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Invertibility of some heat potentials in *BMO* norms

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Invertibility of Some Heat Potentials in BMO Norms.

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0. Introduction.

For C^1-domains D in R^n and L^p boundary data $(1 < p < \infty)$, Fabes and Riviere [1] considered the Initial-Dirichlet Problem for the (linear) heat equation

$$\text{(I.D.P.)} \quad \begin{cases} \Delta_x u - D_t u = 0 & \text{in the cylinder } D \times (0, T), \text{ uniformly} \\ \lim\limits_{t \to 0} u(X, t) = 0 & \text{on compacts in } D\,, \\ u(X, t) \to f(P, s) & \text{a.e. on the surface } \partial D \times (0, T)\,. \end{cases}$$

They proved the existence of a unique solution of (I.D.P.) given by the double-layer heat potential of a suitable transform of the boundary data f. Subsequently, in our paper [4], we began to examine a sort of regularity question arising by considering data f in appropriate BMO spaces on $\partial D \times (0, T)$. Due to the more local nature of these norms and to the higher regularity of BMO functions, two modifications

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were needed. The usual BMO norm had to be replaced by a caloric analogue, $BMOC$, reflecting the mixed homogeneity of the heat equation. Secondly, a kind of compatibility condition (with the constant initial data) was introduced in the form of restricting ourselves to a subspace B_0MOC of those f in $BMOC$ having bounded initial behavior at $t = 0$. On this subspace we proved in [4] the continuity of the boundary integral J, where J is the singular integral operator

$$[Jf](P,t) = \lim_{\varepsilon\to 0} \int^{t-\varepsilon} \int_{\partial D} K(P, Q, t-s) f(Q, s) dQ ds$$

and $K(P, Q, t-s)$ as defined below.

In this present paper, the invertibility of the boundary terms $(cI + J)$ in $B_0MOC(\partial D\times(0, T))$ is established. The technique used differs from [1] and elaborates the ideas in [4]. However, the dyadic decomposition of $\partial D\times(0, T)$ and the local analysis on « short time intervals » is finer than the one needed in [4].

Combining the main results here with those in [1] we deduce the unique solvability of (I.D.P.), by means of double-layer heat potentials, with data in the class B_0MOC.

We wish to thank Prof. Eugene Fabes for some helpful conversations concerning the construction in § 3 here.

1. Definitions and preliminaries.

If $D \subset R^n$, $n \geq 2$, is a bounded C^1 domain, we shall consider in the space-time $R^n\times R^+$, the cylinder $D_T = D\times(0, T)$, $0 < T < +\infty$, with lateral boundary $S_T = \partial D\times(0, T)$. Capital letters X, Y will denote points in D, while P, Q will denote points in ∂D. Letters t and s are used for time variables in R^+. For all $(X, t) \in R^n\times R^+$, we let

$$\Gamma(X,t) = (\pi t)^{-n/2} \exp[-|X|^2/4t]$$

denote the fundamental solution of the heat equation and

$$K(X,t) = \langle N_Q, \nabla_X \Gamma(X,t)\rangle$$

where N_Q is the inner unit normal, denote the kernel of the double layer heat potential. More explicity

$$(1.0)\qquad K(X, t) = c_n \frac{\langle X - Q, N_Q\rangle}{t^{(n/2+1)}} \exp\left[- |X - Q|^2/4t\right] .$$

If $(P, t) \in S_T$ we call

$$\Delta = \Delta_r(P, t) = \{(Q, s) \in S_T\colon |P - Q| < r,\ |s - t| < r^2\}$$

a caloric surface disc with center (P, t) and radius r, and for any $0 \leqslant a < T$

$$\Delta^a = \Delta_r^a(P) = \{(Q, s) \in S_T,\ |P - Q| < r,\ a < s < a + r^2\}$$

the initial caloric surface disc, with center P and radius r, with initial point a. Moreover we call

$$S = S_r(P) = \{Q \in D\colon |P - Q| < r\}$$

the spatial surface disc, with center P and radius r. We introduce the spaces $BMOC(S_T)$ and $B_0MOC(S_T)$ [4].

We say that $f \in BMOC(S_T)$ if

$$(1.1)\qquad \|f\|_* = \sup_{\Delta} \{|\Delta|^{-1}\int_{\Delta} |f - f_\Delta|\, dQ\, ds\} < +\infty$$

where $f_\Delta = |\Delta|^{-1}\int_{\Delta} f$.

With the identification $f_1 \sim f_2$ if $f_1 - f_2 =$ constant, $BMOC$ turns out a complete norm space with norm (1.1).

By the anisotropic John-Niremberg inequality, we have the equivalent norm

$$(1.2)\qquad \|f\|_{*,p} = \sup_{\Delta} \left\{|\Delta|^{-1}\int |f - f_\Delta|^p\right\}^{1/p} .$$

We say that $f \in B_0MOC$ if (1.1) is valid and

$$B_0(f) = \sup_{\Delta^0} \left| \left\{|\Delta^0|^{-1}\int_{\Delta^0} f\, dQ\, ds\right\} \right| < +\infty .$$

B_0MOC turns out a complete norm space if we equipe it with the norm

$$\|f\|_{0,*} = B_0(f) + \|f\|_* .$$

Set

$$C_p(f) = \sup_{\Delta^0} \left\{ |\Delta^0|^{-1} \int_{\Delta^0} |f|^p \, dQ \, ds \right\}^{1/p} .$$

since the finiteness of $B_0(f)$ is equivalent to that of $C_p(f)$ for any $1 \leqslant p < \infty$ (see [4]), it follows that we have also the equivalent norm

$$\|f\|_{p,*} = C_p(f) + \|f\|_{*,p}$$

More generally we shall deal with the space $B_aMOC(\partial D \times (a, b))$, $0 \leqslant a < b \leqslant T$. We say that $f \in B_aMOC(\partial D \times (a, b))$ if $f \in BMOC$ $(\partial D \times (a, b))$ and

$$B_a(f) = \sup_{\Delta^a} \left| \left\{ |\Delta^a|^{-1} \int_{\Delta^a} f \, dQ \, ds \right\} \right| < +\infty .$$

For these spaces, the norms

$$\|f\|_{a,*} = B_a(f) + \|f\|_* \quad \text{and} \quad \|f\|_{a,p,*} = C_{a,p}(f) + \|f\|_{*,p}$$

are equivalent, where $C_{a,p}(f)$ are the corresponding L^p-means relative to initial caloric surface discs Δ^a.

2. Behaviour of the operator J in the strip $\partial D \times (a, b) = S(a, b)$.

We know that the study of the double layer heat potential, give rise to the singular integral operator

$$(2.0) \qquad [Jf](P, t) = \lim_{\varepsilon \to 0} \int_0^{t-\varepsilon} \int_{\partial D} K(P, Q, t - s) f(Q, s) \, dQ \, ds$$

which is a bounded operator on $L^p(S_T)$, $1 < p < +\infty$, see [1]. In addition J is bounded on $B_0MOC(S_T)$, see [4].

Moreover as shown in [1], $cI + J$, $c \neq 0$ and I identity operator, is invertible in $L^p(S_T)$. This fact is obtained by showing that the operator J belongs to the class $\mathfrak{J}(S_T)$ of all bounded operators on $L^p(S_T)$ which satisfy the following two conditions

i) for all a, $0 < a \leq T$, $J\chi_{(a,\infty)} = \chi_{(a,\infty)} J \chi_{(a,\infty)}$

where $\chi_{(a,b)} =$ characteristic function of (a, b),

ii) if $(a, b) \subset (0, T)$, $\|J(\chi_{(a,b)}f)\|_{L^p(S(a,b))} \leq \omega_J(b-a)\|f\|_{L^p(S(a,b))}$

where $\omega_J(\delta) \to 0$ as $\delta \to 0$.
The aim of this work is to prove that J also belongs to a corresponding class $\mathfrak{J}(S_T)$ of bounded operators on $B_0MOC(S_T)$ and that $cI + J$ is invertible in this space, for $c \neq 0$.

Lemma 2.1. If $f \in B_a MOC(\partial D \times (a, b))$, with $(a, b) \subset (0, T)$, then $C_{a,2}(J(\chi_{(a,b)}f)) \leq \gamma(b-a)\, C_{a,2}(f)$, where $\gamma = \gamma_J$ and $\gamma(r) \to 0$ as $r \to 0$.

We observe that, since we work in the strip $\partial D \times (a, b)$, the initial surface disc Δ_r^a are truncated in the time dimension, that is

$$|\Delta_r^a| = cr^{n+1} \quad \text{if } r^2 < b - a\,; \qquad |\Delta_r^a| = cr^{n-1}(b-a) \quad \text{if } r^2 > b - a\,.$$

Now, recalling theorem 1.3 of [1], we have

$$\|J(\chi_{a,b)}f)\|_{L^2(\partial D \times (a,b))} \leq c\omega_J(b-a)\|f\|_{L^2(\partial D \times (a,b))} \tag{2.1}$$

In order to show the Lemma it is enough to examine the case $\omega_J(b-a) > b - a$. In fact if $\omega_J(b-a) \leq b - a$, we may take an $\tilde{\omega}_J$ such that $\tilde{\omega}_J(\delta) > \delta$, $\forall \delta$, and observe that the condition ii) of page 5 holds also for $\tilde{\omega}_J(b-a)$.

Let us fix $\Delta_r^a(P)$ and denote it simple by Δ; set $\omega_J(b-a) = \omega$, and suppose, as we may, $\omega < 1$.

We distinguish three cases:

$$\alpha_1)\ \ \omega > b - a \geq r^2\,, \qquad \alpha_2)\ \ \omega > r^2 > b - a\,, \qquad \alpha_3)\ \ r^2 > \omega > b - a\,.$$

Let us start with α_1).

Let p be an integer such that $\omega^{p/2(n+1)} \leq r^2 < \omega^{(p-1)/2(n+1)}$.

Set $^*\Delta = \Delta^a(2\omega^{p-2/4(n+1)})$, to simplyfy notation, let now $\delta = \omega^{1/4(n+1)}$

so that $^*\Delta = \Delta^a(2\delta^{p-2})$, and consider balls $S_j = S(2^j\delta^{p-2})$ and the discs $\Delta_j = \Delta^a(2^j\delta^{p-2})$. If χ_1 denote the characteristic function of $^*\Delta$, then

$$\left\{|\Delta|^{-1}\int_\Delta |J(\chi_{(a,b)}f)|^2\right\}^{1/2} \leqslant \left\{|\Delta|^{-1}\int_\Delta |J[\chi_{(a,b)}\chi_1 f]|^2\right\}^{1/2} + \\ + \left\{|\Delta|^{-1}\int_\Delta |J(\chi_{(a,b)}(1-\chi_1)f)|^2\right\}^{1/2} = A + B\,.$$

Recalling (2.1) we have

$$A < c\omega\left\{|^*\Delta|\cdot|\Delta|^{-1}\cdot|^*\Delta|^{-1}\int_{^*\Delta}|f|^2\right\}^{1/2} \leqslant c\omega^{3/4}C_{a,2}(f)$$

since $^*\Delta$ is a initial surface disc and

$$|^*\Delta|\cdot|\Delta|^{-1} = c2^{n+1}\delta^{(p-2)(n+1)}r^{-(n+1)} \leqslant c2^{n+1}\delta^{(p-2)(n+1)}\delta^{-p(n+1)}$$
$$\leqslant c2^{n+1}\omega^{-1/2} \quad \text{if } 2^2\delta^{2(p-2)} \leqslant b-a\,;$$
$$|^*\Delta|\cdot|\Delta|^{-1} = c2^{n-1}\delta^{(p-2)(n-1)}(b-a)r^{-(n+1)}$$
$$\leqslant c2^{n+1}\delta^{(p-2)(n-1)}(b-a)2^{-2}\delta^{-p(n+1)}$$
$$\leqslant c2^{n+1}\delta^{-2(n+1)} = c2^{n+1}\omega^{-1/2} \quad \text{if } 2^2\delta^{2(p-2)} > b-a\,.$$

In order to estimate the term B it suffices to show that there is a constant $M > 0$ such that

$$|J[(1-\chi_1)\chi_{(a,b)}f]| \leqslant M\omega C_{a,2}(f)\,.$$

If $(Q, s) \in \Delta_j - \Delta_{j-1}$, and $(P, t) \in \Delta$, $a < s < t$, then $2|P-Q| \geqslant 2^j\delta^{p-2}$. From (1.0) we have the estimate

$$|K(P-Q, t-s)| \leqslant c|P-Q|^{-(n+1)}\exp[-|P-Q|^2/8(t-s)]\,. \tag{2.2}$$

Hence, we can write

$$I \equiv |J((1-\chi_1)\chi_{(a,b)}f)| \leqslant c\sum_{j\geqslant 2}\{\exp[-2^{2j}\delta^{2(p-2)}/8\omega^{2(p-1)}]/\{2^{j(n+1)}\delta^{(n+1)(p-2)}\}\}\cdot \\ \cdot\int_{\Delta_j}|f| = c\sum_{j\geqslant 2}R_j\int_{\Delta_j}|f|$$

and

$$R_j = \begin{cases} c|\Delta_j|^{-1}\exp[-4^j/\delta^2] & \text{if } 2^{2j}\delta^{2(p-2)} \leqslant b-a \\ c\exp[-4^j/2\delta^2]/\{2^{2j}(2^{j(n-1)}\delta^{(p-2)(n-1)})\delta^{2(p-2)})\} & \\ \leqslant c\exp[-4^j/2\delta^2]|S_j|^{-1}(b-a)^{-1} = c\exp[\,\ldots\,]|\Delta_j|^{-1} & \\ & \text{if } 2^{2j}\delta^{2(p-2)} > b-a\,. \end{cases}$$

Then

$$I \leqslant c\sum_{j\geqslant 2}\exp[-4^j/8\delta^2]\,C_{a,1}(f) \leqslant$$
$$\leqslant c\exp[-4^2/8\delta^2]\Big\{\sum_{j\geqslant 3}\exp[-(4^j-4^2)/2\delta^2]\Big\}C_{a,1}(f)\,.$$

Taking in account that $\delta^2 < 1$ the last series is dominated by a constant independent of δ. Since $C_1(f) \leqslant C_2(f)$, the estimate on B is complete.

The proof of Case α_2) is similar to the previous case. However since $\delta^2 > b-a$, one shoud abserve that again

$$|{}^*\Delta|\cdot|\Delta|^{-1} \leqslant c2^{n-1}\delta^{(p-2)(n-1)}(b-a)\delta^{-p(n-1)}(b-a)^{-1} =$$
$$= c2^{n-1}\delta^{-2(n-1)} \leqslant c2^{n-1}\omega^{-1/2}$$

while for the R_j we have the second estimate only. For the Case α_3), let $k \geqslant 2$ be an integer such that $(k-1)\omega \leqslant r^2 < k\omega$. We consider the initial surface disc ${}^*\Delta = \Delta^a(2k^{1/2}\omega^{1/2-1/4n}) = \Delta^a(2\delta)$ with $\delta = k^{1/2}\omega^{1/2-1/4n}$ and for $j \geqslant 2$ we let $\Delta_j = \Delta^a(2^j\delta)$.

With the same meaning for A and B as above we have

$$A \leqslant c\omega^{3/8}C_{a,2}(f)$$

since

$$|{}^*\Delta|\cdot|\Delta|^{-1} = 2^{n-1}\delta^{n-1}(b-a)r^{-(n-1)}(b-a)^{-1} \leqslant$$
$$\leqslant 2^{n-1}\delta^{n-1}(k-1)^{-(n-1)/2}\omega^{-(n-1)/2} =$$
$$= 2^{n-1}(k/k-1)^{(n-1)/2}\omega^{-(n-1)/4n} \leqslant c2^{n-1}\omega^{-1/4}\,.$$

To estimate the term B, reasoning as in the previous cases, we observe that: if $(Q, s) \in \Delta_j - \Delta_{j-1}$ (so $a < s < b$, since $r^2 > b - a$) and $(P, t) \in \Delta$, we have $|P - Q| > c2^j\delta$, for some $c > 0$ independent of j. Hence using estimate (2.2), we obtain

$$I = |J(1-\chi_1)\chi_{(a,b)}f| \leqslant c\sum_{j\geqslant 2}\exp[-c2^{2j}\delta^2/2k\omega]2^{-j(n+1)}\delta^{-(n+1)}\cdot\int_{\Delta_j}|f| \leqslant$$

$$\leqslant c(k-1)\exp[\,\dots\,]\{k2^{2j}|S_j|(b-a)\omega^{-1/2n}\}^{-1}\int_{\Delta_j}|f| \leqslant c\omega^{1/2n}C_{a,1}(f)\,.$$

We note that the proof of Lemma 2.1 yields a function $\gamma = \omega^{1/s}$ for some $s > 1$. Since $0 < \omega < 1$, we have $\omega < \gamma$.

LEMMA 2.2. If $f \in B_aMOC(\partial D\times(a, b))$, $(a, b) \subset (0, T)$, then

$$\|J(\chi_{(a,b)}f)\|_{*,2} \leqslant \psi(b-a)\|f\|_{2,*}$$

where $\psi(r) \to 0$ as $r \to 0$.

PROOF. For any caloric surface disc $\Delta = S_r(P_0)\times(t_0 - r^2, t_0 + r^2)$ with $a < t_0 < b$, if $r^2 \geqslant b - a$ then $t_0 < a + r^2$ and Δ is an initial disc Δ^a in the strip $\partial D\times(a, b)$. Therefore

$$|\Delta|^{-1}\int_{\Delta}|Jf|^2 = |\Delta^a|^{-1}\int_{\Delta^a}|Jf|^2 < \{\gamma(b-a)C_{a,2}(f)\}^2$$

by Lemma 2.1.

Next, if $r^2 < b - a$, and γ is the function of Lemma 2.1, as observed at the beginning of this Lemma, we can examine only the case $\gamma(b-a) > b - a$.

Set $\gamma = \gamma(b-a)$ and we may assume that $\gamma < 1$.

We consider $\gamma \geqslant b - a > r^2$.

Reasoning as in Case α_1) of Lemma 2.1, let p be an integer such that $\delta^p \leqslant r < \delta^{p-1}$, with $\delta = \gamma^{1/4(n+1)}$ and let now

$$\text{(2.3)} \qquad {}^*\Delta = \Delta(\delta^{p-2})(P_0, t_0)\,.$$

We distinguish again two cases: β_1) $t_0 - a \leqslant \delta^{2(p-2)}$ and β_2) $t_0 - a > \delta^{2(p-2)}$.

Case $\beta)$: $t_0 - a \leqslant \delta^{2(p-2)}$.

If we also have $t_0 - a \leqslant r^2$, we can view Δ as an initial disc so that

$$|\Delta|^{-1}\int_{\Delta}|Jf|^2 \leqslant \{\gamma C_{a,2}(f)\}^2 .$$

When $r^2 < t_0 - a \leqslant \delta^{2(p-2)}$, letting $h = (t_0 - a + r^2)^{1/2}$, $\Delta \subset \Delta_h^a$ and hence

$$|\Delta|^{-1}\int_{\Delta}|Jf|^2 \leqslant |\Delta|^{-1}\int_{\Delta_h^a}|Jf|^2 \leqslant |\Delta|^{-1}|\Delta_h^a|\Big\{|\Delta_h^a|^{-1}\int_{\Delta_h^a}|Jf|^2\Big\} .$$

Now

$$|\Delta_h^a|\cdot|\Delta|^{-1} = (t_0 - a + r^2)^{(n+1)/2}(r^{-2})^{(n+1)/2} = \big(1 + (t_0 - a)/r^2\big)^{(n+1)/2} \leqslant$$

$$\leqslant \big((t_0 - a)/r^2 + (t_0 - a)/r^2\big)^{(n+1)/2} \leqslant \{2\delta^{-2p+2(p-2)}\}^{(n+1)/2} = c\gamma^{-1/2}$$

since $\delta = \gamma^{1/4(n+1)}$. Thus, again by Lemma 2.1,

$$|\Delta|^{-1}\int_{\Delta}|Jf|^2 \leqslant c\gamma^{-1/2}\{\gamma C_{a,2}(f)\}^2 = c\gamma^{3/2}C_{a,2}(f) .$$

Case $\beta_2)$: $t_0 - a > \delta^{2(p-2)}$. Here, $t_0 > \delta^{2(p-2)} > r^2$. Thus, if $J_1 = J_1(P, t) = [J(1)](P, t)$, Lemma 2.1 of [4] shows that there exist a constant $C(\Delta)$ such that, for any $(P, t) \in \Delta$

$$|J_1 - C(\Delta)| \leqslant crt_0^{-1/2} . \tag{2.4}$$

Next we let χ_1 be the characteristic function of $^*\Delta$, $f_1 = [f - f_{*\Delta}]\chi_1$ $f_2 = |f - f_{*\Delta}|(1 - \chi_1)$ and choose constant $Jf(\Delta) = J(f_2)(P_0, t_0) - f_{*\Delta}\cdot C(\Delta)$ with $f = f_{*\Delta} + f_1 + f_2$, we have

$$J(f)(P, t) - Jf(\Delta) = f_{*\Delta}|J_1(P, t) - C(\Delta)| + J(f_1) +$$

$$+ [-J(f_2)(P_0, t_0) + J(f_2)] .$$

We note that, by (2.4) and Holder's inequality

$$|f_{*\Delta}[J_1 - C(\Delta)]| \leqslant ct_0^{-1/2} r |{}^*\Delta|^{-1} \int\limits_{*\Delta} |f| \leqslant ct_0^{-1/2} \delta^{p-1} |{}^*\Delta|^{-1} \int\limits_{*\Delta} |f| \leqslant$$

$$\leqslant ct_0^{-1/2} \delta^{p-1} |{}^*\Delta|^{-1} |{}^*\Delta|^{n/n+1} \Big\{ \int\limits_{*\Delta} |f|^{n+1} \Big\}^{1/n+1} =$$

$$= ct_0^{-1/2} \delta \Big\{ \int\limits_{\Delta} |f|^{n+1} \Big\}^{1/n+1} \quad \text{since } |{}^*\Delta| = \delta^{(p-2)(n+1)} .$$

Moreover, since $t_0 > t_0 - a > \delta^{2(p-2)}$, we claim that ${}^*\Delta \subset \Delta^a((2t_0)^{1/2})$. Infact, for $t_0 + \delta^{2(p-2)} \geqslant b$, we see that $2t_0 - a = t_0 + (t_0 - a) > t_0 + \delta^{2(p-2)} > b$ so that $2t_0 > b - a$. For $t_0 + \delta^{2(p-2)} < b$ the initial surface disc of height $t_0 + \delta^{2(p-2)} - a$ and center P_0 contains ${}^*\Delta$. But, since $\delta^{2(p-2)} < t_0$, this height is $< 2t_0$ as desired. Therefore

$$t_0^{-1/2} \Big\{ \int\limits_{*\Delta} |f|^{n+1} \Big\}^{1/n+1} \leqslant \Big\{ t_0^{-(n+1)/2} \int\limits_{\Delta^a((2t_0)^{1/2})} |f|^{n+1} \Big\}^{1/n+1} = cC_{a,q}(f) , \quad \text{with } q = n+1 .$$

Consequently,

$$f_{*\Delta}[J_1 - C(\Delta)] \leqslant c\gamma^{1/4(n+1)} C_{a,q}(f)$$

and hence

$$\Big\{ |\Delta|^{-1} \int\limits_{\Delta} |f_{*\Delta}|^2 |J_1 - C(\Delta)|^2 \Big\}^{1/2} \leqslant c\gamma^{1/4(n+1)} C_{a,2}(f) ,$$

by the equivalence of $C_p(f)$ for various $p \geqslant 1$.

By (2.3) with $\delta = \gamma^{1/4(n+1)}$, we have $\big(|{}^*\Delta| \cdot |\Delta|^{-1}\big) < c_n \gamma^{-1/2}$ as in the proof of Lemma 2.1. Hence, using (2.1), we have

$$\Big\{ |\Delta| \int\limits_{\Delta} |J(f_1)|^2 \Big\}^{1/2} \leqslant c\omega \Big\{ \big(|{}^*\Delta| \cdot |\Delta|^{-1}\big) |{}^*\Delta|^{-1} \int\limits_{*\Delta} |f_1|^2 \Big\}^{1/2} \leqslant c\omega\gamma^{-1/4} \|f\|_{*,2} .$$

Since, as noted above, $\omega < \gamma$,

$$\Big\{ |\Delta|^{-1} \int\limits_{\Delta} |J(f_1)|^2 \Big\}^{1/2} \leqslant c\gamma^{3/4} \|f\|_{*,2} .$$

In order to estimate the term

$$B = \left\{|\Delta|^{-1}\int_{\Delta}|J(f_2)(P,t) - J(f_2)(P_0,t_0)|dP\,dt\right\}$$

let us examine the integrand

$$|J(f_2)(P,t) - J(f_2)(P_0,t_0)|$$

which is majorized by term

$$\int_{S(a,b)\setminus{}^*\Delta}|K(P-Q,t-s) - K(P_0-Q,t_0-s)|\,|f_2(Q,s)|dQ\,ds$$

where $S(a,b) = \partial D\times(a,b)$. Following [4] we add and subctrat $K(P-Q,t_0-s)$ and use the Mean-Value Theorem, to see that

$$|K(P-Q,t-s) - K(P_0-Q,t_0-s)| \leqslant |D_tK(P-Q,\tilde{t}-s)||t-t_0| + \\ + |\nabla_t K(\tilde{P}-Q,t_0-s)|\cdot|P-P_0| = B_1 + B_2$$

for some $\tilde{t}$ between t and t_0 and $\tilde{P}$ some intermediate point between P_0 and P. Interchanging the order of integration

$$B \leqslant \int_{S(a,b)\setminus{}^*\Delta}|f - f_{*\Delta}|\left\{|\Delta|^{-1}\int_{\Delta}(B_1+B_2)dP\,dt\right\}dQ\,ds\,.$$

Let now $\Delta = \Delta_0$, ${}^*\Delta = \Delta_1$, $S_j = S(2^j\delta^{p-2})$ and $\Delta_j = \Delta(2^j\delta^{p-2}) = S_j^X(t_0 - 2^{2j}\delta^{2(p-2)}, t_0 + 2^{2j}\delta^{2(p-2)})$. If $(P,t)\in\Delta$ and $(Q,s)\in\Delta_j - \Delta_{j-1}$, the same estimates of Theorem 2.3 in [4] yield

$$B_1 \leqslant c(t-t_0|\,|\tilde{t}-s|^{-(n+3)/2} \quad \text{for } |P_0-Q| < 2^{j-1}\delta^{p-2}$$

$$B_2 \leqslant c|t-t_0|\,|\tilde{P}-Q|^{-(n+3)} \quad \text{for } |P_0-Q| \geqslant 2^{j-1}\delta^{p-2}$$

Moreover, since $r < \delta^{p-1}$, if $|P_0-Q| < 2^{j-1}\delta^{p-2}$, we have

$$|\tilde{t}-s| \geqslant \|s-t_0| - |t_0-\tilde{t}\| \geqslant 2^{-1}(2^{2(j-1)}\delta^{2(p-2)} - \delta^{2(p-1)}) \geqslant \\ \geqslant 2^{-1}\delta^{2(p-2)}(2^{2(j-1)}-1) \geqslant c\,2^{2j}\delta^{2(p-2)}\,.$$

similary, if $|P_0 - Q| \geqslant 2^{j-1}\delta^{p-2}$ both $|P - Q|$, $|\tilde{P} - Q| \geqslant c2^j\delta^{p-2}$.
Consequently, if $2^j\delta^{2(p-2)} \leqslant b - a$,

$$B_1 \leqslant c\delta^{2(p-1)}2^{-j(n+3)}\delta^{-(p-2)(n+3)} = c\delta^{2(p-1)}2^{-2j}\delta^{-2(p-2)}|\Delta_j|^{-1} = c\delta^2 2^{-2j}|\Delta_j|^{-1}$$

while if $2^{2j}\delta^{2(p-2)} > b - a$

$$B_1 \leqslant c\delta^{2(p-1)}2^{-j(n+3)}\delta^{-(p-2)(n+3)} = c\delta^{2(p-1)}(b-a)[2^{j(n-1)}\delta^{(p-2)(n-1)}2^{4j}\delta^{4(p-2)}]^{-1} \cdot$$
$$\cdot (b-a)^{-1} \leqslant c\delta^{2(p-1)}2^{2j}\delta^{2(p-2)}|\Delta_j|^{-1}2^{-4j}\delta^{-4(p-2)} = c\delta^2|\Delta_j|^{-1}2^{-2j} .$$

So, in both cases, $B_1 \leqslant c2^{-j}|\Delta_j|^{-1}\delta$.
In the same manner, we obtain

$$B_2 \leqslant c\delta 2^{-j}|\Delta_j|^{-1} \quad \text{when } (P,t) \in \Delta \text{ and } (Q,s) \in \Delta_j - \Delta_{j-1} .$$

Therefore, we have

$$B \leqslant \sum_{j\geqslant 2} \int_{\Delta_j - \Delta_{j-1}} |f - f_{*\Delta}| \Big\{ |\Delta|^{-1} \int_{\Delta} (B_1 + B_2)\, dP\, dt \Big\}\, dQ\, ds \leqslant$$
$$\leqslant c \sum_{j\geqslant 2} \Big\{ \delta 2^{-j}|\Delta_j|^{-1} \int_{\Delta_j} |f - f_{*\Delta}|\, dQ\, ds \Big\} \leqslant c \sum_{j\geqslant 2} \delta 2^{-j}\{1 + (j-1)2^{n+1}\} \|f\|_{*,2} \leqslant$$
$$\leqslant c\gamma^{1/4(n+1)}\|f\|_{*,2} \quad \text{with norms on } \partial D \times (a,b) .$$

Combining the two Cases, the conclusion follows at once.

COROLLARY 2.3. Let $f \in B_a MOC(\partial D \times (a,b))$ with $(a,b) \subset (0,T)$. Then

$$\|J(\chi_{(a,b)} f)\|_{2,*} \leqslant \varphi(b-a)\|f\|_{2,*} \tag{2.5}$$

where $\varphi(r) > 0$ and $\varphi(r) \to 0$ as $r \to 0$.
This follows from Lemma 2.1 and 2.2 with $\varphi = \gamma + \psi$.

3. Construction of the global solution of $(J + cI)f = g$ on S_T.

We shall construct the global solution f of $(J + cI)f = g$ for a given $g \in B_0 MOC(S_T)$. First let us verify, given numbers c, d, m such that $c < d < m < T$, if $h \in BMOC(\partial D \times (c, m))$ and $h(P, t) = 0$ on $\partial D \times (c, d)$, then $h \in B_d MOC(\partial D \times (d, m))$.

In fact, if $'\Delta_r = \Delta_r(P, d) = S_r \times (d - r^2, d + r^2)$ and $h_r = |\Delta_r|^{-1}\int_{\Delta} h$, we have

$$\|h\|_{BMOC(\partial D \times (c,m))} \geq |\Delta_r|^{-1}\int_{\Delta r}|h - h_r| \geq |\Delta_r|^{-1}\int_{d-r^2}^{d}\int_{S_r}|h - h_r| =$$

$$= h_r/2 = 2^{-1}\left||\Delta_r|^{-1}\int_{d}^{d+r^2}\int_{S_r} h\right|$$

that is $h \in B_d MOC(\partial D \times (d, m))$.

THEOREM 3.1. The operator $J + cI$ is invertible on

$$B_0 MOC(\partial D \times (0, T)) .$$

PROOF. A standard argument, see [1], [2], shows that the operator is one-to-one. Let $g \in B_0 MOC(\partial D \times (0, T))$. We partition $(0, T)$ in $N = N(\varepsilon)$ subinterval of length $\varepsilon > 0$, so small that $cI + J$ is invertible on each of the spaces $B_{k\varepsilon} MOC\big(\partial D \times (k\varepsilon, (k + 1)\varepsilon)\big)$, $k = 0, 1, \ldots, N - 1$, by Corollary 2.3.

Consider g on $\partial D \times (0, \varepsilon)$ only. Since $g \in B_0 MOC(\partial D \times (0, \varepsilon))$, there exists $f_1 \in B_0 MOC(\partial D \times (0, \varepsilon))$ such that $(J + cI)f_1 = g$. Next let $\tilde{f}_1$ be any $B_0 MOC$ extension of f_1 to $\partial D \times (0, 2\varepsilon)$ for example:

$$\tilde{f}_1(P, t) = \begin{cases} f_1(P, t) & \text{for } t \in (0, \varepsilon) \\ f_1(P, 2\varepsilon - t) & \text{for } t \in (\varepsilon, 2\varepsilon) . \end{cases}$$

Since $\tilde{f}_1 \in B_0 MOC(\partial D \times (0, 2\varepsilon))$ so does $(J + cI)\tilde{f}_1$ by [4]. Clearly $g - (J + cI)\tilde{f}_1$ is in $B_0 MOC(\partial D \times (0, 2\varepsilon))$, and is identically zero in $\partial D \times (0, \varepsilon)$. Thus, by the remark preceding the Theorem $g - (J + cI)\tilde{f}_1$ is in $B_\varepsilon MOC(\partial D \times (\varepsilon, 2\varepsilon))$, and hence there exists an f_2 in $B_\varepsilon MOC$ $(\partial D \times (\varepsilon, 2\varepsilon))$ such that $(J + cI)f_2 = g - (J + cI)\tilde{f}_1$. If we extend f_2 to equal zero on $\partial D \times (0, \varepsilon)$, we obtain

$$(J + cI)(\tilde{f}_1 + f_2) = g \quad \text{on } D \times (0, 2\varepsilon)$$

and $\tilde{f}_1 + f_2$ remains in $B_0 MOC(\partial D \times (0, 2\varepsilon))$. Iterating this process we obtain a function $f \in B_0 MOC(\partial D \times (0, T))$ such that $(J + cI)f = g$.

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