(x, y, \int_0^x \int_0^y w(t, s) dt ds, w(x, y)).$$

Let $z \in \mathcal{X}_0$. Then

$$\|(Tw)(x, y)\| \leq G \left( x, y, \int_0^x \int_0^y \|z(t, s)\| dt ds, \|z(x, y)\| \right) \leq g_0(x, y)$$

for $(x, y) \in Q$. Further, for $\varepsilon > 0$ and $(x', y')$, $(x'', y'') \in P$ such that $|x' - x''| < \delta_p(\varepsilon)$, $|y' - y''| < \delta_p(\varepsilon)$ we have

$$\|(Tw)(x', y') - (Tw)(x'', y'')\| \leq \|f(x', y', \int_0^x \int_0^y z(t, s) dt ds, z(x', y')) - f(x', y', \int_0^x \int_0^y z(t, s) dt ds, z(x'', y''))\| + \|f(x', y', \int_0^x \int_0^y z(t, s) dt ds, z(x', y')) - f(x', y', \int_0^x \int_0^y z(t, s) dt ds, z(x'', y''))\|.$$
Let $n$ be a positive integer and let $Z$ be a nonempty subset of $\mathcal{X}_n$. Put $P_n = [0, n] \times [0, n]$, $k_n = k(P_n)$ and $L_n = L(P_n)$. Now we shall show the basic inequality:

\[
\sup_{(x, y) \in P_n} \exp \left( -p_n(x + y) \right) \alpha(T[Z](x, y)) \leq (p_n^{-2} k_n + 2 L_n) \cdot \sup_{(x, y) \in P_n} \exp \left( -p_n(x + y) \right) \alpha(Z(x, y)),
\]

where $p_n > 0$.

To this end, fix $(x, y)$ in $P_n$. By Lemma, we obtain

\[
\alpha \left( \int_0^x \int_0^y Z(t, s) \, dt \, ds \right) \leq \int_0^x \int_0^y \alpha(Z(t, s)) \, dt \, ds \leq \sup_{(t, s) \in P_n} \exp \left( -p_n(t + s) \right) \alpha(Z(t, s)) \cdot \int_0^x \int_0^y \exp \left( p_n(t + s) \right) \, dt \, ds \leq p_n^{-2} \cdot \exp \left( p_n(x + y) \right) \cdot \sup_{(t, s) \in P_n} \exp \left( -p_n(t + s) \right) \alpha(Z(t, s)).
\]

It is easy to verify (see [11], p. 476) that

\[
\alpha(T[Z](x, y)) \leq k_n \cdot \alpha \left( \int_0^x \int_0^y Z(t, s) \, dt \, ds \right) + 2 L_n \cdot \alpha(Z(x, y)).
\]

Therefore

\[
\exp \left( -p_n(x + y) \right) \alpha(T[Z](x, y)) \leq (p_n^{-2} k_n + 2 L_n) \cdot \sup_{(t, s) \in P_n} \exp \left( -p_n(t + s) \right) \alpha(Z(t, s))
\]

and our inequality is proved.
Let \( p_n^2 > (1 - 2L_n)^{-1}k_n \) (\( n = 1, 2, \ldots \)). Define:

\[
\Phi(Z) = \left( \sup_{(x,y) \in P_1} \exp \left( -p_1(x + y) \right) \alpha(Z(x,y)) \right).
\]

for any nonempty subset \( Z \) of \( \mathcal{X}_a \). Evidently, \( \Phi(Z) \in S_\alpha \). By Ascoli theorem and properties of \( \alpha \) our function \( \Phi \) satisfy conditions (1)-(3) listed in Section 2. From inequality (*) it follows that \( \Phi(T[Z]) \) < \( \Phi(Z) \) whenever \( \Phi(Z) > 0 \), and all assumptions of Sadovskii’s fixed point theorem are satisfied. Consequently, \( T \) has a fixed point in \( \mathcal{X}_a \) which completes the proof.

REFERENCES


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