RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

GIOVANNI DORE
DAVIDE GUIDETTI
ALBERTO VENNI

Complex interpolation for 2^N Banach spaces

Rendiconti del Seminario Matematico della Università di Padova, tome 76 (1986), p. 1-36
<http://www.numdam.org/item?id=RSMUP_1986__76__1_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 76 (1986)

Complex Interpolation for 2^N Banach Spaces.

GIOVANNI DORE - DAVIDE GUIDETTI - ALBERTO VENNI (*)

SUNTO - Si definiscono spazi d'interpolazione tra 2^N spazi di Banach, con il metodo complesso. Gli spazi sono definiti a partire da spazi di funzioni olomorfe sulla polistriscia di $\mathbf{C}^N$ che siano integrali di Poisson di opportuni dati al bordo. Vengono dimostrati un teorema di densità e alcune proprietà che collegano questo metodo con il metodo di Calderón. Infine viene data una caratterizzazione degli spazi duali.

1. Introduction.

The aim of this paper is to give an extension involving 2^N Banach spaces and N parameters of the complex method of interpolation for Banach spaces introduced by A. P. Calderón in [3].

In the subsequent years other generalizations were given by several authors: we recall Favini [9], who employs $N+1$ spaces and N parameters developing a definition of Lions [17]. S. G. Kreĭn and L. I. Nikolova [18] [14] [15] constructed interpolation spaces for finite or infinite families depending on a complex parameter. In this connection we quote also the papers [4] [5] [6] in which they describe a general theory of complex interpolation for infinite families of Banach spaces with one complex parameter varying on a simply or multiply connected domain.

(*) Indirizzo degli AA.: Dipartimento di Matematica, Piazza di Porta S. Donato 5, 40127 Bologna, Italia.

The method involving 2^N spaces and N parameters goes back to the papers of Fernandez who deals with the real case in [10] [11] [12], with the complex case in [13], and with their connection in [2]. In the two last papers they sketch a theory following a pattern very similar to the pattern of Calderón, but the crucial inequality 4.4. (2) of [2] is wrong, as we prove in our counterexample 5.5. It is just the lack of that inequality, without which part of Calderón's theory cannot be developed, which led us to give a new definition of the interpolation spaces, starting from holomorphic functions on the polystrip S^N that are not supposed to be continuous at the boundary but are Poisson integrals of functions belonging to suitably weighted L^q spaces.

The paper is arranged in the following way. In § 2 we obtain some inequalities about Poisson kernels and some technical results about Poisson integrals. In § 3 we define the interpolation spaces for a compatible family (A_j) as the spaces of values in the points of $]0,1[^N$ assumed by holomorphic functions on $(]0,1[+i\mathbb{R})^N$ which can be expressed as Poisson integrals of functions of type $L^q_\varrho(\mathbb{R}^N, A_j)$, where ϱ is a weight function equivalent to the Poisson kernel. In § 4 we prove the density of a certain class of « good functions » in the function spaces introduced in the previous section. This result is repeatedly employed in § 5 to show the density of $\bigcap_j A_j$ in the interpolation spaces and other properties. In particular we study the connection between our method of interpolation and Calderón's; we show that our interpolation spaces are embedded in iterated interpolation spaces of Calderón. A counterexample shows also that in general this inclusion is proper. In § 6 we prove some results of duality.

2. Poisson kernels and Poisson integrals.

In this section we give some definitions, deduce some estimates about Poisson kernels for the strip and study some properties of Poisson integrals. We establish the following notations which we keep from now on.

$$I =]0,1[\,, \qquad J = \{0,1\}\,, \qquad S = I + i\mathbb{R} = \{z \in \mathbf{C};\ \operatorname{Re} z \in I\}\,,$$

N is a positive integer, and when $\alpha \in \mathbb{R}$ $(\alpha \neq 0)$ $\vec{\alpha}$ is the N-tuple $(\alpha, \dots, \alpha)$.

For $z = (z_1, \dots, z_N) \in \mathbf{C}^N$ we set

$$\operatorname{Re} z = (\operatorname{Re} z_1, \dots, \operatorname{Re} z_N), \qquad \operatorname{Im} z = (\operatorname{Im} z_1, \dots, \operatorname{Im} z_N)\,.$$

$$\forall (a, s) \in I \times \mathbf{R} \qquad P_0(a, s) = \frac{\sin \pi a}{2(\cosh \pi s - \cos \pi a)}\,,$$

$$P_1(a, s) = P_0(1 - a, s) = \frac{\sin \pi a}{2(\cosh \pi s + \cos \pi a)}\,.$$

These are the Poisson kernels for the strip S. $\forall (a, s) \in I^N \times \mathbf{R}^N$ $\forall j \in J^N$ $\mathbf{P}_j(a, s) = \prod_{k=1}^{N} P_{j_k}(a_k, s_k)$. Moreover $q \in [1, \infty]$ (unless a more restrictive condition is required) and q' is its conjugate exponent (i.e. $1/q + 1/q' = 1$). The weight function $\varrho\colon \mathbf{R}^N \to \mathbf{R}^+$ is defined by $\varrho(s) = \prod_{k=1}^{N} (\cosh \pi s_k)^{-1}$. Remark that $\int_{\mathbf{R}^N} \varrho(s)\,ds = 1$. When X is a Banach space and $q < \infty$, $L^q_\varrho(\mathbf{R}^N, X)$ is the Banach space of strongly measurable functions $f\colon \mathbf{R}^N \to X$ such that $\|f\|_{L^q_\varrho} = \left(\int_{\mathbf{R}^N} \varrho(s) \|f(s)\|^q_X\, ds\right)^{1/q} < +\infty$. For notational convenience we shall also write L^q_ϱ when q is allowed to reach ∞: however in the case $q = \infty$ it is understood that this symbol denotes simply the space L^∞ (without weight).

We observe that from the well-known identity

$$\cosh(s + t) + \cosh(s - t) = 2 \cosh s \, \cosh t$$

it follows that

$$\frac{\cosh(s - t)}{\cosh s} \leqslant 2 \cosh t \tag{2.1}$$

and later we shall refer to this inequality even without explicit mention. We have also

$$\limsup_{t \to 0} \sup_{s \in \mathbf{R}} \left| \frac{\cosh(s - t)}{\cosh s} - 1 \right| = 0\,. \tag{2.2}$$

In fact

$$\left| \frac{\cosh(s - t) - \cosh s}{\cosh s} \right| = \left| \frac{e^s}{e^s + e^{-s}} (e^{-t} - 1) + \frac{e^{-s}}{e^s + e^{-s}} (e^t - 1) \right| \leqslant$$

$$\leqslant |e^{-t} - 1| + |e^t - 1|\,.$$

PROPOSITION 2.1. *The following statements hold*:

$$\forall a \in I \quad \int_{\mathbf{R}} P_0(a, s)\, ds = 1 - a\,, \tag{2.3}$$

$$\forall \delta \in \mathbf{R}^+,\ j \in \{0, 1\} \quad \lim_{a \to j} \int_{-\delta}^{\delta} P_0(a, s)\, ds = 1 - j \tag{2.4}$$

$$\forall \delta \in \mathbf{R}^+ \quad \lim_{a \to 0} \sup_{|s| \geqslant \delta} P_0(a, s) = 0 \tag{2.5}$$

$$0 < a \leqslant \tfrac{1}{2} \Rightarrow P_1(a, s) \leqslant \tfrac{1}{2} \sin \pi a \tag{2.6}$$

(2.7) *if* $0 \leqslant M \leqslant |s|$, *then* $\forall a \in I,\ j \in J$

$$\frac{\sin \pi a \cosh \pi M}{\cosh \pi M + |\cos \pi a|} \leqslant \frac{P_j(a, s)}{P_j(\frac{1}{2}, s)} \leqslant \frac{\sin \pi a \cosh \pi M}{\cosh \pi M - |\cos \pi a|}$$

(2.8) $\forall s,\ \sigma \in \mathbf{R},\ a \in I,\ j \in J$

$$P_j(a, s - \sigma) \leqslant \frac{2 \cosh \pi s \sin \pi a}{1 - |\cos \pi a|} P_j(\tfrac{1}{2}, \sigma)\,.$$

PROOF. (2.3), (2.4), (2.5), (2.6) are trivial. (2.7) Obviously

$$\frac{\sin \pi a \cosh \pi s}{\cosh \pi s + |\cos \pi a|} \leqslant \frac{\sin \pi a \cosh \pi s}{\cosh \pi s - (-1)^j \cos \pi a} \leqslant \frac{\sin \pi a \cosh \pi s}{\cosh \pi s - |\cos \pi a|}$$

and the middle term is exactly $P_j(a, s)/P_j(\frac{1}{2}, s)$. Now it is sufficient to observe that the left term is an increasing function of $|s|$ and the right one is a decreasing function of $|s|$.

(2.8) follows from (2.7) (with $M = 0$) and (2.1).

Remark that from (2.3) it follows that $\sum_{j \in J^N} \int_{\mathbf{R}^N} \mathbf{P}_j(a, s)\, ds = 1$.

In the remaining part of this section, X is a fixed Banach space. Remark that if $1 \leqslant q \leqslant r \leqslant \infty$, then $L^r_\varrho(\mathbf{R}^N, X) \subseteq L^q_\varrho(\mathbf{R}^N, X)$ (with $\|f\|_{L^q_\varrho} \leqslant \|f\|_{L^r_\varrho}$ $\forall f \in L^r_\varrho(\mathbf{R}^N, X)$).

PROPOSITION 2.2. *Let* $f \in L^1_\varrho(\mathbf{R}^N, X)$. *Then* $\forall j \in J^N$, $a \in I^N$, $s \in \mathbf{R}^N$, $\int_{\mathbf{R}^N} \mathbf{P}_j(a, s - \sigma) f(\sigma)\, d\sigma$ *exists and defines a continuous function of* $a + is \in S^N$

which is harmonic in every couple of variables (a_k, s_k). *Moreover*

$$\forall j \in J^N,\ \forall a \in I^N \qquad \sup_{s\in\mathbf{R}^N} \varrho(s)\, \Big\| \int_{\mathbf{R}^N} \mathbf{P}_j(a, s-\sigma) f(\sigma)\, d\sigma \Big\|_X < +\infty .$$

PROOF. By the estimate (2.8) we have

$$\mathbf{P}_j(a, s-\sigma) \leqslant \prod_{h=1}^{N} \frac{\sin \pi a_h}{1 - |\cos \pi a_h|} \, \frac{\varrho(\sigma)}{\varrho(s)} .$$

This ensures the existence of the integral, gives the required growth estimate, and allows us to employ the dominated convergence theorem to prove continuity. Moreover we can check that the function $(a_h, s_h) \mapsto$ $\mapsto \int_{\mathbf{R}^N} \mathbf{P}_j(a, s-\sigma) f(\sigma)\, d\sigma$ has the mean value property by changing the order of integration and exploiting the same property of $\mathbf{P}_j$.

LEMMA 2.3. *Let* $f\colon \bar{S} \to X$ *be a continuous function, harmonic on* S. *Suppose that* $\exists \varepsilon \in \,]0,1[$, $C > 0$ *such that* $\forall a + is \in \bar{S}$

$$\|f(a+is)\|_X \leqslant C \exp\big(\pi|s|(1-\varepsilon)\big) .$$

Then $\sup_{z\in\bar{S}} \|f(z)\|_X = \sup_{\xi\in\partial S} \|f(\xi)\|_X$.

PROOF. Suppose that f is real-valued. In this case we prove that $\forall z \in \bar{S}$ $f(z) \leqslant \sup_{\xi\in\partial S} f(\xi)$. Since $-f$ has the same properties as f, this will prove the statement when $X = \mathbf{R}$.

Let $0 < \delta < \varepsilon$ and put

$$g(a+is) = \cosh\big(\pi s(1-\delta)\big) \cos\big(\pi(\tfrac{1}{2} - a)(1-\delta)\big).$$

Then g is harmonic, and its g.l.b. on $\bar{S}$ is positive. $\forall \eta \in \mathbf{R}^+$

$$\lim_{|s|\to+\infty} f(a+is) - \eta g(a+is) = -\infty \qquad \text{uniformly for } a \in [0,1]$$

(since $1-\delta > 1-\varepsilon$).

Therefore, if we fix $z = a + is \in \bar{S}$ and M is large enough, by the maximum principle on $[0, 1] + i[-M, M]$,

$$f(a + is) - \eta g(a + is) \leqslant \max_{j=0,1} \sup_{|t| \leqslant M} (f - \eta g)(j + it) \leqslant \max_{j=0,1} \sup_{t \in \mathbf{R}} f(j + it) .$$

Letting $\eta \to 0^+$, we obtain $f(a + is) \leqslant \sup_{\xi \in \partial S} f(\xi)$.

Now we suppose that f is complex-valued. Obviously $\operatorname{Re} f$, $\operatorname{Im} f$ fulfil the assumptions of the theorem. Therefore

$$\forall z \in \bar{S} \ \ |f(z)| = \sup_{|\lambda + i\mu| = 1} \big(\lambda \operatorname{Re} f(z) + \mu \operatorname{Im} f(z)\big) \leqslant$$
$$\leqslant \sup_{|\lambda + i\mu| = 1;\ \xi \in \partial S} \big(\lambda \operatorname{Re} f(\xi) + \mu \operatorname{Im} f(\xi)\big) = \sup_{\xi \in \partial S} |f(\xi)| .$$

Finally, if f has values in the Banach space X, then

$$\forall z \in \bar{S} \ \ \|f(z)\|_X = \sup_{\varphi \in X^*, \|\varphi\| \leqslant 1} |\varphi(f(z))| \leqslant \sup_{\varphi \in X^*, \|\varphi\| \leqslant 1;\ \xi \in \partial S} |\varphi(f(\xi))| = \sup_{\xi \in \partial S} \|f(\xi)\|_X .$$

LEMMA 2.4. *Let $f \colon \mathbf{R}^N \to X$ be a continuous function such that $\exists \varepsilon \in]0, 1]$, $C > 0$ such that $\|f(s)\|_X \leqslant C \exp\left(\pi(1 - \varepsilon) \sum_{k=1}^{N} |s_k|\right)$. Let H be a compact subset of $\mathbf{R}^N$. Then, uniformly for $s \in H$,*

$$\lim_{a \to k} \int_{\mathbf{R}^N} \mathbf{P}_j(a, s - \sigma) f(\sigma)\, d\sigma = \delta_{j,k} f(s)$$

where $j, k \in J^N$ and $\delta_{j,k}$ is Kronecker's symbol. Moreover if f is uniformly continuous on $\mathbf{R}^N$ and $\varepsilon = 1$ (i.e. f is bounded), then the convergence is uniform on $\mathbf{R}^N$.

PROOF. Suppose $k \neq j$. Then

$$\int_{\mathbf{R}^N} \mathbf{P}_j(a, s - \sigma)\ \|f(\sigma)\|\, d\sigma \leqslant$$
$$\leqslant C \prod_{h=1}^{N} \int_{\mathbf{R}} \exp\big(\pi(1 - \varepsilon)\ |s_h - \sigma|\big) P_{j_h}(a_h, \sigma)\, d\sigma \leqslant$$
$$\leqslant C \exp\left(\pi(1 - \varepsilon) \sum_{h=1}^{N} |s_h|\right) \prod_{h=1}^{N} \int_{\mathbf{R}} \exp\big(\pi(1 - \varepsilon)|\sigma|\big) P_{j_h}(a_h, \sigma)\, d\sigma ,$$

so that it is enough to show that each integral in the last side of the inequality is bounded for $a_h \in I$ and converges to 0 when $a_h \to k_h \neq j_h$. But this is shown by the inequality

$$\int_{\mathbf{R}} \exp\big(\pi(1-\varepsilon)|t|\big) P_{j_h}(a_h, t)\, dt \leqslant$$

$$\frac{1}{2}\sin \pi a_h \int_{|t|\geqslant 1} \frac{\exp\big(\pi(1-\varepsilon)|t|\big)}{\cosh \pi t - 1}\, dt + \exp\big(\pi(1-\varepsilon)\big)\int_{-1}^{1} P_{j_h}(a_h, t)\, dt\,.$$

Suppose now $k = j$. Then

$$\Big\| \int_{\mathbf{R}^N} \mathbf{P}_j(a, \sigma) f(s-\sigma)\, d\sigma - f(s)\Big\| \leqslant \int_{\mathbf{R}^N \setminus [-\delta, \delta]^N} \mathbf{P}_j(a, \sigma)\, \|f(s-\sigma) - f(s)\|\, d\sigma +$$

$$+ \int_{[-\delta,\delta]^N} \mathbf{P}_j(a, \sigma)\, \|f(s-\sigma) - f(s)\|\, d\sigma + \Big(1 - \int_{\mathbf{R}^N} \mathbf{P}_j(a, \sigma)\, d\sigma\Big)\, \|f(s)\|\,.$$

When $a \to j$ the last term converges to 0, uniformly on every set on which f is bounded. Having fixed $\varepsilon \in \mathbf{R}^+$, we can take $\delta \in \mathbf{R}^+$ such that the second summand is $\leqslant \varepsilon\ \forall s \in H$ (or $\forall s \in \mathbf{R}^N$ if f is uniformly continuous on $\mathbf{R}^N$). The first summand is dominated by

$$C \int_{\mathbf{R}^N \setminus [-\delta,\delta]^N} \mathbf{P}_j(a, \sigma) \exp\Big(\pi(1-\varepsilon)\sum_{h=1}^{N} |\sigma_h|\Big)\, d\sigma\, \exp\Big(\pi(1-\varepsilon)\sum_{h=1}^{N}|s_h|\Big) +$$

$$+ \|f(s)\| \int_{\mathbf{R}^N \setminus [-\delta,\delta]^N} \mathbf{P}_j(a, \sigma)\, d\sigma\,.$$

Here the second term converges uniformly to 0. Therefore the proof will be complete if we show that when $a \to j$

$$\int_{\mathbf{R}^N} \mathbf{P}_j(a, \sigma) \exp\Big(\pi(1-\varepsilon)\sum_{h=1}^{N}|\sigma_h|\Big)\, d\sigma$$

and

$$\int_{[-\delta,\delta]^N} \mathbf{P}_j(a, \sigma) \exp\Big(\pi(1-\varepsilon)\sum_{h=1}^{N}|\sigma_h|\Big)\, d\sigma$$

have the same finite limit. For this we can suppose $N = 1$ and show that

$$\lim_{a\to j}\int_{-\delta}^{\delta} P_j(a,\sigma)\exp\big(\pi(1-\varepsilon)|\sigma|\big)\,d\sigma = 1\,,$$

$$\lim_{a\to j}\int_{\delta}^{+\infty} P_j(a,\sigma)\exp\big(\pi(1-\varepsilon)|\sigma|\big)\,d\sigma = 0\,.$$

The first equality follows from the fact that for $a \to j$ $P_j(a,\cdot)$ is an approximate identity; the second one is shown by an easy computation.

PROPOSITION 2.5 (see [3] p. 116). *Let $f\colon \bar{S}^N \to X$ be a continuous function, harmonic on S^N in every couple of variables (a_h, s_h). Suppose that* $\exists \varepsilon \in]0,1]$, $C > 0$ *such that* $\forall z \in \bar{S}^N$ $\|f(z)\| \leqslant C\exp\Big(\pi(1-\varepsilon)\sum_{h=1}^{N}|\mathrm{Im}\, z_h|\Big)$. *Then*

$$\forall a + is \in S^N \qquad f(a+is) = \sum_{j\in J^N}\int_{\mathbf{R}^N} \mathbf{P}_j(a, s-\sigma) f(j+i\sigma)\,d\sigma\,.$$

PROOF. We begin by supposing that $N = 1$. By lemma 2.4 and proposition 2.2, the function

$$a + is \mapsto f(a+is) - \sum_{j=0}^{1}\int_{\mathbf{R}} P_j(a, s-\sigma) f(j+i\sigma)\,d\sigma$$

is harmonic on S and has a continuous extension to $\bar{S}$, which vanishes on ∂S. We can get our result through lemma 2.3 if we prove that

$$\Big\|\sum_{j=0}^{1}\int_{\mathbf{R}} P_j(a, s-\sigma) f(j+i\sigma)\,d\sigma\Big\| \leqslant C_1\exp\big(\pi(1-\varepsilon)|s|\big)\,.$$

In fact we have

$$\exp\big(\pi(\varepsilon-1)|s|\big)\sum_{j=0}^{1}\int_{\mathbf{R}} P_j(a, s-\sigma)\,\|f(j+i\sigma)\|\,d\sigma \leqslant$$

$$\leqslant C\sum_{j=0}^{1}\int_{\mathbf{R}} P_j(a, s-\sigma)\exp\big(\pi(1-\varepsilon)|s-\sigma|\big)\,d\sigma\,.$$

This depends only on a, continuously on $[0, 1]$ by lemma 2.4, so that it is bounded.

Suppose now that f is a function of $N+1$ variables which satisfies our assumptions. If we fix the last variable, we obtain a function of N variables satisfying the same assumptions, so that we can proceed by induction.

The aim of our next result is to show that if $f_j \in L^1_\varrho(\mathbf{R}^N, X)$ $\forall j \in J^N$ and

$$\sum_{j\in J} \int_{\mathbf{R}^N} \mathbf{P}_j(a, s-\sigma) f_j(\sigma)\, d\sigma = 0 \qquad \forall a+is \in S^N,$$

then $\forall j \in J^N$ $f_j = 0$. However, since it is possible that

$$s \mapsto \int_{\mathbf{R}^N} \mathbf{P}_j(a, s-\sigma)\, f(\sigma)\, d\sigma$$

does not belong to $L^1_\varrho(\mathbf{R}^N, X)$, we work in a suitable larger space.

PROPOSITION 2.6. *Let* $f \in L^1_\varrho(\mathbf{R}^N, X)$, $j \in J^N$. *Then the function* $s \mapsto \int_{\mathbf{R}^N} \mathbf{P}_j(a, s-\sigma)\, f(\sigma)\, d\sigma$ *belongs to* $L^1_{\varrho^2}(\mathbf{R}^N, X)$, *and when* $a \to k \in J^N$ *it converges to* $\delta_{j,k} f$ *in the norm of* $L^1_{\varrho^2}(\mathbf{R}^N, X)$.

PROOF. $\int_{\mathbf{R}^N} \varrho^2(s) \| \int_{\mathbf{R}^N} \mathbf{P}_j(a, s-\sigma)\, f(\sigma)\, d\sigma \|\, ds \leqslant$ (see (2.8))

$$2^N \prod_{h=1}^{N} \frac{\sin \pi a_h}{1-|\cos \pi a_h|} \int_{\mathbf{R}^N} \varrho(s)\, ds \int_{\mathbf{R}^N} \mathbf{P}_j(\vec{\tfrac{1}{2}}, \sigma)\, \|f(\sigma)\|\, d\sigma < +\infty .$$

Since $\int_{\mathbf{R}^N} \varrho^2(s)\, ds < +\infty$, from lemma 2.4 it follows that we have the asserted convergence whenever f is uniformly continuous and bounded. Since such functions are dense in $L^1_\varrho(\mathbf{R}^N, X)$, it suffices to show that $\exists C > 0$ such that

$$\int_{\mathbf{R}^N} \varrho^2(s) \left\| \int_{\mathbf{R}^N} \mathbf{P}_j(a, s-\sigma)\, f(\sigma)\, d\sigma \right\| ds \leqslant C \|f\|_{L^1_\varrho} \quad \forall f \in L^1_\varrho(\mathbf{R}^N, X) \text{ and } \forall a \in I^N .$$

In fact

$$\int_{\mathbf{R}^N} \varrho^2(s) \left\| \int_{\mathbf{R}^N} \mathbf{P}_j(a, s-\sigma) f(\sigma)\, d\sigma \right\| ds \leqslant \int_{\mathbf{R}^N} \|f(\sigma)\|\, \varrho(\sigma) \int_{\mathbf{R}^N} \frac{\varrho^2(s)}{\varrho(\sigma)} \mathbf{P}_j(a,s-\sigma)\, ds\, d\sigma ,$$

so that it remains to show that

$$\sup_{a\in I^N,\, \sigma\in\mathbf{R}^N} \prod_{k=1}^{N} \left(\cosh \pi\sigma_k \int_{\mathbf{R}} (\cosh \pi t)^{-2} P_{j_k}(a_k, t-\sigma_k)\, dt \right) < +\infty .$$

But

$$\cosh \pi\sigma \int_{\mathbf{R}} (\cosh \pi t)^{-2} P_j(a, t-\sigma)\, dt =$$

$$\cosh \pi\sigma \int_{\mathbf{R}} \Big(\cosh\big(\pi(t-\sigma)\big)\Big)^{-2} P_j(a, t)\, dt \leqslant$$

$$\leqslant \int_{\mathbf{R}} \frac{\cosh \pi t}{\cosh\big(\pi(t-\sigma)\big)} \, \frac{\sin \pi a}{\cosh \pi t - (-1)^j \cos \pi a}\, dt$$

and direct computation of the last integral concludes the proof.

3. Definition of the interpolation spaces.

In this section we introduce the complex interpolation spaces for compatible families of complex Banach spaces. Henceforth every vector space we consider is complex.

A finite family $A = (A_j)_{j\in K}$ of Banach spaces is said to be compatible if there is a Hausdorff vector topological space $\mathcal{A}$ such that $\forall j \in K\ A_j \hookrightarrow \mathcal{A}$ (by « $\hookrightarrow$ » we denote algebraic inclusion with continuous embedding; if X and Y are normed spaces, $X \overset{M}{\hookrightarrow} Y$ means that the embedding norm does not exceed M). When A is a compatible family $\Delta(A)$ is the Banach space $\bigcap_{j\in K} A_j$ with norm $\|x\|_{\Delta(A)} = \max_{j\in K} \|x\|_{A_j}$, and $\Sigma(A)$ is the linear hull of $\bigcup_{j\in K} A_j$ with the norm $\|x\|_{\Sigma(A)} = \inf \left\{ \sum_{j\in K} \|x_j\|_{A_j};\ x_j \in A_j,\ \sum_{j\in K} x_j = x \right\}$ (this too is a Banach space). Evidently $\Delta(A) \overset{1}{\hookrightarrow} A_j \overset{1}{\hookrightarrow} \Sigma(A) \hookrightarrow \mathcal{A}$.

A normed space X is said to be intermediate for the family A if $\Delta(A) \hookrightarrow X \hookrightarrow \Sigma(A)$. When $A = (A_j)_{j\in K}$ and $B = (B_j)_{j\in K}$ are compatible families, $L(A, B)$ is defined as the vector space of the linear mappings $T: \Sigma(A) \to \Sigma(B)$ such that $T|_{A_j} \in \mathfrak{L}(A_j, B_j)$ $\forall j \in K$ (as usual, when X and Y are normed spaces, $\mathfrak{L}(X, Y)$ is the space of continuous linear mappings from X to Y). If X and Y are intermediate spaces for A, B respectively, we say that X, Y are interpolation spaces with respect to A, B when for every $T \in L(A, B)$ $T|_X \in \mathfrak{L}(X, Y)$.

From now on we consider compatible families A, B etc. of Banach spaces whose index set is J^N ($J = \{0, 1\}$). Since for a compatible family A and $1 \leqslant q \leqslant \infty$ $L^q_\varrho(\mathbb{R}^N, A_j) \overset{1}{\hookrightarrow} L^1_\varrho(\mathbb{R}^N, \Sigma(A))$, on the space $\prod_{j\in J^N} L^q_\varrho(\mathbb{R}^N, A_j)$ we can define a linear map $\mathcal{T}$, whose range is contained in the space of continuous functions $g: S^N \to \Sigma(A)$ (see prop. 2.2), by putting

$$\mathcal{T}(f)(a + is) = \sum_{j\in J^N} \int_{\mathbf{R}^N} \mathbf{P}_j(a, s - \sigma) f_j(\sigma)\, d\sigma$$

where $f = (f_j)_{j\in J^N}$, $a \in I^N$, $s \in \mathbb{R}^N$.

From prop. 2.6 it follows at once that is one-to- one, and we call $\tilde{\mathcal{F}}_q(A)$ the image under $\mathcal{T}$ of $\prod_{j\in J^N} L^q_\varrho(\mathbb{R}^N, A_j)$. On $\prod_{j\in J^N} L^q_\varrho(\mathbb{R}^N, A_j)$ the functional

$$(f_j)_{j\in J^N} \mapsto \Big(\sum_{j\in J^N} \|f_j\|^q_{L^q_\varrho(\mathbf{R}^N, A_j)}\Big)^{1/q}$$

(with the usual modification if $q = \infty$) is obviously a Banach norm. If $\forall f = (f_j)_{j\in J^N} \in \prod_{j\in J^N} L^q_\varrho(\mathbb{R}^N, A_j)$, $\forall a \in I^N$ we put

$$|||f|||_{(a;\,q)} = \Big(\sum_{j\in J^N} \int_{\mathbf{R}^N} \mathbf{P}_j(a, t)\, \|f_j(t)\|^q_{A_j} dt\Big)^{1/q} \qquad (1 \leqslant q < \infty)$$

and if

$$q = \infty \quad |||f|||_{(a;\,\infty)} = \max_{j\in J^N} \|f_j\|_{L^\infty(\mathbf{R}^N, A_j)}, \qquad \text{then } |||\cdot|||_{(a;\,q)}$$

is a norm on $\prod_{j\in J^N} L^q_\varrho(\mathbb{R}^N, A_j)$. Remark that $|||\cdot|||_{(a;\,\infty)}$ does not depend on a (but we keep this notation for convenience) and that when $1 \leqslant q < \infty$

$$|||f|||_{(\vec{\frac{1}{2}};\,q)} = 2^{-N/q}\Big(\sum_{j\in J^N} \|f_j\|^q_{L^q_\varrho(\mathbf{R}^N, A_j)}\Big)^{1/q}.$$

$\forall a \in I^N$ we call $|||\cdot|||_{(a;q)}$ also the norm induced by $\mathcal{T}$ on $\tilde{\mathcal{F}}_q(A)$ $(|||\mathcal{T}(f)|||_{(a;q)} = |||f|||_{(a;q)})$.

PROPOSITION 3.1. *For each fixed* $q \in [1, \infty]$, *all the norms* $|||\cdot|||_{(a;q)}$ $(a \in I^N)$ *on* $\tilde{\mathcal{F}}_q(A)$ *are equivalent, uniformly for a in compact subsets of* I^N. *Moreover for* $1 \leqslant q \leqslant r \leqslant \infty$ $\tilde{\mathcal{F}}_r(A) \subseteq \tilde{\mathcal{F}}_q(A)$ *and* $\forall f \in \tilde{\mathcal{F}}_r(A)$

$$|||f|||_{(a;q)} \leqslant |||f|||_{(a;r)}\,.$$

PROOF. The first statement is an obvious consequence of the inequality (2.7) with $M = 0$. The second one follows from Hölder inequality and from

$$\sum_{j \in J^N} \int_{\mathbf{R}^N} \mathbf{P}_j(a, s)\, ds = 1\,.$$

PROPOSITION 3.2. $\forall g \in \tilde{\mathcal{F}}_q(A)$ *and* $\forall a \in I^N$ $\|g(a)\|_{\Sigma(A)} \leqslant |||g|||_{(a;q)}$. *If* K *is a compact subset of* S^N, $\exists C_K > 0$ *such that* $\forall g \in \tilde{\mathcal{F}}_q(A)$ $\sup\limits_{z \in K} \|g(z)\|_{\Sigma(A)} \leqslant C_K |||g|||_{(\vec{\frac{1}{2}};q)}$.

PROOF. Let $g = \mathcal{T}(f) \in \tilde{\mathcal{F}}_q(A)$. Then

$$\|g(a)\|_{\Sigma(A)} = \|\mathcal{T}(f)(a)\|_{\Sigma(A)} \leqslant \sum_{j \in J^N} \int_{\mathbf{R}^N} \mathbf{P}_j(a, s)\, \|f_j(s)\|_{A_j} ds = |||g|||_{(a;1)} \leqslant |||g|||_{(a;q)}\,.$$

If $z \in K$, we have

$$\|g(z)\|_{\Sigma(A)} \leqslant \sum_j \int_{\mathbf{R}^N} \mathbf{P}_j(\operatorname{Re} z, \operatorname{Im} z - \sigma)\, \|f_j(\sigma)\|_{A_j} d\sigma \leqslant$$

$$\leqslant (\text{by } (2.8)) \max_{z \in K} \prod_{h=1}^{N} \frac{2\cosh(\pi \operatorname{Im} z_h)}{1 - |\cos(\pi \operatorname{Re} z_h)|} \sum_j \int_{\mathbf{R}^N} \mathbf{P}_j(\vec{\tfrac{1}{2}}, \sigma)\, \|f_j(\sigma)\|_{A_j} d\sigma \leqslant$$

$$\leqslant C_K |||g|||_{(\vec{\frac{1}{2}};1)} \leqslant C_K |||g|||_{(\vec{\frac{1}{2}};q)}\,.$$

DEFINITION 3.3. $\mathcal{F}_q(A)$ *is the space of the functions* $f\colon S^N \to \Sigma(A)$ *holomorphic and belonging to* $\tilde{\mathcal{F}}_q(A)$. *By the second part of prop.* 3.2, *convergence in* $\tilde{\mathcal{F}}_q(A)$ *implies uniform convergence on compact subsets of* S^N, *with respect to the norm of* $\Sigma(A)$, *so that* $\mathcal{F}_q(A)$ *is a closed subspace of* $\tilde{\mathcal{F}}_q(A)$ *and hence a Banach space.*

When X is a Banach space, we write $\tilde{\mathcal{F}}_q(X)$ and $\mathcal{F}_q(X)$ to mean $\tilde{\mathcal{F}}_q(A)$ and $\mathcal{F}_q(A)$ where $A = (A_j)_{j\in J^N}$ and $A_j = X \ \forall j \in J^N$.

Now we define interpolation spaces $A_{(a;q)}$.

DEFINITION 3.4. $\forall a \in I^N$, $q \in [1, \infty]$ $A_{(a;q)} = \{f(a);\ f \in \mathcal{F}_q(A)\}$.

Since the operator $f \mapsto f(a)$ is continuous from $\mathcal{F}_q(A)$ to $\Sigma(A)$ by prop. 3.2, we can define a Banach norm on $A_{(a;q)}$ by putting $\|x\|_{(a;q)} = \inf\{|||f|||_{(a;q)};\ f \in \mathcal{F}_q(A),\ f(a) = x\}$.

THEOREM 3.5. *$A_{(a;q)}$ is an intermediate space for the family A. More precisely,* $\Delta(A) \overset{1}{\hookrightarrow} A_{(a;q)} \overset{1}{\hookrightarrow} \Sigma(A)$.

PROOF. Let $x \in \Delta(A)$ and $f\colon S^N \to \Sigma(A)$, $f(z) = x\ \forall z \in S^N$. It is obvious that $f \in \mathcal{F}_q(A)$, so that $x \in A_{(a;q)}$ and $\|x\|_{(a;q)} \leqslant |||f|||_{(a;q)} =$ $=$ (when $q < \infty$)

$$\Big(\sum_j \int_{\mathbf{R}^N} \mathbf{P}_j(a, s)\ \|x\|^q_{A_j} ds\Big)^{1/q} \leqslant \|x\|_{\Delta(A)} \Big(\sum_j \int_{\mathbf{R}^N} \mathbf{P}_j(a, s)\, ds\Big)^{1/q} = \|x\|_{\Delta(A)}\,.$$

The proof is even simpler when $q = \infty$, since $|||f|||_{(a;\infty)} = \|x\|_{\Delta(A)}$.

Let $x \in A_{(a;q)}$. Then obviously $x \in \Sigma(A)$ and the estimates on the norms follow at once from the first part of prop. 3.2.

THEOREM 3.6. *If A and B are compatible families of Banach spaces, then $A_{(a;q)}$ and $B_{(a;q)}$ are interpolation spaces with respect to A and B. More precisely,* $\forall T \in L(A, B)\ \|T\|_{\mathfrak{L}(A_{(a;q)}, B_{(a;q)})} \leqslant \max_{j\in J^N} \|T\|_{\mathfrak{L}(A_j, B_j)}$.

PROOF. Let $x \in A_{(a;q}$ and $g = \mathfrak{T}(f) \in \mathcal{F}_q(A)$ with $g(a) = x$. Then $Tg \in \mathcal{F}_q(B)$, so that $Tx \in B_{(a;q)}$ and

$$\|Tx\|_{(a;q)} \leqslant |||Tg|||_{(a;q)} = \Big(\sum_j \int_{\mathbf{R}^N} \mathbf{P}_j(a, s)\ \|Tf_j(s)\|^q_{B_j} ds\Big)^{1/q} \leqslant$$

$$\leqslant \Big(\sum_j \|T\|^q_{\mathfrak{L}(A_j, B_j)} \int_{\mathbf{R}^N} \mathbf{P}_j(a, s)\ \|f_j(s)\|^q_{A_j} ds\Big)^{1/q} \leqslant \max_j \|T\|_{\mathfrak{L}(A_j, B_j)}\ |||f|||_{(a;q)}$$

(and similarly when $q = \infty$). As f is arbitrary, we obtain the inequality.

REMARKS

3.7. When X is a Banach space and $A_j = X\ \forall j \in J^N$, we have $\Delta(A) = X = \Sigma(A)$ with equal norms. Therefore from th. 3.5 it follows that in this case $A_{(a;q)} = X$ with equal norms.

3.8. From prop. 3.1 we have $A_{(a;r)} \overset{1}{\hookrightarrow} A_{(a;q)}$ whenever $1 \leqslant q \leqslant r \leqslant \infty$.

3.9. If $1 < q < \infty$ and $\forall j \in J^N$ A_j is reflexive, then $L^q_\varrho(\mathbf{R}^N, A_j)$ is reflexive ([19] th. 5.7), so that $\tilde{\mathcal{F}}_q(A)$, $\mathcal{F}_q(A)$ and $A_{(a;q)}$ are reflexive, since products, closed subspaces and quotients of reflexive spaces are reflexive.

4. A density theorem.

Let A be a compatible family of Banach spaces. We consider the holomorphic functions $f\colon \mathbf{C}^N \to \Delta(A)$ such that $\lim\limits_{z \in \bar{S}^N, |z| \to \infty} \|f(z)\|_{\Delta(A)} = 0$, and we call $\mathcal{F}_0(A)$ the space of the restrictions to S^N of these functions. Remark that by prop. 2.5 $\mathcal{F}_0(A) \subseteq \mathcal{F}_q(A)$ $\forall q \in [1, \infty]$.

The main result of this section is the following one.

THEOREM 4.1. *If* $1 \leqslant q < \infty$, *then* $\mathcal{F}_0(A)$ *is dense in* $\mathcal{F}_q(A)$.

We shall prove this theorem through a number of lemmas concerning the functions

$$f_n^{(a)}(z) = (n/\sqrt{\pi})^N \int_{\mathbf{R}^N} \exp\left(n^2 \sum_{k=1}^N (z_k - a_k - is_k)^2\right) f(a + is)\, ds \tag{4.1}$$

(where $f \in \mathcal{F}_1(A)$, $z \in \mathbf{C}^N$, $a \in I^N$ and $n \in \mathbf{N}$),

$$f_n^{(j)}(z) = (n/\sqrt{\pi})^N \int_{\mathbf{R}^N} \exp\left(n^2 \sum_{k=1}^N (z_k - j_k - is_k)^2\right) f_j(s)\, ds \tag{4.2}$$

(where $j \in J^N$, $f_j \in L^1_\varrho(\mathbf{R}^N, A_j)$, $z \in \mathbf{C}^N$ and $n \in \mathbf{N}$).

LEMMA 4.2. *The above integrals* (4.1) *and* (4.2) *exist in the spaces* $\Sigma(A)$ *and* A_j *respectively. The function* $f_n^{(j)}$, *as defined by formula* (4.2) *is holomorphic on* $\mathbf{C}^N$ *in the norm of* A_j.

PROOF. Let $f = \mathcal{T}\big((f_j)_{j \in J^N}\big) \in \mathcal{F}_1(A)$. As

$$\int_{\mathbf{R}^N} \exp\left(n^2 \operatorname{Re} \sum_{k=1}^N (z_k - a_k - is_k)^2\right) \|f(a + is)\|_{\Sigma(A)} ds \leqslant$$

$$\leqslant C \int_{\mathbf{R}^N} \exp\left(-n^2 \sum_{k=1}^N (\operatorname{Im} z_k - s_k)^2\right) \sum_j \int_{\mathbf{R}^N} \mathbf{P}_j(a, s - \sigma)\, \|f_j(\sigma)\|_{A_j} d\sigma\, ds \leqslant$$

$$\leqslant (by\ (2.8))\ C_1 \int_{\mathbf{R}^N} \exp\Big(-n^2 \sum_{k=1}^{N} (\operatorname{Im} z_k - s_k)^2\Big) \cdot$$

$$\cdot \prod_{k=1}^{N} \cosh \pi s_k \, ds \sum_j \int_{\mathbf{R}^N} \frac{\|f_j(\sigma)\|_{A_j}}{\prod_{k=1}^{N} \cosh \pi \sigma_k} \, d\sigma < +\infty ,$$

the integral (4.1) exists in $\Sigma(A)$. Proving the existence of (4.2) in A_j is even simpler, as

$$\int_{\mathbf{R}^N} \frac{\|f_j(s)\|_{A_j}}{\prod_{k=1}^{N} \cosh \pi s_k} \, ds$$

and

$$\sup_{s \in \mathbf{R}^N} \exp\Big(-n^2 \sum_{k=1}^{N} (\operatorname{Im} z_k - s_k)^2\Big) \prod_{k=1}^{N} \cosh \pi s_k$$

are both finite. Let K be a compact subset of $\mathbf{C}^N$. Then for $z \in K$ and $s \in \mathbf{R}^N$,

$$\Big|\exp\Big(n^2 \sum_{k=1}^{N} (z_k - j_k - i s_k)^2\Big)\Big| \prod_{k=1}^{N} \cosh \pi s_k \leqslant$$

$$\leqslant C_K \exp\Big(-n^2 \sum_{k=1}^{N} (\operatorname{Im} z_k - s_k)^2\Big) \prod_{k=1}^{N} \cosh \pi s_k \leqslant$$

$$\leqslant C_K \exp\Big(-n^2 |s|^2 + C'_K n^2 \sum_{k=1}^{N} |s_k|\Big) \prod_{k=1}^{N} \cosh \pi s_k \leqslant C''_K .$$

Therefore, by the dominated convergence theorem, $f_n^{(j)}$ is continuous from $\mathbf{C}^N$ to A_j. Moreover, if γ is a closed, piecewise differentiable curve in $\mathbf{C}$, when we compute

$$\int_\gamma f_n^{(j)}(z)\, dz_h = (n/\sqrt{\pi})^N \int_\gamma \int_{\mathbf{R}^N} \exp\Big(n^2 \sum_{k=1}^{N} (z_k - j_k - i s_k)^2\Big) f_j(s)\, ds\, dz_h ,$$

we can change the order of integration, so that it vanishes. This proves the lemma.

LEMMA 4.3. *Let* $f = \mathcal{T}((f_j)) \in \mathcal{F}_1(A)$. *Then* $\forall z \in \mathbf{C}^N$ $\forall j \in J^N$

$$\lim_{a \to j} f_n^{(a)}(z) = f_n^{(j)}(z) \qquad (in\ \Sigma(A)) .$$

PROOF. Without loss of generality we may consider only the case $j = 0$. When we substitute in (4.1) $f(a + is)$ with its expression by means of the Poisson integral, we can change the order of integration, so that

$$f_n^{(a)}(z) = (n/\sqrt{\pi})^N \sum_{j\in J^N} \int_{\mathbf{R}^N} f_j(\sigma) \int_{\mathbf{R}^N} \exp\left(n^2 \sum_{k=1}^N (z_k - a_k - is_k)^2\right) \mathbf{P}_j(a, s - \sigma)\, ds\, d\sigma\,.$$

We prove that in this sum every summand converges to 0, except the one with $j = 0$ which converges to $f_n^{(0)}(z)$, as $a \to 0$. By the dominated convergence theorem it is enough to prove that

$$\left|\int_{\mathbf{R}} \exp\left(n^2(w - a - is)^2\right) P_j(a, s - \sigma)\, ds\right| \leqslant \frac{C(n, w)}{\cosh \pi\sigma} \tag{4.3}$$

and that

$$\int_{\mathbf{R}} \exp\left(n^2(w - a - is)^2\right) P_j(a, s - \sigma)\, ds \xrightarrow[a\to 0]{} \delta_{j0} \exp\left(n^2(w - i\sigma)^2\right). \tag{4.4}$$

In fact the left side of (4.3) is dominated by

$$\left(\int_{-1}^{1} + \int_{|s|\geqslant 1}\right) \exp\left(n^2\left((\operatorname{Re} w - a)^2 - (\operatorname{Im} w - s - \sigma)^2\right)\right) P_j(a, s)\, ds$$

The first integral is

$$\leqslant \sup_{|s|\leqslant 1} \exp\left(n^2\left((\operatorname{Re} w - a)^2 - (\operatorname{Im} w - s - \sigma)^2\right)\right) \leqslant$$
$$\leqslant C'(n, w) \exp\left(n^2\left(-(\operatorname{Im} w - \sigma)^2 + 2|\operatorname{Im} w - \sigma|\right)\right) \leqslant \frac{C''(n, w)}{\cosh \pi\sigma}\,,$$

while the second one is $\leqslant$ (see (2.7))

$$\frac{\sin \pi a \cosh \pi}{\cosh \pi - |\cos \pi a|} \int_{\mathbf{R}} \exp\left(n^2\left((\operatorname{Re} w - a)^2 - (\operatorname{Im} w - s - \sigma)^2\right)\right) \frac{1}{2 \cosh \pi s}\, ds \leqslant$$
$$\leqslant \text{(see (2.1))}$$

$$C_1'(n, w) \int_{\mathbf{R}} \exp\left(-n^2(\operatorname{Im} w - \tau)^2\right) \cosh \pi\tau\, d\tau\, \frac{1}{\cosh \pi\sigma} \leqslant \frac{C_1''(n, w)}{\cosh \pi\sigma}\,.$$

For $j = 1$, (4.4) follows at once from (2.6). Finally

$$\left|\int_{\mathbf{R}} \exp\left(n^2(w-a-is)^2\right) P_0(a, s-\sigma)\, ds - \exp\left(n^2(w-i\sigma)^2\right)\right| \leqslant$$

$$\leqslant \int_{\mathbf{R}} \left|\exp\left(n^2(w-a-is)^2\right) - \exp\left(n^2(w-is)^2\right)\right| P_0(a, s-\sigma)\, ds +$$

$$+ \left|\int_{\mathbf{R}} \exp\left(n^2(w-is)^2\right) P_0(a, s-\sigma)\, ds - \exp\left(n^2(w-i\sigma)^2\right)\right|.$$

Here the second summand converges to 0 by lemma 2.4. The first one is dominated by

$$(1-a) \sup_{s\in\mathbf{R}} \left|\exp\left(n^2(w-a-is)^2\right) - \exp\left(n^2(w-is)^2\right)\right| \xrightarrow[a\to 0]{} 0 .$$

LEMMA 4.4. *For fixed* $f \in \mathcal{F}_1(A)$, $z \in \mathbf{C}^N$, $f_n^{(a)}(z)$ *does not depend on* $a \in I^N$.

PROOF. We prove that for fixed $\bar{a} = (a_2, \dots, a_N) \in I^{N-1}$, $f_n^{(a)}(z)$ does not depend on a_1. For this it is enough to show that

$$\int_{\mathbf{R}} \exp\left(n^2(z_1 - a_1 - it_1)^2\right) f(a + it)\, dt_1$$

does not depend on a_1 (the existence of this integral is guaranteed by prop. 2.2). Being holomorphic the function

$$\xi \mapsto \exp\left(n^2(z_1 - \xi)^2\right) f(\xi, \bar{a} + i\bar{t}) \qquad (\bar{t} = (t_2, \dots, t_N))$$

it suffices to prove that as $|M| \to +\infty$

$$\int_{a'}^{a''} \exp\left(n^2(z_1 - \alpha - iM)^2\right) f(\alpha + iM, \bar{a} + i\bar{t})\, d\alpha \to 0 \qquad \forall a', a'' \in I .$$

But this follows from prop. 2.2.

Thus we have proved the following proposition:

PROPOSITION 4.5. $f_n^{(a)}(z) \in \Delta(A)$ *and the function* $f_n^{(a)}$ *is holomorphic in the norm of* $\Delta(A)$.

Lemma 4.4 allows us to write henceforth f_n instead of $f_n^{(a)}$ or $f_n^{(j)}$. Moreover, by lemmas 4.3 and 4.4

$$f_n(a+is) = f_n^{(a)}(a+is) = (n/\sqrt{\pi})^N \int_{\mathbf{R}^N} \exp\left(-n^2|s-t|^2\right) f(a+it)\,dt$$

and

$$f_n(j+is) = (n/\sqrt{\pi})^N \int_{\mathbf{R}^N} \exp\left(-n^2|s-t|^2\right) f_j(t)\,dt\,.$$

LEMMA 4.6. *If* $f = \mathcal{T}((f_j)) \in \mathcal{F}_1(A)$, *then* $\forall n \in \mathbf{N}\ \exists C_n > 0$ *such that*

$$\forall z \in \bar{S}^N \qquad \|f_n(z)\|_{\Delta(A)} \leqslant C_n \prod_{k=1}^N \cosh(\pi \operatorname{Im} z_k)\,.$$

PROOF

$$\|f_n(z)\|_{A_j} = \|f_n^{(j)}(z)\|_{A_j} \leqslant C_n' \int_{\mathbf{R}^N} \exp\left(-n^2 \sum_{k=1}^N (\operatorname{Im} z_k - s_k)^2\right) \|f_j(s)\|_{A_j} ds \leqslant$$
$$\leqslant C_n' \|f_j\|_{L^1_\varrho(\mathbf{R}^N, A_j)} \sup_{s \in \mathbf{R}^N} \exp\left(-n^2 \sum_{k=1}^N (\operatorname{Im} z_k - s_k)^2\right) \prod_{k=1}^N \cosh \pi s_k \leqslant$$
$$C_n \prod_{k=1}^N \cosh(\pi \operatorname{Im} z_k)$$

(recall that $\cosh \alpha \leqslant 2 \cosh(\alpha - \beta) \cosh \beta$), and this proves the lemma.

LEMMA 4.7. *If* $f = \mathcal{T}((f_j)) \in \mathcal{F}_q(A)$ $(1 \leqslant q \leqslant \infty)$, *then* $f_n \in \mathcal{F}_q(A)$.

PROOF. As f_n is holomorphic, it is enough to prove that the function $s \mapsto f_n(j+is) = f_{n,j}(s)$ belongs to $L^q_\varrho(\mathbf{R}^N, A_j)$ and that $f_n = \mathcal{T}((f_{n,j}))$. The first statement is trivial when $q = \infty$ and is easily proved when $q < \infty$ by means of Minkowski integral inequality, through the expression of $f_{n,j}(s)$ given before lemma 4.6. For the second one we have

$$\mathcal{T}((f_{n,j}))(a+is) = (n/\sqrt{\pi})^N \sum_j \int_{\mathbf{R}^N} \mathbf{P}_j(a, s-\sigma) \int_{\mathbf{R}^N} \exp\left(-n^2|\xi|^2\right) f_j(\sigma - \xi)\,d\xi\,d\sigma =$$
$$(n/\sqrt{\pi})^N \int_{\mathbf{R}^N} \exp\left(-n^2|\xi|^2\right) \sum_j \int_{\mathbf{R}^N} \mathbf{P}_j(a, s-\sigma-\xi) f_j(\sigma)\,d\sigma\,d\xi =$$
$$(n/\sqrt{\pi})^N \int_{\mathbf{R}^N} \exp\left(-n^2|s-\xi|^2\right) f(a+i\xi)\,d\xi = f_n(a+is)\,.$$

PROPOSITION 4.8. *Let* $1 \leqslant q < \infty$. *If* $f = \mathfrak{F}((f_j)) \in \mathcal{F}_q(A)$, *then in* $\mathcal{F}_q(A)$ $f_n \to f$ *as* $n \to \infty$.

PROOF

$$|||f_n - f|||_{(\vec{\frac{1}{2}};q)} \leqslant$$

$$\leqslant 2^{-N/q}(n/\sqrt{\pi})^N \Big(\sum_j \int_{\mathbf{R}^N} \varrho(s) \Big(\int_{\mathbf{R}^N} \exp(-n^2|t|^2) \, \|f_j(s-t) - f_j(s)\|_{A_j} dt\Big)^q ds\Big)^{1/q} \leqslant$$

(by Minkowski integral inequality)

$$\leqslant 2^{-N/q}(n/\sqrt{\pi})^N \int_{\mathbf{R}^N} \exp(-n^2|t|^2) \Big(\sum_j \int_{\mathbf{R}^N} \varrho(s) \, \|f_j(s-t) - f_j(s)\|_{A_j}^q ds\Big)^{1/q} dt\,.$$

But

$$\Big(\sum_j \int_{\mathbf{R}^N} \varrho(s) \, \|f_j(s-t) - f_j(s)\|_{A_j}^q ds\Big)^{1/q} \leqslant \Big(\sum_j \int_{\mathbf{R}^N} \varrho(s) \, \|f_j(s-t)\|_{A_j}^q ds\Big)^{1/q} +$$

$$+ \Big(\sum_j \int_{\mathbf{R}^N} \varrho(s) \, \|f_j(s)\|_{A_j}^q ds\Big)^{1/q} \leqslant (\text{by } (2.1)) \; 2^{N/q}\Big(\frac{2^{N/q}}{\varrho(t)^{1/q}} + 1\Big) |||f|||_{(\vec{\frac{1}{2}};q)}\,.$$

Therefore $\forall \delta \in \mathbf{R}^+$

$$2^{-N/q}(n/\sqrt{\pi})^N \int_{|t|\geqslant\delta} \exp(-n^2|t|^2) \Big(\sum_j \int_{\mathbf{R}^N} \varrho(s) \, \|f_j(s-t) - f_j(s)\|_{A_j}^q ds\Big)^{1/q} dt \leqslant$$

$$\leqslant (n/\sqrt{\pi})^N \, |||f|||_{(\vec{\frac{1}{2}};q)} \int_{|t|\geqslant\delta} \exp(-n^2|t|^2) \Big(\frac{2^{N/q}}{\varrho(t)^{1/q}} + 1\Big) dt \xrightarrow[n\to\infty]{} 0$$

as it can be seen by putting $\tau = nt$ and remembering that $\varrho(\tau/n) \geqslant \varrho(\tau)$. On the other hand

$$2^{-N/q}(n/\sqrt{\pi})^N \int_{|t|\leqslant\delta} \exp(-n^2|t|^2) \Big(\sum_j \int_{\mathbf{R}^N} \varrho(s) \, \|f_j(s-t) - f_j(s)\|_{A_j}^q ds\Big)^{1/q} dt \leqslant$$

$$\leqslant 2^{-N/q} \sup_{|t|\leqslant\delta} \Big(\sum_j \int_{\mathbf{R}^N} \|\varrho(s-t)^{1/q} f_j(s-t) - \varrho(s)^{1/q} f_j(s)\|_{A_j}^q ds\Big)^{1/q} +$$

$$+ 2^{-N/q} \sup_{|t|\leqslant\delta} \Big(\sum_j \int_{\mathbf{R}^N} |\varrho(s) - \varrho(s-t)| \, \|f_j(s-t)\|_{A_j}^q ds\Big)^{1/q}.$$

Here the first summand is small when δ is small because $\varrho^{1/q} f_j \in L^q(\mathbb{R}^N, A_j)$. The second one is dominated by

$$|||f|||_{(\vec{\frac{1}{2}};q)} \sup_{s\in\mathbb{R}^N, |t|\leq\delta} \left|\frac{\varrho(s)-\varrho(s-t)}{\varrho(s-t)}\right|^{1/q},$$

and this also is small when δ is small by (2.2).

We have thus proved that the space of the restrictions to S^N of the functions $f\colon \mathbb{C}^N \to \Delta(A)$, holomorphic and satisfying the growth condition of lemma 4.6 and belonging to $\mathcal{F}_q(A)$ is dense in $\mathcal{F}_q(A)$. Therefore the following result completes the proof of th. 4.1.

LEMMA 4.9. *Let $f\colon \mathbb{C}^N \to \Delta(A)$ be a holomorphic function such that*

$$\sup_{z\in\bar{S}^N} \left(\prod_{k=1}^N \cosh(\pi \operatorname{Im} z_k)\right)^{-1} \|f(z)\|_{\Delta(A)} < +\infty,$$

and that $f|_{S^N} \in \mathcal{F}_q(A)$, with $1 \leq q < \infty$. $\forall\delta\in\mathbb{R}^+$ we set $g_\delta\colon \mathbb{C}^N \to \Delta(A)$, $g_\delta(z) = \exp\left(\delta \sum_{k=1}^N z_k^2\right) f(z)$. Then $g_\delta|_{S^N} \in \mathcal{F}_0(A)$ and $g_\delta \xrightarrow[\delta\to 0+]{} f$ in $\mathcal{F}_q(A)$.

PROOF. It is obvious that $g_\delta \in \mathcal{F}_0(A)$. In order to prove that $|||g_\delta - f|||_{(\vec{\frac{1}{2}};q)} \xrightarrow[\delta\to 0+]{} 0$ it is enough to apply the dominated convergence theorem.

5. Some properties of the spaces $A_{(a;q)}$.

We prove some consequences of th. 4.1.

THEOREM 5.1. *$\forall a \in I^N$ $\forall q \in [1, \infty[$, $\Delta(A)$ is dense in $A_{(a;q)}$.*

PROOF. Trivial consequence of th. 4.1.

THEOREM 5.2. *$\forall j \in J^N$ let A_j^0 be the closure of $\Delta(A)$ in A_j, with the norm of A_j, so that $A^0 = (A_j^0)_{j\in J^N}$ is a compatible family. Then $\forall q \in [1, \infty[$ $\mathcal{F}_q(A) = \mathcal{F}_q(A^0)$ and $A_{(a;q)} = A^0_{(a;q)}$, in both cases with equal norms.*

PROOF. Since $\Delta(A) = \Delta(A^0)$ (with equal norms), we have that $\mathcal{F}_0(A) = \mathcal{F}_0(A^0)$ and that the restriction to $\mathcal{F}_0(A)$ of the norm $|||\cdot|||_{(a;q)}$ of $\mathcal{F}_q(A)$ coincides with the restriction of the homonymous norm of $\mathcal{F}_q(A^0)$. Then our statements follow easily from th. 4.1.

THEOREM 5.3. *Suppose that* $\forall k = (k_2, \dots, k_N) \in J^{N-1}$ $A_{0,k} \hookrightarrow A_{1,k}$. *Then* $\forall b = (b_2, \dots, b_N) \in I^{N-1}$ *and for* $0 < a' < a'' < 1$, $1 \leqslant q < \infty$,

$$A_{(a',b;q)} \hookrightarrow A_{(a'',b;q)}.$$

PROOF. $\forall f \in \mathcal{F}_0(A)$ we set

$$g_f\colon \mathbf{C}^N \to \Delta(A),\ g_f(z) = f\left(\frac{a'}{a''} z_1, z_2, \dots, z_N\right).$$

Then it is obvious that $g_f \in \mathcal{F}_0(A)$. We prove that $\exists C > 0$ such that $\forall f \in \mathcal{F}_0(A)$

$|||g_f|||_{(a'';b;q)} \leqslant C\ |||f|||_{(a',b;q)}$. In fact $|||g_f|||_{(a'',b;q)} \leqslant$

$$\leqslant \Bigg(\sum_{k\in J^{N-1}} \int_{\mathbf{R}^N} P_0\left(a'', \frac{a''}{a'} s_1\right) \prod_{h=2}^{N} P_{k_h}(b_h, s_h)\ \|f(is_1, k_2 + is_2, \dots, k_N + is_N)\|^q_{A_{0,k}} \cdot$$

$$\cdot \frac{a''}{a'} ds + \sum_{k\in J^{N-1}} \int_{\mathbf{R}^N} P_1(a'', s_1) \prod_{h=2}^{N} P_{k_h}(b_h, s_h) \cdot$$

$$\left\| \sum_{l=0}^{1} \int_{\mathbf{R}} P_l\left(\frac{a'}{a''}, \frac{a'}{a''} s_1 - \theta\right) f(l + i\theta, k_2 + is_2, \dots, k_N + is_N)\, d\theta \right\|^q_{A_{1,k}} ds \Bigg)^{1/q} \leqslant$$

$$\leqslant C_1 \Bigg(\sum_{k\in J^{N-1}} \int_{\mathbf{R}^N} P_0\left(a'', \frac{a''}{a'} s_1\right) \prod_{h=2}^{N} P_{k_h}(b_h, s_h) \left\| f(is_1, k_2 + is_2, \dots) \right\|^q_{A_{0,k}} ds +$$

$$+ \sum_{k\in J^{N-1}} \int_{\mathbf{R}^N} P_1(a'', s_1) \prod_{h=2}^{N} P_{k_h}(b_h, s_h) \cdot$$

$$\cdot \left(\sum_{l=0}^{1} \int_{\mathbf{R}^N} P_l\left(\frac{a'}{a''}, \frac{a'}{a''} s_1 - \theta\right) \|f(l + i\theta, k_2 + is_2, \dots)\|_{A_{l,k}} d\theta \right)^q ds \Bigg)^{1/q}.$$

But

$$P_0\left(a'', \frac{a''}{a'} s_1\right) \leqslant C' P_0(a', s_1) \left(\text{because } \frac{a''}{a'} > 1\right) \text{ and } P_l\left(\frac{a'}{a''}, \frac{a'}{a''} s_1 - \theta\right) \leqslant$$

$$C'' \cosh\left(\pi \frac{a'}{a''} s_1\right) P_l(a', \theta)$$

by (2.7) and (2.8) (the constants depending only on a', a''). Therefore

$$|||g_f|||_{(a'',b;q)} \leqslant C_2 \Bigg(\sum_{k \in J^{N-1}} \int_{\mathbf{R}^N} \mathbf{P}_{0,k}((a',b),s) \, \|f(is_1, k_2 + is_2, \ldots)\|^q_{A_{0,k}} ds +$$

$$+ \sum_{k \in J^{N-1}} \int_{\mathbf{R}^{N-1}} \prod_{h=2}^{N} P_{k_h}(b_h, s_h) \int_{\mathbf{R}} P_1(a'', s_1) \cosh\left(\pi \frac{a'}{a''} s_1\right) ds_1 \cdot$$

$$\cdot \left(\sum_{l=0}^{1} \int_{\mathbf{R}} P_l(a', \theta) \, \|f(l + i\theta, k_2 + is_2, \ldots)\|_{A_{l,k}} d\theta \right)^q ds_2 \ldots ds_N \Bigg)^{1/q}.$$

Here

$$\int_{\mathbf{R}} P_1(a'', s_1) \cosh\left(\pi \frac{a'}{a''} s_1\right) ds_1 < \infty \text{ because } \frac{a'}{a''} < 1$$

and by Hölder inequality

$$\left(\sum_{l=0}^{1} \int_{\mathbf{R}} P_l(a', \theta) \, \|f(l + i\theta, k_2 + is_2, \ldots)\|_{A_{l,k}} d\theta \right)^q \leqslant$$

$$\leqslant \sum_{l=0}^{1} \int_{\mathbf{R}} P_l(a', \theta) \, \|f(l + i\theta, k_2 + is_2, \ldots)\|^q_{A_{l,k}} d\theta .$$

This allows us to get the inequality.

Let $x \in A_{(a',b;q)}$ and $f \in \mathcal{F}_q(A)$ such that $f(a', b) = x$. Let $(f_n)_{n \in \mathbf{N}}$ be a sequence in $\mathcal{F}_0(A)$ converging to f in the norm $|||\cdot|||_{(a',b;q)}$: then

$$\|f_n(a', b) - x\|_{(a',b;q)} \xrightarrow[n \to \infty]{} 0 .$$

The above inequality shows that $(f_n(a', b))_{n \in \mathbf{N}}$ is a Cauchy sequence in $A_{(a'',b;q)}$, because $g_{f_n}(a'', b) = f_n(a', b)$, and as $A_{(a',b;q)}$ and $A_{(a'',b;q)}$ are both continuously embedded in $\Sigma(A)$, it follows that $x \in A_{(a'',b;q)}$ and $\|f_n(a', b) - x\|_{(a'',b;q)} \xrightarrow[n \to \infty]{} 0$. Therefore $\|x\|_{(a'',b;q)} \leqslant C |||f|||_{(a',b;q)}$, and as f is arbitrary $\|x\|_{(a'',b;q)} \leqslant C \|x\|_{(a',b;q)}$.

REMARK 5.4. When $N=1$, our interpolation spaces $A_{(a;q)}$ ($a \in]0, 1[$, $1 \leqslant q \leqslant \infty$) coincide (with equal norms) with the space $[A_0, A_1]_a$ defined by A. P. Calderón in [3]. In fact it is obvious that Calderón's space $\mathcal{F}(A_0, A_1)$ is contained in our $\mathcal{F}_q(A)$ and that $\forall f \in \mathcal{F}(A_0, A_1)$ $|||f|||_{(a;q)} \leqslant$

$\leqslant \max_{j=0,1} \sup_{s\in\mathbf{R}} \|f(j+is)\|_{A_j} = \|f\|_{\mathcal{F}(A_0,A_1)}$, and this implies that $[A_0, A_1]_a \overset{1}{\hookrightarrow}$ $\overset{1}{\hookrightarrow} A_{(a;q)}$. Conversely, let $f\in\mathcal{F}_0(A)$. Then $f\in\mathcal{F}(A_0,A_1)$ and the inequality $\|f(a)\|_{[A_0,A_1]_a} \leqslant |||f|||_{(a;1)}$ holds (see [3], p. 117). By an argument similar to that of the proof of th. 5.3, we obtain the embedding $A_{(a;1)} \overset{1}{\hookrightarrow} [A_0, A_1]_a$, which proves our statement by remark 3.8.

When $N\geqslant 2$ the situation is no more so clear, because of the lack of the inequality (9.4 iii) of [3] p. 117, as we see in the subsequent counterexample. As a matter of fact the asserted proof of that inequality, given in [2] pp. 199-200 is wrong, since the argument is based on the false assertion that, given 2^N bounded infinitely differentiable real-valued functions g_j $(j\in J^N)$, there exists a function $f\colon \bar{S}^N \to \mathbf{C}$, continuous on $\bar{S}^N$ and holomorphic on S^N, such that $\forall s\in\mathbf{R}^N$ $\forall j\in J^N$ $\operatorname{Re} f(j+is) = g_j(s)$.

COUNTEREXAMPLE 5.5. We exhibit a compatible quadruple $A=(A_{jk})_{j,k=0,1}$ and a sequence of functions $f_n\colon \bar{S}^2\to\Sigma(A)$, continuous and bounded on $\bar{S}^2$, holomorphic on S^2, such that $f_n(j+is, k+it)\in A_{jk}$ and is bounded and continuous in the norm of A_{jk}, for which the norm of $f_n(\frac{1}{2},\frac{1}{2})$ in the interpolation space defined in [13] and dealed with in [2] is equal to 1, while $|||f_n|||_{(\frac{1}{2},\frac{1}{2};1)} \xrightarrow[n\to\infty]{} 0$. This proves that the inequality 4.4(2) of [2] (corresponding to (9.4 iii) of [3]) fails, and also that the interpolation space of [13] relative to the point $(\frac{1}{2},\frac{1}{2})$ is different from $A_{(\frac{1}{2},\frac{1}{2};1)}$ (actually it is easily checked that it is smaller).

Let l^∞ have the usual meaning and let l be the Banach space of complex-valued sequences $(x_n)_{n\in\mathbf{N}}$ such that $\|x\|_l = \sup_n |x_n|/n < +\infty$, so that $l^\infty \overset{1}{\hookrightarrow} l$. We set $A_{00}=A_{11}=l^\infty$, $A_{01}=A_{10}=l$. Then $\Delta(A)=l^\infty$, $\Sigma(A)=l$ (with equal norms). If $[A;\frac{1}{2},\frac{1}{2}]$ is the interpolation space defined in [13], we have $l^\infty \overset{1}{\hookrightarrow} [A;\frac{1}{2},\frac{1}{2}]$. On the other hand, let f: $\bar{S}^2\to\Sigma(A)=l$ be a bounded continuous function, holomorphic on S^2 and such that $f(j+is, k+it)\in A_{jk}$ and is bounded and continuous (with respect to (s,t)) in A_{jk}. Put $g(z)=f(z,z)$ $\forall z\in\bar{S}$. As g is bounded in the norm of l, we can represent it as the Poisson integral of its boundary values, and these values are bounded and continuous in l^∞, so that $g(z)\in l^\infty$ $\forall z\in\bar{S}$ (see [3] p. 116). Therefore $f(\frac{1}{2},\frac{1}{2})\in l^\infty$, and this proves that $l^\infty=[A;\frac{1}{2},\frac{1}{2}]$. Moreover, from

$$\|f(\tfrac{1}{2},\tfrac{1}{2})\|_{l^\infty} = \|g(\tfrac{1}{2})\|_{l^\infty} \leqslant \max_{j=0,1}\sup_{t\in\mathbf{R}} \|g(j+it)\|_{l^\infty} \leqslant$$

$$\leqslant \max_{j,k=0,1}\sup_{(s,t)\in\mathbf{R}^2} \|f(j+is, k+it)\|_{A_{jk}},$$

it follows that $\forall x \in l^\infty \|x\|_{l^\infty} \leqslant \|x\|_{[A;\frac{1}{2},\frac{1}{2}]}$. Therefore we have also equality of norms.

Let $a^{(n)} = (\delta_{kn})_{k\in N}$ (Kronecker's symbol) and $\forall \varepsilon > 0$ $f_{n,\varepsilon}: \bar{S}^2 \to l^\infty$, $f_{n,\varepsilon}(w, z) = \exp\big(\varepsilon(w - z)^2\big)\, a^{(n)}$. Obviously

$$\|f_{n,\varepsilon}(\tfrac{1}{2}, \tfrac{1}{2})\|_{[A;\frac{1}{2},\frac{1}{2}]} = 1 \text{ and } f_{n,\varepsilon} \in \mathcal{F}_1(A) \text{ with } |||f_{n,\varepsilon}|||_{(\frac{1}{2},\frac{1}{2};1} =$$

$$= \sum_{j,k=0,1} \int_{\mathbf{R}^2} (4 \cosh \pi s \cosh \pi t)^{-1} \exp\Big(\varepsilon\big((j-k)^2 - (s-t)^2\big)\Big)\, ds\, dt \|a^{(n)}\|_{A_{jk}} =$$

$$= \tfrac{1}{2}\Big(1 + \frac{e^\varepsilon}{n}\Big) \int_{\mathbf{R}^2} (\cosh \pi s \cosh \pi t)^{-1} \exp\big(-\varepsilon(s-t)^2\big)\, ds\, dt\,.$$

If we fix $\eta > 0$, we can find $\varepsilon > 0$ such that the last integral is $\leqslant \eta$, and if we take $n > e^\varepsilon$ we obtain $|||f_{n,\varepsilon}|||_{(\frac{1}{2},\frac{1}{2};1)} \leqslant \eta$.

THEOREM 5.6. *Let $A = (A_j)_{j\in J^N}$ be a compatible family of Banach spaces and $N \geqslant 2$. Put $B = (A_{k,0})_{k\in J^{N-1}}$, $C = (A_{k,1})_{k\in J^{N-1}}$. $\forall a \in I^{N-1}$, $b \in I$, $q \in [1, \infty[$, $A_{(a,b;q)} \overset{1}{\hookrightarrow} [B_{(a;q)}, C_{(a;q)}]_b$.*

PROOF. We shall prove that $\forall f \in \mathcal{F}_0(A)$ $\|f(a, b)\|_{[B_{(a;q)}, C_{(a;q)}]_b} \leqslant |||f|||_{(a,b;q)}$ and this will prove the theorem by an argument similar to that of the proof of th. 5.3. In fact $\|f(a, b)\|_{[B_{(a;q)}, C_{(a;q)}]_b} \leqslant$ (by (9.4 iii) of [3] and Hölder inequality)

$$\Big(\int_{\mathbf{R}} P_0(b, t)\, \|f(a, it)\|^q_{B_{(a;q)}} dt + \int_{\mathbf{R}} P_1(b, t)\, \|f(a, 1 + it)\|^q_{C_{(a;q)}} dt\Big)^{1/q} \leqslant$$

$$\leqslant \Big(\sum_{j=0}^{1} \int_{\mathbf{R}} P_j(b, t) \sum_{k\in J^{N-1}} \int_{\mathbf{R}^{N-1}} \prod_{h=1}^{N-1} P_{k_h}(a_h, s_h)\, \|f(k + is, j + it)\|^q_{A_{k,j}} ds\, dt\Big)^{1/q} =$$

$$= |||f|||_{(a,b;q)}\,.$$

THEOREM 5.7. *Let $N \geqslant 2$ and $A = (A_j)_{j\in J^N}$ be a compatible family of Banach spaces such that $\forall k \in J^{N-1}$ $A_{k,0} = A_{k,1} = B_k$. Then $\forall (a, b) \in I^{N-1} \times I$ $\forall q \in [1, \infty[$ $A_{(a,b;q)} = B_{(a;q)}$.*

PROOF. By theorem 5.6 $A_{(a,b;q)} \overset{1}{\hookrightarrow} [B_{(a;q)}, B_{(a;q)}]_b = B_{(a;q)}$. Let $f \in \mathcal{F}_q(B)$ and put $g: S^N \to \Sigma(A) = \Sigma(B)$, $g(z_1, \dots, z_N) = f(z_1, \dots, z_{N-1})$. It is an easy computation to check that $g \in \mathcal{F}_q(A)$ and that $|||g|||_{(a,b;q)} = |||f|||_{(a;q)}$. This proves the converse embedding.

We conclude this section by showing that, in general, the inclusion of th. 5.6 is not an equality. For this we need two preliminary lemmas.

LEMMA 5.8. *Let A_0, A_1, B_0, B_1 be Banach spaces such that (A_0, A_1) is a compatible family, $A_0 \hookrightarrow B_0 \hookrightarrow A_0 + A_1$, and suppose that $\exists \theta_0 \in]0, 1[$ such that $[A_0, A_1]_{\theta_0} \hookrightarrow B_0$. Let $T \in \mathfrak{L}(B_0, B_1)$ and endow* $\ker T|_{A_0}$ *with the norm of A_0. Then $\forall \theta \in]0, \theta_0]$ $[\ker T|_{A_0}, A_1]_\theta \subseteq \ker T$.*

PROOF. Let $f \in \mathcal{F}(\ker T|_{A_0}, A_1) \subseteq \mathcal{F}(A_0, A_1)$ (Calderón's function spaces, see [3]). Then $f(it) \in B_0$, $f(\theta_0 + it) \in B_0$ $\forall t \in \mathbf{R}$. For $0 < \theta < \theta_0$ $f(\theta + it)$ is the Poisson integral (in $A_0 + A_1$) of its values for $\operatorname{Re} z = 0$ and $\operatorname{Re} z = \theta_0$. But this is also a Poisson integral of B_0-valued, B_0-bounded and continuous functions, so that $f(\theta + it) \in B_0$. Moreover f is a B_0-continuous function which, for $0 < \theta < \theta_0$ coincides with its Cauchy integral in the norm of $A_0 + A_1$ and hence in the norm of B_0. This proves that Tf is holomorphic for $0 < \operatorname{Re} z < \theta_0$ and continuous for $0 \leqslant \operatorname{Re} z \leqslant \theta_0$ in the norm of B_1. But $T(f(it)) = 0$ $\forall t \in \mathbf{R}$, so that $Tf(z) = 0$ for each z such that $0 \leqslant \operatorname{Re} z \leqslant \theta_0$. Since f is arbitrary, this concludes the proof.

We recall that if v_0 and v_1 are positive weight sequences, then $[l^1_{v_0}, l^1_{v_1}]_\theta = l^1_{v_\theta} (0 < \theta < 1)$, where $v_\theta(n) = v_0(n)^{1-\theta} v_1(n)^\theta$ (see [1] th. 5.5.3).

LEMMA 5.9. *Let $v_j : \mathbf{Z} \to \mathbf{R}^+$ $(j = 0, 1)$ be non-decreasing sequences. We suppose that* $\inf v_0 > 0 = \inf v_1$. *Let $A_0 = \left\{\lambda \in l^1_{v_0};\ \sum_{n \in \mathbf{Z}} \lambda_n = 0\right\}$, $A_1 = l^1_{v_1}$. Then $\forall \theta \in]0, 1[$ $[A_0, A_1]_\theta = l^1_{v_\theta}$ with equivalent norms.*

PROOF. First of all from $A_0 \overset{1}{\hookrightarrow} l^1_{v_0}$, $A_1 = l^1_{v_1}$ it follows by interpolation $[A_0, A_1]_\theta \overset{1}{\hookrightarrow} [l^1_{v_0}, l^1_{v_1}]_\theta = l^1_{v_\theta}$.

$\forall n \in \mathbf{Z}$ set $a^{(n)} = (\delta_{kn})_{k \in \mathbf{Z}}$. Then $a^{(n)} - a^{(m)} \in A_0 \cap A_1$. We fix n, $m \in \mathbf{Z}$, $n > m$, and we put

$$f: \bar{S} \to A_0 + A_1\,, \qquad f(z) = v_0(n)^{z-\theta} v_1(n)^{\theta - z}(a^{(n)} - a^{(m)})\,.$$

It is obvious that the function $z \mapsto \exp\left(\delta(z-\theta)^2\right) f(z)$ belongs to $\mathcal{F}(A_0, A_1)$ $\forall \delta \in \mathbf{R}^+$, so that

$$a^{(n)} - a^{(m)} = f(\theta) \in [A_0, A_1]_\theta$$

and

$$\|a^{(n)} - a^{(m)}\|_\theta \leqslant \max_{j=0,1} \sup_{t\in\mathbf{R}} \|f(j+it)\|_{A_j} \text{ (see [16] Ch. IV §1 p. 217)} =$$

$$= \max_{j=0,1} \sup_{t\in\mathbf{R}} |v_0(n)^{j+it-\theta} v_1(n)^{\theta-j-it}| \, \|a^{(n)} - a^{(m)}\|_{A_j} =$$

$$= \max_{j=0,1} v_0(n)^{j-\theta} v_1(n)^{\theta-j} \big(v_j(n) + v_j(m)\big) \leqslant$$

$$\leqslant 2 \max_{j=0,1} v_0(n)^{j-\theta} v_1(n)^{\theta-j} v_j(n) = 2 v_\theta(n) .$$

Thus in general $\|a^{(n)} - a^{(m)}\|_\theta \leqslant 2v_\theta(\max\{n, m\})$. Therefore, as

$$m, p \to -\infty \quad \|(a^{(n)} - a^{(m)}) - (a^{(n)} - a^{(p)})\|_\theta \leqslant 2v_\theta(\max\{m, p\}) \to 0 ,$$

since $\lim\limits_{m\to-\infty} v_\theta(m) = 0$ as it follows easily from the assumptions. This proves that $\forall n \in \mathbb{Z}$ $(a^{(n)} - a^{(m)})_{m\in\mathbf{Z}}$ converges in $[A_0, A_1]_\theta$ as $m \to -\infty$. But $\|a^{(m)}\|_{A_1} = v_1(m) \xrightarrow[m\to-\infty]{} 0$, and as both A_1 and $[A_0, A_1]_\theta$ are continuously embedded in $A_0 + A_1$, we have that $a^{(n)} \in [A_0, A_1]_\theta$ and $\|a^{(n)}\|_\theta = \lim\limits_{m\to-\infty} \|a^{(n)} - a^{(m)}\|_\theta \leqslant 2v_\theta(n)$. Hence it follows easily that each « finite » sequence λ belongs to $[A_0, A_1]_\theta$ with $\|\lambda\|_\theta \leqslant 2\|\lambda\|_{l^1_{v_\theta}}$, and so we get at once that $l^1_{v_\theta} \overset{2}{\hookrightarrow} [A_0, A_1]_\theta$.

Counterexample 5.10. We show an example of a compatible quadruple $(A_{jk})_{j,k=0,1} = A$ where $A_{(\frac12,\frac12;1)}$ is strictly smaller than $[[A_{00}, A_{10}]_{\frac12}, [A_{01}, A_{11}]_{\frac12}]_{\frac12}$. We set $A_{00} = A_{11} = \left\{\lambda \in l^1(\mathbf{Z});\ \sum_{n\in\mathbf{Z}} \lambda_n = 0\right\}$ with the l^1-norm,

$$A_{10} = \left\{\lambda\colon \mathbf{Z} \to \mathbf{C};\ \sum_{n\in\mathbf{Z}} 2^n |\lambda_n| < +\infty\right\},$$

$$A_{01} = \left\{\lambda\colon \mathbf{Z} \to \mathbf{C};\ \sum_{n\in\mathbf{Z}} 2^{-n} |\lambda_n| < +\infty\right\}$$

with the natural norms. By applying lemma 5.9 we see that $[A_{00}, A_{10}]_{\frac12}$ is equal to l^1 with weight $2^{n/2}$ and that $[A_{01}, A_{11}]_{\frac12}$ is equal to l^1 with weight $2^{-n/2}$, so that the iterated interpolation space is equal to l^1 (without weight). But $\Delta(A)$ is contained in the proper closed subspace of l^1 defined by the condition $\sum_{n\in\mathbf{Z}} \lambda_n = 0$, and by th. 5.1 $A_{(\frac12,\frac12;1)} \neq l^1$.

COUNTEREXAMPLE 5.11. We show an example of a compatible quadruple $A = (A_{jk})_{j,k=0,1}$ where for some $(\vartheta, \varrho) \in I^2$

$$[[A_{00}, A_{10}]_\theta, [A_{01}, A_{11}]_\theta]_\varrho \neq [[A_{00}, A_{01}]_\varrho, [A_{10}, A_{11}]_\varrho]_\theta .$$

We define $v_{jk}: \mathbf{Z} \to \mathbf{C}$ in the following way: $v_{00}(n) = \max\{1, 2^n\}$, $v_{10}(n) = \max\{1, 2^{-n}\}$, $v_{01}(n) = \min\{1, 2^n\}$, $v_{11}(n) = \min\{1, 2^{-n}\}$. We put $A_{j1} = l^1_{v_{j1}}$ and $A_{j0} = \left\{\lambda \in l^1_{v_{j0}};\ \sum_{n \in \mathbf{Z}} \lambda_n = 0\right\}$ (in every case with the natural norm). Then, by lemma 5.9, $[A_{00}, A_{01}]_\varrho = l^1_{v_{0\varrho}}$ $(v_{0\varrho} = v_{00}^{1-\varrho} v_{01}^\varrho)$ and $[A_{10}, A_{11}]_\varrho = l^1_{v_{1\varrho}}$, so that

$$[[A_{00}, A_{01}]_\varrho, [A_{10}, A_{11}]_\varrho]_\theta = l^1_{v_{\theta\varrho}} \qquad (v_{\theta\varrho} = v_{0\varrho}^{1-\theta} v_{1\varrho}^\theta) .$$

By th. 1.17.1/1 of [21] $[A_{00}, A_{10}]_\theta = \left\{\lambda;\ \lambda \in l^1_{v_{\theta 0}},\ \sum_{n \in \mathbf{Z}} \lambda_n = 0\right\}$, and we know that $[A_{01}, A_{11}]_\theta = l^1_{v_{\theta 1}}$. An application of lemma 5.8 shows that if $\varrho \leqslant \theta$ and $\theta + \varrho \leqslant 1$ then $\lambda \in [[A_{00}, A_{10}]_\theta, [A_{01}, A_{11}]_\theta]_\varrho \Rightarrow \sum_{n \in \mathbf{Z}} \lambda_n = 0$.

6. Duals

In this section we study the duals of our interpolation spaces.

When we are given a compatible family $A = (A_j)_{j \in J^N}$ of Banach spaces, the assumption that $\Delta(A)$ is dense in each A_j (which we suppose satisfied throughout this section) ensures that the dual space A_j^* of A_j can be identified, in the usual way, with a subspace A_j' of $\Delta(A)^*$. Thus $A' = (A_j')_{j \in J^N}$ is another compatible family and we can put the question whether the space $(A_{(a;q)})'$, identified with the dual of $A_{(a;q)}$, is an intermediate space for the family A'. When $N=1$, from the fact that $\Delta(A)^* = \Sigma(A')$ and $\Sigma(A)' = \Delta(A')$ (see [1] th. 2.7.1) it follows at once that $(A_{(a;q)})'$ is intermediate for A', and moreover in that case Calderón [3] showed that it is an interpolation space. But as soon as $N \geqslant 2$ the equality $\Sigma(A)' = \Delta(A')$ fails (see [8] § 4) and in fact the following counterexample shows that $(A_{(a;q)})'$ may also fail to be an intermediate space for the family A'.

COUNTEREXAMPLE 6.1. We set $A_{00} = A_{10} = \left\{\lambda \in l^1(\mathbf{Z});\ \sum_{n \in \mathbf{Z}} \lambda_n = 0\right\}$, $A_{01} = l^1_{v_0}(\mathbf{Z})$, $A_{11} = l^1_{v_1}(\mathbf{Z})$, where $v_0(n) = \min\{1, 2^n\}$, $v_1(n) = \min\{1, 2^{-n}\}$. Then $\Delta(A) = A_{00} = A_{10}$ is dense in each A_{jk} (see [5] appendix 1).

We set $\Phi\colon \Delta(A) \to \mathbf{C}$, $\Phi(\lambda) = \sum_{n=0}^{+\infty} \lambda_n = -\sum_{n=-\infty}^{-1} \lambda_n$. Then Φ can be continuously extended to each A_{jk}, and so $\Phi \in \Delta(A')$. Now we fix $(\theta, \varrho) \in I^2$ such that $\theta + \varrho > 1$, $\theta < \varrho$ and we show that $\Phi \notin (A_{(\theta,\varrho;q)})'$. To do this, we put $a^{(n)} = (\delta_{k,n})_{k\in\mathbf{Z}} (n \in \mathbf{Z})$, and we show that in $A_{(\theta,\varrho;q)}$ $a^{(n)} - a^{(-n)} \xrightarrow[n\to+\infty]{} 0$: since $\Phi(a^{(n)} - a^{(-n)}) = 1$ this will prove our assertion.

For $n, m > 0$ we set $b^{(m,n)} = (a^{(n)} - a^{(m)}) - (a^{(-n)} - a^{(-m)})$. We fix n, m with $0 < n < m$ and we set

$$f\colon \mathbf{C}^2 \to \Delta(A), \qquad f(w, z) = v_1(n)^{\theta+\varrho-w-z}(a^{(n)} - a^{(m)}) .$$

Then $f \in \mathcal{F}_q(A)$ and $f(\theta, \varrho) = a^{(n)} - a^{(m)}$. Since $\|a^{(n)} - a^{(m)}\|_{A_{jk}} \leqslant 2$ for $(j, k) \neq (1, 1)$ and $\|a^{(n)} - a^{(m)}\|_{A_{11}} \leqslant 2^{1-n}$, we have that $|||f|||_{(\theta,\varrho;q)} \leqslant |||f|||_{(\theta,\varrho;\infty)} \leqslant 2^{1+n(1-\theta-\varrho)}$: therefore $\|a^{(n)} - a^{(m)}\|_{(\theta,\varrho;q)} \leqslant 2^{1+n(1-\theta-\varrho)}$. By employing the function $g(w, z) = v_0(-n)^{w-z-\theta+\varrho}(a^{(-n)} - a^{(-m)})$, we can prove that $\|a^{(-n)} - a^{(-m)}\|_{(\theta,\varrho;q)} \leqslant 2^{1+n(\theta-\varrho)}$. Thus in general

$$\|b^{(m,n)}\|_{(\theta,\varrho;q)} \leqslant 2^{1+\min\{n,m\}(1-\theta-\varrho)} + 2^{1+\min\{n,m\}(\theta-\varrho)} .$$

But $\|b^{(m,n)} - b^{(p,n)}\|_{(\theta,\varrho;q)} = \|b^{(m,p)}\|_{(\theta,\varrho;q)} \to 0$ as $\min\{m, p\} \to +\infty$. Therefore as $m \to +\infty$ $b^{(m,n)}$ converges in $A_{(\theta,\varrho;q)}$. Since in $\Sigma(A) = l^1_{\min\{v_0,v_1\}}$ $b^{(m,n)}$ converges to $a^{(n)} - a^{(-n)}$, we have proved that $b^{(m,n)}$ converges to $a^{(n)} - a^{(-n)}$ in $A_{(\theta,\varrho;q)}$, so that $\|a^{(n)} - a^{(-n)}\|_{(\theta,\varrho;q)} \leqslant 2^{1+n(1-\theta-\varrho)} + 2^{1+n(\theta-\varrho)}$.

In the sequel we have to consider dual spaces of spaces like $L^q_\sigma(\mathbf{R}^N, X)$, where X is a Banach space, $\sigma\colon \mathbf{R}^N \to \mathbf{R}^+$ is a measurable weight function and $1 \leqslant q < \infty$. We shall assume that for $1 < q < \infty$ $(L^q_\sigma(\mathbf{R}^N, X))^*$ is isometrically isomorphic to $L^{q'}_\sigma(\mathbf{R}^N, X^*)((1/q) + (1/q') = 1)$ and that $(L^1_\sigma(\mathbf{R}^N, X))^*$ is isometrically isomorphic to $L^\infty(\mathbf{R}^N, X^*)$, in both cases with respect to the duality $\langle f, g\rangle = \int_{\mathbf{R}^N} \sigma(s)\langle f(s), g(s)\rangle\, ds$

These assumptions are fulfilled when X^* has the Radon-Nikodym property, and in particular when X^* is reflexive or separable (see [7] cor. 5 p. 117 and § 6 p. 118).

We shall also employ functions defined on the polydisk D^N $(D = \{z \in \mathbf{C};\ |z| < 1\})$ (or on $(\partial D)^N$) which we obtain from functions defined on S^N (or on $(\partial S)^N$) by means of changes of variables given by direct products of conformal mappings of $\overline{S}$ into $\overline{D}$. More precisely, let $w \in S$ and $\mu_w\colon \overline{S} \to \overline{D}$ be the map defined by

$$\mu_w(z) = \frac{\exp(i\pi z) - \exp(i\pi w)}{\exp(i\pi z) - \exp(-i\pi \overline{w})} .$$

Then μ_w is a homeomorphism of $\bar{S}$ onto $\bar{D}\setminus\{1, \exp(2\pi i \operatorname{Re} w)\}$ such that $\mu_w(w) = 0$ and its restriction to S is a biholomorphic diffeomorphism of S onto D.

When $w = (w_1, \dots, w_N) \in S^N$, we define $\vec{\mu}_w : \bar{S}^N \to \bar{D}^N$ by $\vec{\mu}_w(z) = (\mu_{w_1}(z_1), \dots, \mu_{w_N}(z_N))$. Since $\mu_{w_h}(\partial S) = \partial D \setminus \{1, \exp(2\pi i \operatorname{Re} w_h)\}$, we obtain that the N-dimensional measure of $(\partial D)^N \setminus \vec{\mu}_w((\partial S)^N)$ is zero.

The functions $\vec{\mu}_w (w \in S^N)$ act as changes of variables, giving rise to one-to-one correspondances between functions defined on S^N and functions defined on D^N, and between (classes of) functions defined a.e. on $(\partial S)^N$ and (classes of) functions defined a.e. on $(\partial D)^N$. Moreover these changes of variables « commute » with the Poisson integrals (on $(\partial D)^N$ and on $(\partial S)^N$), in the following sense.

Let $1 \leqslant q \leqslant \infty$, $\psi \in L^q((\partial D)^N, X)$ and $\varphi = \psi \circ \vec{\mu}_w$. Let g be the Poisson integral of ψ on D^N (here the Poisson kernel for D^N is the tensor product of Poisson kernels for D) and $f = g \circ \vec{\mu}_w$. Then $\varphi \in L^q_\varrho((\partial S)^N, X)$ and f is the Poisson integral of φ

$$\left(\text{i.e. } f(\xi + i\eta) = \sum_{j \in J^N} \int_{\mathbf{R}^N} \mathbf{P}_j(\xi, \eta - s)\, \varphi(j + is)\, ds\right).$$

The computations are straightforward and we omit them. As a particular case (with $\xi + i\eta = w$ and $\|\varphi(j + is)\|^q$ instead of $\varphi(j \leqslant is)$) we get

$$\sum_{j \in J^N} \int_{\mathbf{R}^N} \mathbf{P}_j(\operatorname{Re} w, \operatorname{Im} w - s)\ \|\varphi(j + is)\|^q\, ds =$$
$$= (2\pi)^{-N} \int_{[0, 2\pi]^N} \|\psi(\exp(it_1), \dots, \exp(it_N))\|^q\, dt .$$

Hence, if $\varphi \in L^q_\varrho((\partial S)^N, X)$, then $\varphi \circ \vec{\mu}_w^{-1} \in L^q((\partial D)^N, X)$, so that the correspondance is onto.

It is well-known ([20] th. 2.1.4) that a necessary and sufficient condition that g be holomorphic on D^N is that the Fourier coefficients c_α $(\alpha \in \mathbf{Z}^N)$ of ψ are zero whenever $\exists j \in \{1, \dots, N\}$ such that $\alpha_j < 0$. Thus for a function $\varphi \in L^q_\varrho((\partial S)^N, X)$ we are interested in looking for the vanishing Fourier coefficients of its corresponding function ψ on $(\partial D)^N$. In connection with this, remark that if $\vec{\mu}, \vec{\nu} : S^N \to D^N$ are direct products of conformal mappings such that $\vec{\mu}(w) = \vec{\nu}(w) = 0$ (where $w \in S^N$), then, by well-known properties of the conformal mappings of the disk, there are $\lambda_1, \dots, \lambda_N \in \mathbf{C}$ with $|\lambda_k| = 1$ such that $\nu_k = \lambda_k \mu_k$

($k = 1, \dots, N$); therefore after we have chosen the point $w \in S^N$ which we map into 0, the fact that a fixed Fourier coefficient vanishes does not depend on the product of conformal mappings we employ.

THEOREM 6.2. *Let* $1 \leqslant q < \infty$, q' *be its conjugate exponent,* X *be a Banach space and* $\forall j \in J^N$ *let* $f_j \in L_\varrho^{q'}(\mathbf{R}^N, X)$. *We put* $f(j + is) = f_j(s)$ $\forall s \in \mathbf{R}^N$, $j \in J^N$, *and* $g = f \circ \vec{\mu}_w^{-1}$ (*where* $w \in S^N$). *Then the following statements are equivalent*:

(*a*) *the Fourier coefficient of* g

$$c_\alpha = (2\pi)^{-N} \int_{[0,2\pi]^N} \exp\left(-i \sum_{k=1}^{N} \alpha_k t_k\right) g\big(\exp(it_1), \dots, \exp(it_N)\big)\, dt$$

is zero whenever $\alpha \neq 0$ *and* $\max\limits_{1 \leqslant k \leqslant N} \alpha_k \leqslant 0$

(*b*) $\forall \varphi \in \mathcal{F}_q(\mathbf{C})$, $\varphi = \mathcal{T}((\varphi_j))$, *if* $\varphi(w) = 0$ *then*

$$\sum_{j \in J^N} \int_{\mathbf{R}^N} \mathbf{P}_j(\operatorname{Re} w, Im\, w - s)\, \varphi_j(s) f_j(s)\, ds = 0\,.$$

In particular, if X *is the dual space of a Banach space* Y, *then* (*a*) *and* (*b*) *are equivalent to*

(*c*) $\forall \varphi = \mathcal{T}((\varphi_j)) \in \mathcal{F}_q(Y)$ *such that* $\varphi(w) = 0$

$$\sum_{j \in J^N} \int_{\mathbf{R}^N} \mathbf{P}_j(\operatorname{Re} w, \operatorname{Im} w - s) < f_j(s), \varphi_j(s) > ds = 0\,.$$

PROOF. We show that (*b*) $\Rightarrow$ (*a*) and that (*a*) $\Rightarrow$ (*c*). Analogously it can be shown that (*a*) $\Rightarrow$ (*b*). Moreover it is obvious that (*c*) $\Rightarrow$ (*b*).

(*b*) $\Rightarrow$ (*a*). Let $\alpha \in \mathbf{Z}^N$, $\alpha \neq 0$, $\min\limits_k \alpha_k \geqslant 0$, and let $\varphi = \mathcal{T}((\varphi_j)) \in \mathcal{F}_q(\mathbf{C})$ such that $\forall z \in D^N$ $z^\alpha = \varphi \circ \vec{\mu}_w^{-1}(z)$. Then $\varphi(w) = 0$, so that

$$0 = \sum_{j \in J^N} \int_{\mathbf{R}^N} \mathbf{P}_j(\operatorname{Re} w, \operatorname{Im} w - s)\, \varphi_j(s) f_j(s)\, ds =$$

$$= (2\pi)^{-N} \int_{[0,2\pi]^N} \exp\left(i \sum_{k=1}^{N} \alpha_k t_k\right) g\big(\exp(it_1), \dots, \exp(it_N)\big)\, dt = c_{-\alpha}\,.$$

$(a) \Rightarrow (c)$. Let $\varphi = \mathfrak{T}((\varphi_j)) \in \mathcal{F}_a(Y)$ with $\varphi(w) = 0$ and let $\psi = \varphi \circ \vec{\mu}_w^{-1}$. We also call ψ the function on $(\partial D)^N$ obtained by the change of variable $\vec{\mu}_w$ from the function $j + is \mapsto \varphi_j(s)$ on $(\partial S)^N$. Since

$$(2\pi)^{-N}\int\limits_{[0,2\pi]^N} \langle g(\exp(it_1), \ldots, \exp(it_N)), \psi(\exp(it_1), \ldots, \exp(it_N))\rangle\, dt =$$

$$= \sum_{j \in J^N} \int\limits_{\mathbf{R}^N} \mathbf{P}_j(\operatorname{Re} w, \operatorname{Im} w - s)\, \langle f_j(s), \varphi_j(s)\rangle\, ds$$

we have to prove that the first integral vanishes. Set

$$\psi_m(\exp(it_1), \ldots, \exp(it_N)) =$$

$$= \sum_{|\alpha_1|, \ldots, |\alpha_N| \leqslant m} \left(\prod_{k=1}^{N} \left(1 - \frac{|\alpha_k|}{m+1}\right) \exp\left(i \sum_{k=1}^{N} a_k t_k\right)\right) b_\alpha$$

where $m \in \mathbf{N}$ and b_α is the α-th Fourier coefficient of ψ. By means of the Fourier coefficients $(c_\alpha)_{\alpha \in \mathbf{Z}^N}$ of g we define analogously g_m. As ψ is holomorphic on D^N and $\psi(0) = 0$, we have that $b_\alpha = 0$ when $\alpha = 0$ and whenever $\min\limits_k \alpha_k < 0$. Therefore

$$\forall m, n \in \mathbf{N} \quad (2\pi)^{-N}\int\limits_{[0,2\pi]^N} \langle g_m, \psi_n\rangle\, dt =$$

$$= (2\pi)^{-N} \sum_{|\alpha_1|, \ldots, |\alpha_N| \leqslant m} \; \sum_{|l_1|, \ldots, |l_N| \leqslant n} \; \prod_{k=1}^{N} \left(1 - \frac{|\alpha_k|}{m+1}\right)\left(1 - \frac{|l_k|}{n+1}\right) \cdot$$

$$\cdot \int\limits_{[0,2\pi]^N} \exp\left(i \sum_{k=1}^{N} (\alpha_k + l_k)\, t_k\right) dt\, \langle c_\alpha, b_l\rangle =$$

$$= \sum_{|\alpha_1|, \ldots, |\alpha_N| \leqslant \min\{m,n\}} \; \prod_{k=1}^{N} \left(1 - \frac{|\alpha_k|}{m+1}\right)\left(1 - \frac{|\alpha_k|}{n+1}\right) \langle c_\alpha, b_{-\alpha}\rangle = 0\,.$$

We now remark that $\psi_n \in L^\infty((\partial D)^N, Y)$, as it is a finite sum of functions in L^∞, and that $g \in L^{q'}((\partial D)^N, X) \subseteq L^1((\partial D)^N, X)$, so that $g_m \xrightarrow[m \to \infty]{} g$ in $L^1((\partial D)^N, X)$ ([22] Ch. XVII th. (1.23)). Therefore $\forall n \in \mathbf{N}$ $(2\pi)^{-N} \int\limits_{[0,2\pi]^N} \langle g, \psi_n\rangle\, dt = 0$. But $g \in L^{q'}((\partial D)^N, X)$, $\psi_n \to \psi$ in $L^q((\partial D)^N, Y)$ ([22] same theorem) and so $(2\pi)^{-N} \int\limits_{[0,2\pi]^N} \langle g, \psi\rangle\, \mathrm{d}t = 0$.

REMARK 6.3. Suppose that $f \in \tilde{\mathcal{F}}_{q'}(X)$ $(1 < q' \leqslant \infty)$. Then f is holomorphic if and only if the equivalent properties of th. 6.2 hold $\forall w \in S^N$. In fact, if f is holomorphic, then condition (a) is fulfilled as we have already remarked. Conversely, if the condition (b) is satisfied $\forall w \in S^N$, then it is easy to check that

$$\forall \varphi \in \mathcal{F}_q(\mathbf{C}) \quad \varphi(w) f(w) = \sum_j \int_{\mathbf{R}^N} \mathbf{P}_j(\operatorname{Re} w, \operatorname{Im} w - s) \varphi_j(s) f_j(s)\, ds\,,$$

so that φf is harmonic in each complex variable $z_k = \xi_k + i\eta_k$. In particular if we set $\varphi(z) = z_k$, we get

$$0 = \frac{1}{2} \Delta_{(\xi_k, \eta_k)} \big(z_k f(z)\big) = \frac{\partial f}{\partial \xi_k} + i \frac{\partial f}{\partial \eta_k}\,,$$

so that it is proved that f is holomorphic.

DEFINITION 6.4. *Let $A = (A_j)_{j \in J^N}$ be a compatible family of Banach spaces, $a \in I^N$, $1 \leqslant q < \infty$, and let q' be the conjugate exponent to q. We call $\mathcal{G}(A)_{(a;q')}$ the closed subspace of $\tilde{\mathcal{F}}_{q'}(A)$ whose elements are the functions f with boundary values $(f_j)_{j \in J^N}$ fulfilling the conditions (a)-(b) of th. 6.2 (where $X = \Sigma(A)$ and $w = a$). We endow $\mathcal{G}(A)_{(a;q')}$ with the norm $|||\cdot|||_{(a;q')}$ of $\tilde{\mathcal{F}}_{q'}(A)$.*

By prop. 3.2 $f \mapsto f(a)$ is continuous on $\mathcal{G}(A)_{(a;q')}$, so that we can define a Banach space in the following way:

DEFINITION 6.5. $A^{(a;q')} = \{f(a);\ f \in \mathcal{G}(A)_{(a;q')}\}$.

$$\forall x \in A^{(a;q')} \qquad \|x\|^{(a;q')} = \inf \{|||f|||_{(a;q')};\ f \in \mathcal{G}(A)_{(a;q')},\ f(a) = x\}\,.$$

From remark 6.3 it follows at once that $\mathcal{F}_{q'}(A)$ is a closed subspace of $\mathcal{G}(A)_{(a;q')}$, so that $A_{(a;q')} \overset{1}{\hookrightarrow} A^{(a;q')}$ $(1 < q' \leqslant \infty)$.

THEOREM 6.6. *$A^{(a;q')}$ is an intermediate space for the family A, and actually $\Delta(A) \overset{1}{\hookrightarrow} A^{(a;q')} \overset{1}{\hookrightarrow} \Sigma(A)$. Moreover, if A and B are compatible families of Banach spaces, then $A^{(a;q')}$ and $B^{(a;q')}$ are interpolation spaces with respect to A and B.*

PROOF. From the remark preceding the theorem and from prop. 3.2 and th. 3.5 it follows that $\Delta(A) \overset{1}{\hookrightarrow} A_{(a;q')} \overset{1}{\hookrightarrow} A^{(a;q')} \overset{1}{\hookrightarrow} \Sigma(A)$. It is easily checked that whenever $T \in L(A, B)$ and $f \in \mathcal{G}(A)_{(a;q')}$, then $Tf \in \mathcal{G}(B)_{(a;q')}$: hence the theorem follows at once.

As above, when Y is a Banach space such that $\Delta(A)$ is a dense subspace of Y (we recall that we have made this assumption for $Y = A_j$, and this is not restrictive by th. 5.2), we denote by Y' the vector subspace of $\Delta(A)^*$ which can be identified in the usual way with the dual space Y^* of Y.

THEOREM 6.7. *If* $1 \leq q < \infty$, *then* $(A_{(a;q)})'$ *is a closed subspace of* $A'^{(a;q')}$ *with the same norm.*

PROOF. Let $\varphi' \in (A_{(a;q)})'$ and let φ^* be its continuous extension to $A_{(a;q)}$. Then $f \mapsto \varphi^*(f(a))$ is a continuous linear functional on $\mathcal{F}_q(A)$ and (if $\mathcal{F}_q(A)$ has the norm $|||\cdot|||_{(a;q)}$) it has the same norm as φ'. By the Hahn-Banach theorem we get a continuous linear functional on $\tilde{\mathcal{F}}_q(A)$ and by a composition with $\mathcal{T}$ we get a continuous linear functional φ_1 on $\prod\limits_{j\in J^N} L^q_\varrho(\mathbf{R}^N, A_j)$ such that $\|\varphi_1\| = \|\varphi'\|$ $\Big($if $\prod\limits_{j\in J^N} L^q_\varrho(\mathbf{R}^N, A_j)$ has the norm $|||\cdot|||_{(a;q)}$, see § 3$\Big)$ and that

$$\varphi_1\big((f_j)\big) = \varphi^*\Bigg(\sum_{j\in J^N}\int_{\mathbf{R}^N}\mathbf{P}_j(a, s) f_j(s)\, ds\Bigg)$$

whenever $\mathcal{T}\big((f_j)\big) \in \mathcal{F}_q(A)$. In connection with φ_1 there are $\psi_j \in L^{q'}_\varrho(\mathbf{R}^N, A_j')$ $(j \in J^N)$ such that

$$\varphi_1\big((f_j)\big) = \sum_{j\in J^N}\int_{\mathbf{R}^N}\mathbf{P}_j(a, s)\, \langle\psi_j(s), f_j(s)\rangle\, ds$$

(recall the assumptions that we have made above and the equivalence between the weights ϱ and $\mathbf{P}_j(a,\cdot)$) and $|||\psi|||_{(a;q')} = \|\varphi'\|$ $(\psi = \mathcal{T}\big((\psi_j)\big))$. As the constant $\Delta(A)$-valued functions belong to $\mathcal{F}_q(A)$,

$$\forall x \in \Delta(A) \quad \langle\psi(a), x\rangle = \sum_{j\in J^N}\int_{\mathbf{R}^N}\mathbf{P}_j(a, s)\, \langle\psi_j(s), x\rangle\, ds =$$

$$= \varphi^*\Bigg(\sum_{j\in J^N}\int_{\mathbf{R}^N}\mathbf{P}_j(a, s)\, x\, ds\Bigg) = \varphi'(x)\,,$$

so that $\psi(a) = \varphi'$. To show that $\psi \in \mathcal{G}(A')_{(a;q')}$, since $\Sigma(A') = \Delta(A)^*$, we can verify that ψ fulfils the condition (c) of th. 6.2 with $Y = \Delta(A)$. But if $f \in \mathcal{F}_q(\Delta(A))$ and $f(a) = 0$, then

$$\sum_{j \in J^N} \int_{\mathbf{R}^N} \mathbf{P}_j(a, s) \langle \psi_j(s), f_j(s) \rangle ds = \varphi'(f(a)) = 0 .$$

Thus we have proved that $\varphi' \in A'^{(a;q')}$ and that $\|\varphi'\|^{(a;q')} \leqslant \||\psi\||_{(a;q')} = \|\varphi'\|_{(A_{(a;q)})'}$. It remains to show that $\forall \varphi' \in (A_{(a;q)})'$, $\|\varphi'\|_{(A_{(a;q)})'} \leqslant \|\varphi'\|^{(a;q')}$. Let $g \in \mathcal{G}(A')_{(a;q')}$ such that $g(a) = \varphi'$ and let $f \in \mathcal{F}_0(A)$. Then

$$\sum_{j \in J^N} \int_{\mathbf{R}^N} \mathbf{P}_j(a, s) \langle g_j(s), f_j(s) \rangle ds =$$

$$= \sum_{j \in J^N} \int_{\mathbf{R}^N} \mathbf{P}_j(a, s) \langle g_j(s), f_j(s) - f(a) \rangle ds + \langle g(a), f(a) \rangle = \langle g(a), f(a) \rangle$$

(by condition (c) of th. 6.2, with $Y = \Delta(A)$ and $X = \Sigma(A') = \Delta(A)^*$). Hence $|\langle \varphi', f(a) \rangle| \leqslant \||g\||_{(a;q')} \||f\||_{(a;q)}$, so that $|\langle \varphi', f(a) \rangle| \leqslant \|\varphi'\|^{(a;q')} \||f\||_{(a;q)}$ $\forall f \in \mathcal{F}_0(A)$. Let $x \in \Delta(A)$, $f \in \mathcal{F}_q(A)$ with $f(a) = x$ and let $(f_n)_{n \in \mathbf{N}}$ be a sequence in $\mathcal{F}_0(A)$ such that $\||f_n - f\||_{(a;q)} \xrightarrow[n \to \infty]{} 0$. Then

$$|\langle \varphi', x \rangle| = \lim_{n \to \infty} |\langle \varphi', f_n(a) \rangle| \ \left(\text{because } \varphi' \in (A_{(a;q)})'\right) \leqslant$$

$$\leqslant \lim_{n \to \infty} \|\varphi'\|^{(a;q')} \||f_n\||_{(a;q)} = \|\varphi'\|^{(a;q')} \||f\||_{(a;q)}, \text{ and thus } \|\varphi'\|_{(A_{(a;q)})'} \leqslant \|\varphi'\|^{(a;q')}.$$

REMARK 6.8. Along the lines of the proof of th. 6.7, it can be proved that the dual space $(A_{(a;q)})'$ can be obtained as the space of vectors $h(a)$, where h runs over the space $\mathcal{H}(A')_{(a;q')}$ whose elements are the functions $h \in \tilde{\mathcal{F}}_{q'}(A')$ such that $\forall f \in \mathcal{F}_q(A)$ with $f(a) = 0$,

$$\sum_{j \in J^N} \int_{\mathbf{R}^N} \mathbf{P}_j(a, s) \langle h_j(s), f_j(s) \rangle ds = 0 .$$

We note that the definition of $\mathcal{H}(A')_{(a;q')}$ differs from the definition of $\mathcal{G}(A')_{(a;q')}$ because in the latter only functions $f \in \mathcal{F}_q(\Delta(A))$ are employed (see 6.2(c) and the definition 6.4). However for $\mathcal{H}(A')_{(a;q')}$ we do not know any characterization in terms of Fourier coefficients analogous to condition 6.2(a).

REFERENCES

[1] J. Bergh - J. Löfström, *Interpolation Spaces. An Introduction*, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[2] J. I. Bertolo - D. L. Fernandez, *On the connection between the real and the complex interpolation method for several Banach spaces*, Rend. Sem. Mat. Univ. Padova, **66** (1982), pp. 193-209.

[3] A. P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math., **24** (1964), pp. 113-190.

[4] R. R. Coifman - M. Cwikel - R. Rochberg - Y. Sagher - G. Weiss, *The complex method for interpolation of operators acting on families of Banach spaces*, Lecture Notes in Math., no. 779, Springer-Verlag, Berlin-Heidelberg-New York, 1980, pp. 123-153.

[5] R. R. Coifman - M. Cwikel - R. Rochberg - Y. Sagher - G. Weiss, *A theory of complex interpolation for families of Banach spaces*, Adv. in Math., **43** (1982), pp. 203-229.

[6] M. Cwikel - S. Fisher, *Complex interpolation spaces on multiply-connected domains*, Adv. in Math., **48** (1983), pp. 286-294.

[7] N. Dinculeanu, *Linear operations on L^p-spaces*, in *Vector and Operator Valued Measures and Applications* (ed. by D. H. Tucker and H. B. Maynard), Academic Press Inc., New York-London, 1973, pp. 109-124.

[8] G. Dore - D. Guidetti - A. Venni, *Some properties of the sum and the intersection of normed spaces*, Atti Sem. Mat. Fis. Univ. Modena, **31** (1982), pp. 325-331.

[9] A. Favini, *Su una estensione del metodo d'interpolazione complesso*, Rend. Sem. Mat. Univ. Padova, **47** (1972), pp. 243-298.

[10] D. L. Fernandez, *Interpolation for 2^n-tuples of Banach spaces*, Studia Math., **65** (1979), pp. 175-201.

[11] D. L. Fernandez, *On the interpolation of 2^n Banach spaces. - I: The Lions-Peetre interpolation method*, Bul. Inst. Politehn. Iași Secț., I **26**, no. 1-2 (1980), pp. 49-54.

[12] D. L. Fernandez, *On the duality of interpolation spaces of several Banach spaces*, Acta Sci. Math. (Szeged), **44** (1982), pp. 43-51.

[13] D. L. Fernandez, *An extension of the complex method of interpolation*, Boll. Un. Mat. Ital., (5), B-**18** (1981), pp. 721-732.

[14] S. G. Kreĭn - L. I. Nikolova, *Holomorphic functions in a family of Banach spaces, and interpolation*, Soviet Math. Dokl., **21** (1980), pp. 131-134 (= Dokl. Akad. Nauk SSSR, **250** (1980), pp. 547-550).

[15] S. G. Kreĭn - L. I. Nikolova, *Complex interpolation for families of Baanch spaces*, Ukrainian Math. J., **34** (1982), pp. 26-36 (= Ukrain. Mat˙ Ž., **34** (1982), pp. 31-42).

[16] S. G. Kreĭn - Ju. I. Petunin - E. M. Semenov, *Interpolation of linear operators,* Amer. Math. Soc., Providence, R.I., 1982.

[17] J. L. Lions, *Une construction d'espaces d'interpolation,* C. R. Acad. Sci. Paris Sér. A-B, **251** (1960), pp. 1853-1855.

[18] L. I. Nikolova, *Kompleksnyĭ metod interpoljacii dlja neskol'kih prostranstv,* C. R. Acad. Bulgare Sci., **31** (1978), pp. 1523-1526.

[19] R. S. Phillips, *On weakly compact subsets of a Banach space,* Amer. J. Math., **65** (1943), pp. 108-136.

[20] W. Rudin, *Function Theory in Polydiscs,* W. A. Benjamin Inc., New York-Amsterdam, 1969.

[21] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators,* North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978.

[22] A. Zygmund, *Trigonometric Series,* 2nd ed., Cambridge University Press, Cambridge, 1968.

Note added in proof.

After the sending of this paper for publication, a paper was published by J. Peetre (Math. Nachr., **119** (1984), pp. 231-238), whose results and methods are very close to the ones of our last section.

Manoscritto pervenuto in redazione il 7 ottobre 1984.