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L^2 Vector Bundle Valued Forms and the Laplace-Beltrami Operator.

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0. Introduction.

The theory of vector bundle valued forms has been introduced first in complex geometry, both for compact and non compact manifolds. Later, J. Eells and J. H. Sampson used the real theory, on compact manifolds mainly, in the study of harmonic maps [ES].

This paper deals with some questions that naturally arise in the study of the Laplace-Beltrami operator on the Hilbert space of square integrable bundle valued forms, on a complete Riemannian manifold.

In section 1, after reviewing the basic facts on vector bundle valued differential forms, we introduce the relevant Hilbert spaces of forms and we prove the essentially self-adjointness of the Laplacian, which yields the uniqueness of the selfadjoint extension. We establish also a condition for the unique selfadjoint extension of the Laplacian equals $\boldsymbol{d\delta} + \boldsymbol{\delta d}$, where $\boldsymbol{d}$ and $\boldsymbol{\delta}$ are the weak extension of the differential and codifferential operator respectively.

In section 2 we are concerned with some spectral problems of the Laplace operator and we examine the Hodge orthogonal decomposition for vector valued differential forms.

Section 3 is devoted to some examples. We prove a vanishing condition for harmonic 1-forms with values in the tangent bundle to complete Riemannian manifolds with non negative constant sectional curvature at every point. A result of non existence is also

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proved for selfadjoint harmonic 1-forms. Furthermore, we establish a lower bound for the first eigenvalue for the Laplacian acting on square integrable 1-forms with values in the tangent bundle to a surface with positive Gauss curvature, for which the Poincaré inequality with compact support holds.

1. Some basic facts.

Let (M, g) be a smooth orientable, connected Riemannian manifold of dimension n. Let $\xi\colon V \to M$ be a smooth vector bundle of finite rank m. Suppose a (positive defined) C^∞ scalar product is defined on the vector bundle ξ and a connection ∇ on ξ is given such that $X\langle\omega, \varphi\rangle = \langle\nabla_X\omega, \varphi\rangle + \langle\omega, \nabla_X\varphi\rangle$, for all vector fields X and all sections ω, φ of ξ. The triple ξ, $\langle\ ,\ \rangle$, ∇ is called a Riemannian vector bundle [EL], [Ee].

We shall denote by $C(\xi)$ or $C(V)$ the vector space of smooth sections of ξ. The smooth sections of $\Lambda^p T^*M \otimes V \to M$ are called p-forms on M with values in ξ, and we shall denote the space of ξ-valued p-forms by $E^p(\xi)$ or $E^p(V)$.

If (M, g) is a Riemannian manifold and $\xi\colon V \to M$ is a Riemannian vector bundle, a canonical structure is defined on $\Lambda^p T^*M \otimes \otimes V \to M$, which satisfies the following equation:

$$(\nabla_X\omega)(X_1, \dots, X_p) = \nabla_X\omega(X_1, \dots, X_p) - \sum_j \omega(X_1, \dots, \nabla^M_X X_j, \dots, X_p)\,.$$

Here ∇^M is the Levi-Civita connection on $TM \to M$, and

$$\langle\omega, \varphi\rangle_x = 1/p!\sum_{i_1\dots i_p}\langle\omega(e_{i_1}, \dots, e_{i_p}), \varphi(e_{i_1}, \dots, e_{i_p})\rangle_x\,,$$

where $x \in M$ and $\{e_1, \dots, e_n\}$ is an orthonormal base of T_xM.

In terms of the connection ∇ on ξ an exterior differential operator $d\colon E^p(V) \to E^{p+1}(V)$ is defined by

$$(d\omega)(X_1, \dots, X_{p+1}) = \sum_j (-1)^{j+1}\nabla_{X_j}\omega(X_1, \dots, \hat{X}_j, \dots, X_{p+1}) + \\ + \sum_{i,j}(-1)^{i+j}\omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{p+1})\,.$$

We can define the exterior product of forms with values in a vector bundle too. For this purpose, let $\xi_i\colon V_i \to M$, $i = 1, 2, 3$, be three Riemannian vector bundles with connections ∇^i, and $\circ\colon V_1 \times V_2 \to V_3$ a vector bundle pairing such that:

$$\nabla^3_X(\omega_1 \circ \omega_2) = (\nabla^1_X \omega_1) \circ \omega_2 + \omega_1 \circ (\nabla^2_X \omega_2)$$

for all $X \in C(TM)$ and all $\omega_i \in C(V_i)$. The exterior product of a V_1-valued p-form ω_1 by a V_2-valued q-form ω_2 is a V_3-valued $p+q$-form $\omega_1 \wedge \omega_2$, defined by

$$(\omega_1 \wedge \omega_2)(X_1, \ldots, X_{p+q}) = \\ = \frac{1}{p!}\frac{1}{q!} \sum_{\pi \in S_{p+q}} \operatorname{sgn} \pi \, \omega_1(X_{\pi(1)}, \ldots, X_{\pi(p)}) \circ \omega_2(X_{\pi(p+1)}, \ldots, X_{\pi(p+q)}) ,$$

S_{p+q} being the permutation group on $p+q$ letters.

Let $v_g \in C(\Lambda^n T^* M) = E^n(\mathbf{R})$ be the volume element associated with g. The Hodge isomorphism $*\colon \Lambda^p T^* M \otimes V \to \Lambda^{n-p} T^* M \otimes V^*$ can be defined setting $\varphi \wedge * \omega = \langle \varphi, \omega \rangle v_g$ for all $\varphi, \omega \in E^p(V)$. As in the case of scalar valued p-forms: $** = (-1)^{p(n-p)} id$.

The codifferential operator $\delta\colon E^{p+1}(V) \to E^p(V)$ is the formal adjoint to d and is defined by $\delta = (-1)^{p+1} *^{-1} d *$.

The Laplace-Beltrami operator $\Delta\colon E^p(V) \to E^p(V)$, $\Delta = d\delta + \delta d$ is elliptic and formally selfadjoint [EL].

If ξ is a Riemannian vector bundle then the curvature $R^V \in E^2 \cdot (V \otimes V^*)$ is the tensor field for which

$$R^V(X, Y)\omega = \nabla_X \nabla_Y \omega - \nabla_Y \nabla_X \omega - \nabla_{[X,Y]}\omega ,$$
$$\omega \in C(V) \text{ and } X, Y \in C(TM) .$$

The curvature of $\Lambda^p T^* M \otimes V \to M$ satisfies the following identity

$$(R(X, Y)\omega\big((X_1, \ldots, X_p) = R^V(X, Y)\omega(X_1, \ldots, X_p) - \\ - \sum_j \omega\big(X_1, \ldots, R^M(X, Y)X_j, \ldots, X_p\big) ,$$

where R^M is the curvature of the Levi-Civita connection.

Let $S_x \in \operatorname{End}(\Lambda^p T^*_x M \otimes V_x)$ be the generalized Ricci endomorphism:

$$(S_x\omega)(X_1, \ldots, X_p) = \sum_{k,j} (-1)^k \big(R(e_j, X_k)\omega\big)(e_j, X_1, \ldots, \hat{X}_k, \ldots, X_p) ,$$

where $p > 0$, $\{e_1, \ldots, e_n\}$ is an orthonormal base of T_xM and $X_i \in \in T_xM$. If $p = 0$ set $S_x = 0$. The following formula (Weitzemböck formula [EL]) holds:

$$\Delta\omega = -\operatorname{Trace}\nabla^2\omega + S\omega , \qquad \omega \in E^p(V) . \tag{1.1}$$

Let $D^p(V)$ be the space of C^∞ V-valued p-forms with compact support and let $(\omega, \varphi) = \int_M \langle\omega, \varphi\rangle v_g = \int_M \omega\wedge *\varphi$. The formula (1.1) implies, for each $\omega \in D^p(V)$,

$$(\Delta\omega, \omega) = (S\omega, \omega) + (\nabla\omega, \nabla\omega) . \tag{1.2}$$

Note $d^2\omega = R^V\wedge\omega$ for all $\omega \in E^p(V)$. Working on a coordinate domain it is easy to show that for any $x \in M$ and for any $\omega^0 \in \Lambda^p \cdot \cdot T_x^* M$ there exists $\omega \in E^p(\mathbb{R})$ such that $d\omega = 0$ in a neighbourhood of x and $\omega^0 = \omega_x$, where ω_x is the value of ω in x. In the vector case we have:

PROPOSITION 1.1. Let $\varphi_1^0, \ldots, \varphi_k^0 \in (\Lambda^p T^* M \otimes V)_x$ be linearly independent, where $p < n - 1$ and $k = m\binom{n}{p}$. If for $j = 1, \ldots, k$ the problem

$$\begin{cases} d\varphi = 0 \\ \varphi_x = \varphi_j^0 \end{cases} \tag{1.3,j}$$

has a solution $\varphi = \varphi_j$ defined in some neighbourhood of x, then $R^V = 0$ in a neighbourhood of x.

In the proof we need the following lemmas.

LEMMA 1.2. Let W be a vector space and

$$\Omega \in \Lambda^r W^* , \qquad r < \dim W = n .$$

Then $\Omega = 0$ iff there are $r + 1$ linearly independent 1-forms $\omega_1, \ldots \ldots, \omega_{r+1}$, such that $\Omega\wedge\omega_i = 0$ for $i = 1, \ldots, r + 1$.

PROOF. Let $\{\varphi_1, \ldots, \varphi_n\}$ be a base of W^* such that $\varphi_i = \omega_i$, $i = 1, \ldots, r+1$.

$$\Omega = \sum_{1 \leqslant i_1 < \ldots < i_r \leqslant n} \Omega_{i_1 \ldots i_r} \varphi_{i_1} \wedge \ldots \wedge \varphi_{i_r},$$

but then $\Omega \wedge \omega_i = 0$ implies that either

$$\Omega_{i_1 \ldots i_r} = 0 \qquad \text{or} \qquad i \in \{i_1, \ldots, i_r\}.$$

Then necessarily $\Omega_{i_1 \ldots i_r} = 0$ for all multindices, i.e., $\Omega = 0$.

LEMMA 1.3. Let W be a vector space, $\Omega \in \Lambda^s W^*$, and let $\{\omega_1, \ldots$ $\ldots, \omega_n\}$ be a base of W^* and $t + s \leqslant n$. Then Ω is 0 iff $\Omega \wedge \omega_{i_1} \wedge \ldots$ $\ldots \wedge \omega_{i_t} = 0$ for all multindices $I = (i_1, \ldots, i_t)$.

PROOF. By Lemma 1.2 this fact holds if $t = 1$. Assuming it to hold for $t-1$, we have $(\Omega \wedge \omega_{i_1} \wedge \ldots \wedge \omega_{i_{t-1}}) \wedge \omega_j = 0$ for all multindices $I = (i_i, \ldots, i_{t-1})$ and all j. Then by Lemma 1.2 $\Omega \wedge \omega_{i_1} \wedge \ldots \wedge \omega_{i_{t-1}} = 0$, and the inductive hypothesis yields $\Omega = 0$.

PROOF OF THE PROPOSITION 1.1. Let φ_j be a solution of the problem (1.3.j). The φ_j are linearly independent in x, so they are linearly independent in a neighbourhood U of x too. Therefore $d^2\varphi_j|_U = R^V \wedge \varphi_j|_U = 0$ and the operator $\varphi \to R^V \wedge \varphi$ is zero in U, which we can assume to be a chart domain as well as a frame domain for a frame $f = (f_1, \ldots, f_m)$ on ξ. Then

$$R^V|_U = \sum_{\alpha,\beta} R^{V\alpha}_{\ \beta} f_\alpha \otimes f^\beta$$

where $f^* = (f^1, \ldots, f^m)$ is the dual frame. We have

$$R^{V\alpha}_{\ \beta} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} = 0,$$

for all $\alpha, \beta = 1, \ldots, m$ and all $I = (i_1, \ldots, i_p)$, thus by Lemma 1.3 $R^{V\alpha}_{\ \beta} = 0$, i.e., $R^V|_U = 0$.

The Stampacchia inequality is a basic tool in the study of the Laplace-Beltrami operator on complete manifolds. It provides a way to estimate the behaviour of $d\omega$ and $\delta\omega$ knowing that of ω and $\Delta\omega$. We do not give a proof of this inequality, because it proceeds as in the scalar case [AV].

From now onwards (M, g) will be always complete (even if some results are still valid more in general). Let $\varrho\colon M \to R_+$ be the geodesic distance from a fixed point $x_0 \in M$.

Set $B_r = \{x \in M\colon \varrho(x) < r\}$, by Höpf-Rinow-de Rham theorem, B_r is relatively compact.

Proposition 1.4 (Stampacchia inequality). Let $\xi\colon V \to M$ be a Riemannian vector bundle on a complete manifold M. There is a constant $A > 0$, such that for all $\omega \in E^p(V)$ and for all positive real numbers σ, r, R, with $R > r$:

$$\|d\omega\|^2_{B_r} + \|\delta\omega\|^2_{B_r} \leqslant \left(1/\sigma + A/(R-r)^2\right)\|\omega\|^2_{B_R} + \sigma\|\Delta\omega\|^2_{B_R},$$

where $\|\omega\|^2_{B_r} = \int\limits_{B_r} \omega \wedge * \omega$.

Corollary 1.5. If $\omega \in E^p(V)$, $\|\omega\| < +\infty$ and $\|\Delta\omega\| < +\infty$ then $\|d\omega\| < +\infty$, $\|\delta\omega\| < +\infty$ and $\|d\omega\|^2 + \|\delta\omega\|^2 < 1/\sigma\|\omega\|^2 + \sigma\|\Delta\omega\|^2$ for all $\sigma > 0$. Hence if $\Delta\omega = 0$ then $d\omega = \delta\omega = 0$.

We shall denote by $L^p_2(V)$ the completion of $D^p(V)$ with respect to the product $(\omega, \varphi) = \int\limits_M \omega \wedge * \varphi$. $L^p_2(V)$ is the Hilbert space of L_2 p-forms with values in the Riemannian vector bundle ξ. The differential operators we have introduced before live in $L^p_2(V)$, e.g., $d\colon L^p_2(V) \to L^{p+1}_2(V)$ and the domain of d, $D(d)$, is the space of those C^1 p-forms ω such that $\omega \in L^p_2(V)$ and $d\omega \in L^{p+1}_2(V)$, similarly $D(\Delta) = \{\omega \in L^p_2(V)\colon \omega$ is C^2 and $\Delta\omega \in L^p_2(V)\}$.

In the case of ordinary p-forms, if M has negligible boundary, i.e., d and δ are adjoint for all p, and if the Riemann curvature tensor is of class C^5, then the Laplace-Beltrami operator is essentially selfadjoint [Ga]. Since Gaffney's proof does not seem to be easily extendible to vector bundle valued forms, we shall follow another way, proving that the restriction Δ_c to forms with compact support of the Laplace-Beltrami operator Δ, has selfadjoint closure.

Let d_c and δ_c be the restrictions of d and δ to $D^p(V)$. Let $\tilde{d}_c$ and d^*_c be respectively the closure and the adjoint of d_c in the Hilbert space $L^p_2(V)$ (similarly $\tilde{\delta}_c$ and $\tilde{\delta}^*_c$ denote respectively the closure and the adjoint of δ_c). We have [Ve]

Proposition 1.6. The following statements hold

i) $D^p(V)$ is dense in $D(\delta^*_c)$ with the norm

$$\|\omega\|_d = \left(\|\omega\|^2 + \|\delta^*_c\omega\|^2\right)^{\frac{1}{2}}.$$

ii) $D^p(V)$ is dense in $D(d_c^*)$ with the norm

$$\|\omega\|_\delta = \left(\|\omega\|^2 + \|d_c^*\omega\|^2\right)^{\frac{1}{2}}.$$

iii) $D^p(V)$ is dense in $D(\delta_c^*) \cap D(d_c^*)$ with the norm

$$w(\omega) = \left(\|\omega\|^2 + \|\delta_c^*\omega\|^2 + \|d_c^*\omega\|^2\right)^{\frac{1}{2}}.$$

COROLLARY 1.7. $d_c^* = \tilde{\delta}_c$ and $\delta_c^* = \tilde{d}_c$.

From now onwards we shall set $\tilde{d}_c = \delta_c^* = \boldsymbol{d}$ and $\tilde{\delta}_c = d_c^* = \boldsymbol{\delta}$. If $\omega \in D(d)$ and $\varphi \in D^{p+1}(V)$ then from the formal adjointness of d and δ we get $(\omega, \delta\varphi) = (d\omega, \varphi)$, i.e., $\omega \in D(\delta_c^*)$ and $d\omega = \delta_c^*\omega$ so $d \subset \delta_c^*$. Similarly $\delta \subset d_c^*$. Therefore by Corollary 1.7 $d \subset \tilde{d}_c$ and $\delta \subset \tilde{\delta}_c$. So the adjointness of $\tilde{d}_c$ and $\tilde{\delta}_c$ implies that of d and δ. From $d \subset \delta_c^*$ we get $d^* \supset \tilde{\delta}_c$ and $d \supset d_c$ yields $d^* \subset d_c^* = \tilde{\delta}_c$. Thus $d^* = \boldsymbol{\delta}$, and analogously $\delta^* = \boldsymbol{d}$, proving thereby

COROLLARY 1.8. The operators d and δ are adjoint and we have

$$\delta^* = \boldsymbol{d}, \qquad d^* = \boldsymbol{\delta}.$$

Let $t\colon L_2^p(V) \times L_2^p(V) \to \mathbf{R}$ be the symmetric non-negative bilinear form defined on $D(t) = D^p(V)$ by $t[\omega, \varphi] = (\Delta\omega, p)$. This form is closable [Ka, p. 318]. The domain $W_2^p(V)$ of the closure $\tilde{t}$ is the Hilbert space of those $\omega \in L_2^p(V)$ such that there is a sequence $(\omega_n)_{n\in N}$ in $D^p(V)$ with $\omega_n \to \omega$, $d\omega_n \to \tilde{d}_c\omega$ and $\delta\omega_n \to \tilde{\delta}_c\omega$. We have

$$\tilde{t}[\omega, \varphi] = \lim_n t[\omega_n, \varphi_n] = (\boldsymbol{d}\omega, \boldsymbol{d}\varphi) + (\boldsymbol{\delta}\omega, \boldsymbol{\delta}\varphi),$$

and the norm of $W_2^p(V)$ is given by $w(\omega) = \left(\|\omega\|^2 + \tilde{t}[\omega, \omega]\right)^{\frac{1}{2}}$. By Proposition 1.6 we have $W_2^p(V) = D(\boldsymbol{d}) \cap D(\boldsymbol{\delta})$. Now we are able to prove

THEOREM 1.9. The Laplace-Beltrami operator, restricted to forms with compact support, is essentially selfadjoint in $L_2^p(V)$.

PROOF. Setting $\Delta' = \Delta_c + I$, then $\tilde{\Delta}' = \tilde{\Delta}_c + I$ and $\tilde{\Delta}_c$ is selfadjoint iff $\tilde{\Delta}'$ is. We shall prove that $\tilde{\Delta}'$ is selfadjoint. If $\omega \in D(\tilde{\Delta}_c)$ we have

$$(\tilde{\Delta}'\omega, \omega) = (\boldsymbol{d}\omega, \boldsymbol{d}\omega) + (\boldsymbol{\delta}\omega, \boldsymbol{\delta}\omega) + (\omega, \omega) \geqslant (\omega, \omega).$$

Therefore by the Schwarz inequality $\|\tilde{\Delta}'\omega\| \geqslant \|\omega\|$, i.e., $(\tilde{\Delta}')^{-1}$ exists as a bounded operator. So $R(\tilde{\Delta}') = D(\tilde{\Delta}'^{-1})$ is closed. We want to show $R(\tilde{\Delta}') = L_2^p(V)$. If $\eta \in R(\tilde{\Delta}')^\perp$ then $(\Delta'\varphi, \eta) = 0$ for all $\varphi \in D^p(V)$. The linear differential operator $\Delta'\colon E^p(V) \to E^p(V)$ is elliptic and formally selfadjoint as well Δ, therefore by regularity theorems, ([EL] or [We]), $\eta \in E^p(V)$ and $\Delta'\eta = 0$, i.e., $\Delta\eta = -\eta$. By Corollary 1.5 $\|d\eta\| < +\infty$ and $\|\delta\eta\| < +\infty$ so $\eta \in D(\boldsymbol{d}) \cap D(\boldsymbol{\delta}) = W_2^p(V)$. Then we can find a sequence $(\eta_n)_{n\in N}$ in $D^p(V)$ with $\eta_n \to \eta$, $d\eta_n \to \boldsymbol{d}\eta$ and $\delta\eta_n \to \boldsymbol{\delta}\eta$. We have

$$0 \geqslant -(\eta, \eta) = (\Delta\eta, \eta) = \lim_n (\Delta\eta, \eta_n) = \lim_n [(d\eta, d\eta_n) + (\delta\eta, \delta\eta_n)] =$$
$$= (d\eta, d\eta) + (\delta\eta, \delta\eta) \geqslant 0 .$$

Therefore $\eta = 0$ and $D(\tilde{\Delta}'^{-1}) = L_2^p(V)$, so $\tilde{\Delta}'^{-1}$ is selfadjoint, $\tilde{\Delta}'$ is selfadjoint and Δ_c is essentially selfadjoint.

REMARK 1.10. The conclusion of Theorem 1.9 holds if the hypothesis that the Riemannian metric of M be complete is replaced by the assumption that Δ be non negative.

Theorem 1.9 implies that there is only one selfadjoint extension of Δ_c. We denote this extension by $\boldsymbol{\Delta}$; we have $\boldsymbol{\Delta} = \tilde{\Delta}_c = \Delta_c^* = \tilde{\Delta}_c^*$. In the same way as for Corollary 1.8 one can show that Δ is symmetric and $\Delta^* = \tilde{\Delta} = \boldsymbol{\Delta}$.

In [Ga] Gaffney shows that if M has negligible boundary the operator $\tilde{d}_c d_c^* + \tilde{\delta}_c \delta_c^*$ is selfadjoint and equals $\tilde{\Delta}$. Let now V be a Riemannian vector bundle on the complete manifold M, for which a unique selfadjoint extension of Δ exists. The question arise whether $\boldsymbol{\Delta} = \boldsymbol{d\delta} + \boldsymbol{\delta d}$. Here is a particular result. Let T be the linear operator $T\colon L_2^p(V) \to L^{p+2}(V)$ defined by $\omega \mapsto R^V \wedge \omega$ for all $\omega \in D^p(V)$.

THEOREM 1.11. If the linear operator T is bounded then

$$\boldsymbol{\Delta} = \boldsymbol{d\delta} + \boldsymbol{\delta d} .$$

PROOF. *a*) We begin by proving that $\tilde{\Delta}_c \subset (d_c\delta_c)^\sim + (\delta_c d_c)^\sim$. If $\omega \in D(\tilde{\Delta}_c)$ let $(\omega_n)_{n\in N}$ be a sequence in $D^p(V)$ such that $\omega_n \to \omega$ and $\Delta\omega_n \to \tilde{\Delta}_c\omega$. Then

$$\|\Delta\omega_n - \Delta\omega_m\|^2 = \|d\delta(\omega_n - \omega_m) + \delta d(\omega_n - \omega_m)\|^2 =$$
$$= 2\big(d\delta(\omega_n - \omega_m), \delta d(\omega_n - \omega_m)\big) + \|d\delta\omega_n - d\delta\omega_m\|^2 + \|\delta d\omega_n - \delta d\omega_m\|^2 .$$

Furthermore

$$|(d\delta(\omega_n-\omega_m), \delta d(\omega_n-\omega_m))| = |(d^2\delta(\omega_n-\omega_m), d(\omega_n-\omega_m))| \leqslant$$
$$\leqslant \|R^V\wedge\delta(\omega_n-\omega_m)\| \|d\omega_n-d\omega_m\| \leqslant \text{const } \|\delta\omega_n-\delta\omega_m\| \|d\omega_n-d\omega_m\| .$$

In view of our hypothesis, $(\delta\omega_n)_{n\in N}$ and $(d\omega_n)_{n\in N}$ are convergent, for

$$\|d\omega_n-d\omega_m\|^2+\|\delta\omega_n-\delta\omega_m\|^2 = (\Delta(\omega_n-\omega_m), \omega_n-\omega_m) \leqslant$$
$$\leqslant \|\Delta\omega_n-\Delta\|\omega_m\|\omega_n-\omega_m\| .$$

Therefore

$$\|d\delta\omega_n-d\delta\omega_m\|^2+\|\delta d\omega_n-\delta d_m\|^2 =$$
$$= \|\Delta\omega_n-\Delta\omega_m\|^2-2(d\delta(\omega_n-\omega_m), \delta d(\omega_n-\omega_m)) \to 0 ,$$

as $n, m\to\infty$. So $(d\delta\omega_n)$ and $(\delta d\omega_n)$ are Cauchy sequence and

$$d\delta\omega_n\to(d_c\delta_c)^{\sim}\omega , \qquad \delta d\omega_n\to(\delta_c d_c)^{\sim}\omega .$$

b) We show now that $(d_c\delta_c)^{\sim}\subset\tilde{d}_c\tilde{\delta}_c$. For $\omega\in D((d_c\delta_c)^{\sim})$, let $(\omega_n)_{n\in N}$ be a sequence in $D^p(V)$ such that $\omega_n\to\omega$ and $d\delta\omega_n\to(d_c\delta_c)^{\sim}\omega$. We have

$$\|\delta\omega_n-\delta\omega_m\| = (d\delta(\omega_n-\omega_m), \omega_n-\omega_m) \leqslant \|d\delta(\omega_n-\omega_m)\| \|\omega_n-\omega_m\| ,$$

so $\delta\omega_n\to\tilde{\delta}_c\omega$, $\omega\in D(\tilde{\delta}_c)$ and this entails that $\omega\in D(\tilde{d}_c\tilde{\delta}_c)$ and $\tilde{d}_c\tilde{\delta}_c\omega = (d_c\delta_c)^{\sim}\omega$. Similarly one shows that $(\delta_c d_c)^{\sim}\subset\tilde{\delta}_c\tilde{d}_c$.

c) In view of the completeness of (M, g) *a*) and *b*) imply that $\boldsymbol{d\delta}+\boldsymbol{\delta d}\supset\boldsymbol{\Delta}$. Now by a theorem of von Neumann [Ka, p. 275] the operators $\boldsymbol{\delta d}$ and $\boldsymbol{d\delta}$ are selfadjoint, so $\boldsymbol{d\delta}+\boldsymbol{\delta d}$ is symmetric. Therefore $\boldsymbol{\delta d}+\boldsymbol{d\delta}\subset(\boldsymbol{d\delta}+\boldsymbol{\delta d})^*\subset\boldsymbol{\Delta}^*=\boldsymbol{\Delta}$, that is $\boldsymbol{d\delta}+\boldsymbol{\delta d}=\boldsymbol{\Delta}$.

LEMMA 1.12. There is a constant $c\geqslant 0$, which depends only on $\dim M$, such that for all $\varphi\in E^p(V\otimes V^*)$ and $\psi\in E^q(V)$

$$\langle\varphi\wedge\psi, \varphi\wedge\psi\rangle\leqslant c\langle\varphi, \varphi\rangle\langle\psi, \psi\rangle .$$

PROOF. Let $\{e_1, \ldots, e_n\}$ be an orthonormal base of T_xM.

$$\langle\varphi\wedge\psi, \varphi\wedge\psi\rangle =$$
$$= 1/(p+q)! \sum_{i_1 \ldots i_{p+q}} \langle(\varphi\wedge\psi)(e_{i_1}, \ldots, e_{i_{p+q}}), (\varphi\wedge\psi)(e_{i_1}, \ldots, e_{i_{p+q}})\rangle_x \leqslant$$
$$\leqslant c(n)/p!q! \sum_{I,J} \langle\varphi(e_I)\circ\psi(e_J), \varphi(e_I)\circ\psi(e_J)\rangle_x$$

where $c(n)$ is a constant depending only on $\dim M$,

$$I = (i_1, \ldots, i_p), \quad J = (j_1, \ldots, j_q) \quad \text{and} \quad \varphi(e_I) = \varphi(e_{i_1}, \ldots, e_{i_p}).$$

If we set

$$|\alpha|_x = \sup_{v \in V_x \sim \{0\}} \langle\alpha\circ v, \alpha\circ v\rangle_x / \langle v, v\rangle_x \qquad \text{for all } \alpha \in V_x \otimes V_x^*,$$

we get

$$\langle\varphi\wedge\psi, \varphi\wedge\psi\rangle_x \leqslant c(n) 1/p! \sum_I \langle\varphi(e_I), \varphi(e_I)\rangle_x 1/q! \sum_J \langle\psi(e_J), \psi(e_J)\rangle_x =$$
$$= c(n)\langle\varphi, \varphi\rangle_x\langle\psi, \psi\rangle_x.$$

REMARK 1.13. Lemma 1.12 implies that if $\langle R^V, R^V\rangle$ is a bounded function on M then the operator T is bounded. That is the case, e.g., if M is compact or if M is locally symmetric and ξ is a tensor product of tensor power and exterior power of the tangent bundle and cotangent bundle of M. Indeed we have $\nabla_X R^V = 0$ and $\langle R^V, R^V\rangle$ is constant.

2. $W_2^p(V)$-ellipticity and $R_2^p(V)$-ellipticity.

A Riemannian vector bundle is called $W_2^p(V)$-elliptic ([AV]) if there is a constant $c > 0$ such that $\|\omega\|^2 \leqslant c(\|d\omega\|^2 + \|\delta\omega\|^2)$, for all $\omega \in W_2^p(V)$. The smallest constant for which the above inequality holds is called the constant of $W_2^p(V)$-ellipticity of ξ. $W_2^p(V)$-ellipticity implies there are no non zero L_2 harmonic forms. Following [Gi] we introduce another notion, weaker than $W_2^p(V)$-ellipticity. We shall say that a Riemannian vector bundle ξ is $R_2^p(V)$-elliptic if there is a positive constant c' such that

$$\|\omega\|^2 \leqslant c'(\|d\omega\|^2 + \|\delta\omega\|^2) \qquad \text{for all } \omega \in W_2^p(V) \cap \overline{R(\Delta)} = R_2^p(V)\,,$$

and we shall call the smallest constant, for which the inequality holds, the constant of $R_2^p(V)$-ellipticity.

Note that we have $L_2^p(V) = N(\Delta) \oplus \overline{R(\Delta)}$ and by regularity theorems $N(\Delta) = \{\omega \in E^p(V)\colon \varDelta\omega = 0 \text{ and } \omega \in L_2^p(V)\}$. The space $N(\Delta)$, which we also denote by $H_2^p(V)$, is the space of L_2 harmonic p-forms with values in the vector bundle ξ. The square root of Δ, which will be need in the proof of Theorem 2.1, is the operator $\Delta^{\frac{1}{2}}\colon L_2^p(V) \to L_2^p(V)$, selfadjoint and non negative as well as Δ, such that $(\Delta^{\frac{1}{2}})^2 = \Delta$. Recall that $\Delta^{\frac{1}{2}}$ has the following properties [Ka]:

i) $D(\Delta^{\frac{1}{2}}) = W_2^p(V) = D(\tilde{t})$ and

$$\tilde{t}[\omega, \varphi] = (d\omega, d\varphi) + (\delta\omega, \delta\varphi) = (\Delta^{\frac{1}{2}}\omega, \Delta^{\frac{1}{2}}\varphi)\,.$$

ii) If $\omega \in W_2^p(V)$, $\varphi \in L_2^p(V)$ and $(\Delta^{\frac{1}{2}}\omega, \Delta^{\frac{1}{2}}\theta) = (\varphi, \theta)$ for all $\theta \in D^p(V)$, then $\omega \in D(\Delta)$ and $\Delta\omega = \varphi$.

THEOREM 2.1. Let $\xi\colon V \to M$ be a Riemannian vector bundle on a complete manifold. Then ξ is $W_2^p(V)$-elliptic iff 0 belongs to the resolvent of Δ, i.e., Δ has a bounded inverse. The $W_2^p(V)$-ellipticity constant is $\|\Delta^{-1}\|$ and (by selfadjointness of Δ^{-1}) is also equal to the spectral radius of Δ^{-1}.

PROOF. If ξ is $W_2^p(V)$-elliptic, then $H_2^p(V) = \{0\}$ and Δ is invertible. Let c be the $W_2^p(V)$-ellipticity constant and $\omega \in D(\Delta)$: we have $(\Delta\omega, \omega)c \geqslant (\omega, \omega)$, therefore $\|\Delta\omega\|c \geqslant \|\omega\|$, so for all $\eta \in R(\Delta)$,

$$\|\Delta^{-1}\eta\| \leqslant c\|\eta\|\,,$$

i.e., Δ^{-1} is bounded and $\|\Delta^{-1}\| \leqslant c$. Conversely suppose Δ^{-1} is bounded. Then Δ^{-1} is selfadjoint and non negative as well Δ, and $(\Delta^{-\frac{1}{2}}\omega, \Delta^{-\frac{1}{2}}\omega) = (\Delta^{-1}\omega, \omega) \leqslant \|\Delta^{-1}\|\,\|\omega\|^2$. So if $\eta \in D(\Delta^{\frac{1}{2}}) = W_2^p(V)$ we have

$$\|\eta\|^2 \leqslant \|\Delta^{-1}\|\,\|\Delta^{\frac{1}{2}}\eta\|^2 = \|\Delta^{-1}\|(\|d\eta\|^2 + \|\delta\eta\|^2)\,,$$

i.e., ξ is $W_2^p(V)$-elliptic and $c \leqslant \|\Delta^{-1}\|$.

REMARK 2.2. If S: $L_2^p(V) \to L_2^p(V)$ is the generalized Ricci endomorphism and if there is a positive constant k such that $(S\omega, \omega) \geqslant k(\omega, \omega)$ for all $\omega \in D^p(V)$, then we have $\|d\omega\|^2 + \|\delta\omega\|^2 \geqslant k\|\omega\|^2$, see (1.2), and by density this inequality holds for all $\omega \in W_2^p(V)$.

In order to characterize the $R_2^p(V)$-ellipticity on complete manifolds, we need the following lemma [Hö].

LEMMA 2.3. Let H and H' be two Hilbert spaces and T: $H \to H'$ be a densely defined closed operator. Further, let F be a closed subspace of H' such that $F \supset R(T)$.

i) If $R(T) = F$ then $\|u\|_{H'} \leqslant c\|T^* u\|_H$, for all $u \in R(T) \cap D(T^*)$, where c is a positive constant.

ii) If $\|u\|_{H'} \leqslant c\|T^* u\|_H$ for all $u \in F \cap D(T^*)$, then the equation $Tv = u$, with $u \in F$, has a solution v such that $\|v\| \leqslant c\|u\|$.

THEOREM 2.4. Let ξ: $V \to M$ be a Riemannian vector bundle on a complete manifold. The following statements are equivalent:

a) $R(\Delta)$ is closed.

b) $R(\Delta^{\frac{1}{2}})$ is closed.

c) ξ is $R_2^p(V)$-elliptic.

PROOF. *a*) $\Rightarrow$ *b*). By property ii) of $\Delta^{\frac{1}{2}}$ we get $N(\Delta^{\frac{1}{2}}) = N(\Delta)$, therefore

$$L_2^p(V) = N(\Delta^{\frac{1}{2}}) \oplus \overline{R(\Delta^{\frac{1}{2}})} = H_2^p(V) \oplus \overline{R(\Delta)}\,, \qquad \text{and} \qquad \overline{R(\Delta)} = \overline{R(\Delta^{\frac{1}{2}})}\,.$$

We conclude noting $R(\Delta^{\frac{1}{2}}) \supset R(\Delta)$ holds.

b) $\Rightarrow$ *c*). By Lemma 2.3 i) there is a constant c such that $\|\omega\|^2 \leqslant c\|\Delta^{\frac{1}{2}}\omega\|^2$, for all $\omega \in D(\Delta^{\frac{1}{2}}) \cap R(\Delta^{\frac{1}{2}})$, i.e., for all $\omega \in R_2^p(V)$

$$\|\omega\|^2 \leqslant c(\|d\omega\|^2 + \|\delta\omega\|^2)\,.$$

c) $\Rightarrow$ *a*). If $\omega \in D(\Delta)$ then there is a sequence $(\omega_n)_{n\in N}$ in $D^p(V)$ such that $\omega_n \to \omega$ and $\Delta\omega_n \to \Delta\omega$. Further by Corollary 1.5 $(d\omega_n)_{n\in N}$ and $(\delta\omega_n)_{n\in N}$ are Cauchy sequences and

$$\|d\omega\|^2 + \|\delta\omega\|^2 \leqslant 1/\sigma\|\omega\|^2 + \sigma\|\Delta\omega\|^2\,, \qquad \sigma > 0\,.$$

By hypothesis there is a constant $c > 0$ such that

$$\|\omega\|^2 \leqslant c(\|d\omega\|^2 + \|\delta\omega\|^2)$$

for all $\omega \in R_2^p(V)$, therefore we have

$$\|\omega\|^2 \leqslant c\sigma/(1 - c/\sigma)\|\Delta\omega\|^2 .$$

If $\sigma > c$ we get $\|\omega\|^2 \leqslant c'\|\Delta\omega\|^2$, for all $\omega \in D(\Delta) \cap \overline{R(\Delta)}$, and by Lemma 2.3 ii) we conclude $\overline{R(\Delta)}$ is closed.

REMARK 2.5. $R_2^p(V)$-ellipticity is equivalent to the possibility of solving the equation $\Delta\omega = f$, for all $f \in R(\Delta)$ coupled with the existence of a constant $c > 0$ such that the solution ω can be so chosen that $\|\omega\| \leqslant c\|f\|$. The $R_2^p(V)$-ellipticity is also equivalent to the Hodge orthogonal decomposition $L_2^p(V) = H_2^p(V) \oplus R(\Delta)$.

REMARK 2.6. Vesentini [Ve] proved in the scalar case that, if there is a compact $K \subset M$ and a constant $c > 0$ such that for all $x \notin K$ we have $\langle S\omega, \omega\rangle_x \leqslant c\langle\omega, \omega\rangle_x$, for all $\omega \in D^p(\mathbb{R})$, then $M \times \mathbb{R} \to M$ is $R_2^p(\mathbb{R})$-elliptic and every eigenspace relatively to a proper value $\lambda < c$ is finite dimensional. This result is easily estensible to the general case of forms with values in a Riemannian vector bundle.

3. Particular cases and examples.

Some of the results established will now be applied to the tangent bundle of an oriented connected complete Riemannian manifold. For the sake of simplicity we shall be concerned only with 1-forms.

The Weitzemböck formula (1.1) for $\omega \in E^1(TM) = C(\text{End}\,(TM))$ takes the local expressions on coordinate domain

$$(3.1) \qquad (\Delta\omega)^i_j = -g^{rs}(\nabla_r\nabla_s\omega)^i_j + R^{M\,i\,\,s}_{\,\,\,\,\,r\,j}\,\omega^r_s + \text{Ric}^r{}_j\,\omega^i_r .$$

If either $n > 2$ and M has constant sectional curvature at every point, or M is a surface, then the Riemann and Ricci curvature tensors are expressed by,

$$(3.2) \qquad R^M{}_{ijkl} = k(g_{ik}g_{jl} - g_{il}g_{jk})\,, \qquad \text{Ric}_{ij} = k(n-1)g_{ij}$$

where k is a real C^∞ function on M, which is constant if $k > 2$.

The operator S': $\omega^i_j \mapsto S'(\omega)^i_j = R^{M i\ s}_{\ \ \ r\ j}\,\omega^r_s$ is selfadjoint on every fiber, i.e., $\langle S'\omega, \varphi\rangle_x = \langle \omega, S'\varphi\rangle_x$ for all $x \in M$. So, if we fix a point $x \in M$ and we choose a Riemann normal coordinate system at x, we get, the proper values of S' in x are determined by the equation

$$k\Big(\omega^i_j - \delta^j_i\big(\mathrm{Trace}\,(\omega)\big)\Big) = \lambda\omega^j_i\,.$$

Solving this equation we decompose every fiber $T^*_x M \otimes T_x M$ into three mutually orthogonal eigenspaces:

1) $\lambda_1 = -k(n-1)$, $A_{1,x} = \{\omega_x \in T^*_x M \otimes T_x M\colon \omega_x = \mathrm{const}\,\delta_x\}$, where $\delta_x(X) = X$, for all $X \in T_x M$.

2) $\lambda_2 = -k$, $A_{2,x} = \{\omega_x \in T^*_x M \otimes T_x M;\ \omega_x$ is skew adjoint).

3) $\lambda_3 = k$, $A_{3,x} = \{\omega_x \in T^*_x M \otimes T_x M\colon\ \omega_x$ is selfadjoint with null trace$\}$.

Setting $A_i = \bigcup_{x\in M} A_{i,x}$, we define three sub bundles of $T^*M \otimes TM$.

Now using (1.2) we get for all $\omega \in D^1(TM)$

$$(3.4)\qquad (\Delta\omega, \omega) = (\nabla\omega, \nabla\omega) + (n-2)\int_M k\langle\omega_2, \omega_2\rangle v_g + n\int_M k\langle\omega_3, \omega_3\rangle v_g$$

and $\omega = \omega_1 + \omega_2 + \omega_3$, with $\omega_i \in C(A_i)$, is the orthogonal decomposition. If $k \geqslant 0$ and if the manifold is complete then:

$$(3.5)\qquad (\boldsymbol{\Delta}^{\frac12}\omega, \boldsymbol{\Delta}^{\frac12}\omega) = (\tilde\nabla\omega, \tilde\nabla\omega) + (n-2)\int_M k\langle\omega_2, \omega_2\rangle v_g + n\int_M k\langle\omega_3, \omega_3\rangle v_g$$

for all $\omega \in W^1_2(TM)$.

Here are a few consequences of this equation (for similar results on scalar valued 1-forms see [Do1], [Do2], [Do3], [GW]).

PROPOSITION 3.1. Let $\omega \in E^1(TS^n)$, $n > 2$, where S^n is the n-sphere. $\Delta\omega = 0$ iff $\omega = \mathrm{const}\,\delta$, where δ is the 1-form with values in TS^n defined by $\delta(X) = X$, for all $X \in C(TM)$.

PROOF. Equation (3.4) implies both $\omega_3 = \omega_2 = 0$ and $\nabla\omega = 0$, then $\omega = f\delta$, where f is a smooth function, and $\nabla\omega = \nabla(f\delta) = df \otimes \delta = 0$, i.e., $df = 0$ and f is a constant.

If M is a surface, let $\sigma \in E^1(TM)$ be the skew adjoint operator, $\sigma \in C(A_2)$, with eigenfunctions $\pm\sqrt{-1}$, σ exists since M is orientable.

PROPOSITION 3.2. Let M be a complete surface with non negative Gauss curvature. If $\omega \in E^1(TM) \cap L_2^1(TM)$ and $\Delta\omega = 0$ then $\nabla\omega = 0$. If the Gauss curvature is positive at some point and $\Delta\omega = 0$ then $\omega = \text{const}\, \delta + \text{const}\, \sigma$. Hence, if $\text{Vol}\, M = \infty$ there are no L_2 harmonic forms with values in TM.

PROOF. Equation (3.5) implies the first statement. If $k > 0$ at some point then $\omega_3 = 0$ everywhere and so $\omega = f\delta + g\sigma$, where f and g are smooth functions on M. $\nabla\sigma = \nabla\delta = 0$ implies $0 = \nabla\omega = df \otimes \delta + dg \otimes \sigma$. But $df \otimes \delta$ and $dg \otimes \sigma$ are mutually orthogonal, then $df \otimes \delta = dg \otimes \sigma = 0$, $df = dg = 0$ and $f = \text{const}$, $g = \text{const}$.

PROPOSITION 3.3. Let M be a complete Riemannian manifold with non negative sectional curvature. If $\omega \in E^1(TM) \cap L_2^1(TM)$ is selfadjoint, and $\Delta\omega = 0$ then $\nabla\omega = 0$. Thus if $\text{Vol}\, M = \infty$ there are no selfadjoint L_2 harmonic 1-forms with values in TM.

PROOF. Choose a normal coordinate system (y, U) at $x \in M$, such that $\{\partial/\partial y_i|_x : i = 1, \ldots, n\}$ is an orthonormal base of T_xM, made up by eigenvectors of ω_x. Since $\omega_x = \sum_{i,j} \lambda_i \delta_j^i (\partial/\partial y_i)|_x \otimes dy^j(x)$, where the λ_i's are the eigenvalues of ω_x, the Weitzemböck formula yields

$$\langle S\omega, \omega\rangle_x = -\sum_{r,i} R^M{}_{irir} \lambda_i \lambda_r + \sum_i \text{Ric}_{ii} \lambda_i^2 .$$

The equation $\text{Ric}_{ii} = \sum_j R^M{}_{jiji}$ implies

$$\langle S\omega, \omega\rangle_x = -\sum_{i,r} R^M{}_{irir} \lambda_i \lambda_r + \sum_{i,r} R^M{}_{irir} \lambda_i^2 = \sum_{i<r} R^M{}_{irir} (\lambda_i - \lambda_r)^2 .$$

But this yields the conclusion since $R^M{}_{irir}$ is the sectional curvature of the oriented plane $(\partial/\partial y_i)|_x \wedge (\partial/\partial y_r)|_x$.

Let M be now a surface and $\omega \in E^1(TM)$. The equations

$$\omega = \omega_1 + \omega_2 + \omega_3 = f\delta + g\sigma + \omega_3 ,$$

$$\nabla_X \omega_3 \in C(A_3) , \qquad \nabla_X(f\delta) = df(X)\delta \qquad \text{and} \qquad \nabla_X g\sigma = dg(X)\sigma$$

imply $\nabla\omega_i$'s are mutually orthogonal as well as the ω_i's. Furthermore $\|f\delta\| = \|f\|$, $\|g\sigma\| = \|g\|$ and $\|\nabla(f\delta)\| = \|df\|$, $\|\nabla(g\sigma)\| = \|dg\|$. Thus (3.4) yields for all $\omega \in D^1(TM)$

$$(3.6) \qquad (\Delta\omega, \omega) = \|\nabla\omega_3\|^2 + \|df\|^2 + \|dg\|^2 + 2\int_M k\langle\omega_3, \omega_3\rangle v_g .$$

Now if the Poincaré inequality for functions with compact support holds (for some examples related to the isoperimetric inequality and the eigenvalue problem for the Laplacian see [Ya]), i.e., if a positive constant c exists, such that $c\int_M f^2 v_g \leqslant \int_M \langle df, df\rangle v_g$, for all smooth functions f with compact support, then

$$(\Delta\omega, \omega) \geqslant c\int_M \langle\omega_1, \omega_1\rangle v_g + c\int_M \langle\omega_2, \omega_2\rangle v_g + 2\int_M k\langle\omega_3, \omega_3\rangle v_g .$$

So if k has a positive lower bound, say k_0, then

$$(\Delta\omega, \omega) \geqslant \min\{c, 2k_0\}\|\omega\|^2 \qquad \text{for all } \omega \in D^1(TM)$$

and, by a density argument, for all $\omega \in W_2^1(TM)$. We have the following proposition.

Proposition 3.4. Let M be a complete Riemannian surface with Gauss curvature $k \geqslant k_0 > 0$ and for which the Poincaré inequality for compactly supported C^1 functions holds, with constant c. Then TM is $W_2^1(TM)$-elliptic and $\min\{c, 2k_0\}$ is a lower bound for the first eigenvalue of the Laplace-Beltrami operator on L_2 TM-valued 1-forms.

Remark 3.5. The constant c that appears in the Poincaré inequality is also the first eigenvalue for the Laplace operator on functions with compact support [Ya], and this extablishes a link between the Laplacian on functions and the Laplacian on L_2 TM-valued 1-forms in the case of a compact orientable connected surface.

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