RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

ALBERTO VALLI

On the integral representation of the solution to the Stokes system

Rendiconti del Seminario Matematico della Università di Padova, tome 74 (1985), p. 85-114

<http://www.numdam.org/item?id=RSMUP_1985__74__85_0>

Rend. Sem. Mat. Univ. Padova, Vol. 74 (1985)

On the Integral Representation of the Solution to the Stokes System.

Alberto Valli (*)

1. Introduction.

In this paper we want to present in a detailed form the methods and the calculations which permit to obtain the representation formulas for the solution to the Stokes system

$$
\begin{cases}
\Delta v - \nabla p = f & \text{in } \Omega\,, \\
\operatorname{div} v = g & \text{in } \Omega\,, \\
v|_{\partial\Omega} = \varphi & \text{on } \partial\Omega\,.
\end{cases}
\tag{1.1}
$$

The method which we shall follow is well-known since the second half of the nineteenth century, and is called the Green's method. It was already applied to the Stokes system long time ago (indeed, always for $g = 0$; in this case (1.1) describes the stationary « slow » motion of an incompressible homogeneous viscous fluid), and one can find in several papers the calculations which lead to the representation formulas. However, despite these numerous results, the situation doesn't appear really clarified, at least for a reader which is not a specialist in this field.

In fact, excepting for the case $g = 0$, $\varphi = 0$, for which the formnlas

(*) Indirizzo dell'A.: Dipartimento di Matematica, Università di Trento, 38050 Povo (Trento), Italy.
Work partially supported by G.N.A.F.A. of C.N.R.

are effectively well-known (see [25]; [17]; [32] pag. 162, 281; [15] pag. 65; [29]; [28]; [5]; [2] pag. 272; but formulas $(25)_2$ in [32], pag. 281 and (31) in [14] are not correct), the relations obtained for $f = 0$, $g = 0$ don't seem to be exact, and moreover they are usually stated in the literature without proof, simply by replacing the fundamental singular solutions with the Green's functions, which nevertheless do not satisfy all the properties required by the calculations performed.

For instance, the formula for $v(x)$ in [25], [32] pag. 162, 281, [2] pag. 271 and [16] is given in terms of H_k^i and H_k, while the correct expression needs G_k^i and G_k (compare with § 3, (3.4) and (3.18); to our knowledge, formula (3.4) was obtained for the first time by Oseen [27], pag. 27). Moreover the problem of finding a representation for $p(x)$ seems to be completely not clarified, since it is apparent that the formulas given in [25], [32] pag. 162, 281 are not correct (check the sketch of the proof given there, and compare the result with § 3, (3.28) or (3.24); in the book of Oseen a formula for the pressure $p(x)$ is not obtained, excepting when Ω is a ball; see [27], pag. 106). To our knowledge, the « general » case $f \neq 0$, $g \neq 0$, $\varphi \neq 0$ is considered only in the paper [5], but equation (26) obtained there is not correct (put for instance $f = \nabla g$, $\varphi = 0$ in that formula).

In our presentation we want to follow closely the classical methods introduced for the Stokes system by Odqvist [25] (and reproduced also in the book of Ladyzhenskaya [15]), despite this procedure is not the most direct one for getting the representation formula for $p(x)$. However, we give also an alternative proof which is more natural (see Remark 3.3), and we analyse in detail the relations between these two approachs (see Remark 3.4) (1).

One must observe moreover that the calculations employed to get (3.4) and (3.24) are quite simple (as we already said, the way for obtaining (3.19) or (3.28) is a little more complicated). It is appropriate to recall again that in this paper we obtain *expected* results by *classical* methods, following the approach given by the Green's method to get the representation formulas. Nevertheless, we repeat that these formulas don't seem to have been yet explicitly presented in a correct way in the current literature.

Let us spend now some words about the Green's method (and its application to Stokes system), which is one of the most classical

(1) Nevertheless, it is clear that the approachs presented here don't exhaust all the possible methods to get the representation formulas.

methods for showing the existence of a solution to partial differential equations. It is well-known that it consists essentially in these steps: (i) write a Green's formula, obtained by considering the differential operator and its adjoint and by integrating by parts in the domain Ω; (ii) find the fundamental singular solution of the adjoint equation; (iii) insert this solution in the Green's formula, getting in this way a representation for any regular solution of the equation. This representation formula usually contains an integral on the boundary $\partial\Omega$ which doesn't depend explicitly on the data of the problem. Hence one is led to determine a Green's function, that is a fundamental solution whose value on $\partial\Omega$ makes this additional term to be zero.

This method was applied in the first three decades of the century also to the Stokes system (1.1) (indeed, as we already said, for $g = 0$). In 1896 Lorentz [19] (reproduced in [20], pag. 23-42) found the fundamental tensor

$$u_k^i(x, y) = -\frac{1}{8\pi}\left[\frac{\delta_{ki}}{|x-y|} + \frac{(x_k - y_k)(x_i - y_i)}{|x-y|^3}\right], \tag{1.2}$$

$$q_k(x, y) = -\frac{1}{4\pi}\frac{x_k - y_k}{|x-y|^3}, \tag{1.3}$$

which satisfies for a fixed $x \in \mathbb{R}^3$

$$\begin{cases} \Delta_y u_k^i(x, y) + \nabla_{y_i} q_k(x, y) = \delta_{ki}\delta(x-y)\,, & y \in \mathbb{R}^3, \\ \nabla_{y_i} u_k^i(x, y) = 0\,, & y \in \mathbb{R}^3, \end{cases} \tag{1.4}$$

and for a fixed $y \in \mathbb{R}^3$

$$\begin{cases} \Delta_x u_k^i(x, y) - \nabla_{x_i} q_k(x, y) = \delta_{ki}\delta(x-y)\,, & x \in \mathbb{R}^3, \\ \nabla_{x_i} u_k^i(x, y) = 0\,, & x \in \mathbb{R}^3, \end{cases} \tag{1.5}$$

Here and in the sequel we adopt the Einstein convention about summation over repeated indices; δ_{ki} is the Kronecker's symbol; ∇_x and ∇_y mean that the differentiation is taken in the first three variables or in the second three variables, respectively; $\delta(x-y)$ is the Dirac delta « function ». One sees in particular that

$$q_k(x, y) = \nabla_{x_k} E(x, y) = -\nabla_{y_k} E(x, y)\,, \tag{1.6}$$

where

$$E(x, y) = \frac{1}{4\pi|x-y|} \tag{1.7}$$

is the fundamental solution for $-\Delta$.

By writing the Stokes system in the following way

$$L(\boldsymbol{v}, p) \equiv (\Delta v - \nabla p, -\operatorname{div} v) , \tag{1.8}$$

the adjoint is given by

$$L^*(\boldsymbol{u}, q) \equiv (\Delta u + \nabla q, \operatorname{div} u) , \tag{1.9}$$

and the fundamental Lorentz tensor satisfies in this notation

$$L_y^*(\boldsymbol{u}_k, q_k) = \big(\delta(x-y)\, \boldsymbol{e}_k, 0\big) , \tag{1.10}$$

and

$$L_x(\boldsymbol{u}_k, q_k) = \big(\delta(x-y)\, \boldsymbol{e}_k, 0\big) , \tag{1.11}$$

where $\boldsymbol{e}_k$ is the unit vector directed along the k-th coordinate axis (we write the index denoting the component of a vector *over* the vector itself).

By means of this fundamental tensor it is easy to get representation formulas for the solution of (1.1) (see § 2, (2.8) and (2.9)). By looking at these formulas, it is clear that, if one wants to express v in terms of Ω, f, g and φ solely, one needs to find a fundamental solution $(\boldsymbol{G}_k, G_k)$ which satisfies $L_y^*(\boldsymbol{G}_k, G_k)(x, y) = \big(\delta(x-y)\boldsymbol{e}_k, 0\big)$ and such that $\boldsymbol{G}_k$ takes value zero for $y \in \partial\Omega$. This can be done by solving the problem

$$\begin{cases} \Delta_y g_k^i(x, y) + \nabla_{y_i} g_k(x, y) = 0 , & y \in \Omega, \\ \nabla_{y_i} g_k^i(x, y) = 0 , & y \in \Omega, \\ g_k^i(x, y)|_{y\in\partial\Omega} = u_k^i(x, y)|_{y\in\partial\Omega} , & y \in \partial\Omega, \end{cases} \tag{1.12}$$

(here x is a fixed point in Ω), and by choosing $(\boldsymbol{G}_k, G_k) \equiv (\boldsymbol{u}_k - \boldsymbol{g}_k, q_k - g_k)$. However, it is clear from the divergence theorem that, for solving (1.12), it is necessary that

$$\int_{\partial\Omega} u_k^i(x, y)\, n^i(y)\, d\sigma_y = 0 \quad \text{for each } k, \quad x \in \Omega\,, \tag{1.13}$$

where $n(y)$ is the unit outward normal vector to $\partial\Omega$ in y.

This condition is obviously satisfied, by $(1.4)_2$, but one has to prove that it is also sufficient for having the solution of (1.12).

In 1908 Korn [11] (see also [12]) showed, by a method of successive approximations, the existence of a unique solution to (1.1) (with $g = 0$, $\operatorname{div} f = 0$; however, for Stokes problem this last condition is not restrictive). He assumed that the data of the problem were regular enough, and that the (necessary) condition

$$\int_{\partial\Omega} \varphi(y) \cdot n(y)\, d\sigma_y = 0 \tag{1.14}$$

was satisfied. In particular, he showed the existence of a solution to (1.12), that is the existence of the Green's functions

$$G_k^i(x, y) \equiv u_k^i(x, y) - g_k^i(x, y)\,, \tag{1.15}$$

$$G_k(x, y) \equiv q_k(x, y) - g_k(x, y)\,. \tag{1.16}$$

In 1928-1930 Odqvist [24], [25] (see also the contribution of Faxén [8], Villat [32], chap. IV, V, IX) proved the same result by following the Green's method that we have explained up to now. Moreover, he studied the properties of the functions g_k^i and g_k, solutions of (1.12) (more precisely, he considered the functions h_k^i and h_k; see § 3, (3.13) and (3.14)), obtaining in this way representation formulas for the solution to (1.1) (with $g = 0$) depending only on the data Ω, f, φ (2) (3).

(2) However, as we said, these formulas seem to be correct only for $f \neq 0$, $\varphi = 0$.

(3) Another method for proving the existence theorem (always for $g = 0$) was introduced by Crudeli [7], who however completed the calculations only when Ω is a ball. Lichtenstein [18] extended the result to any regular bounded domain $\Omega \subset \mathbf{R}^3$. Other partial results about this problem were proved by Boggio [3] (Ω a ball); Oseen [27] (explicit construction of the Green's

It is appropriate to observe now that the Gıeen's method gives indeed a representation formula for a regular solution which we suppose to exist. Hence, for completing the argument, one has also to show in some way that the solution does exist, for instance by verifying directly that the functions expressed by the formula that we have obtained really satisfy the equation. (This procedure is usually called the « synthesis of the solution »: see for instance Krzyżański [13], pag. 239). We shall perform these calculations in § 4, proving in this way the existence theorem in the regular case by a direct approach.

Some remarks are now appropriate.

1) The case $g \neq 0$, which was not considered in the classical papers, since it has not a clear physical meaning, it is interesting for the study of compressible problems, both in the stationary and in the non-stationary case (see Matsumura-Nishida [21]; Valli [31]).

2) The existence theorem is well-known also by a variational approach, and anyway the « general » case $f \neq 0$, $g \neq 0$, $\varphi \neq 0$ can be reduced to the case $f \neq 0$, $g = 0$, $\varphi = 0$ by a standard argument (see for instance Temam [30], pag. 23, 31). Moreover a regularity theory can be developed by means of the a-priori estimates of Cattabriga [5] (see also § 4, Remark 4.6). Hence one can perform the synthesis of the solution also in this way (see for instance, in another context, the procedure adopted by Folland [9], pag. 109-112, 342-345). However, we think that a simple proof by a direct argument is interesting on its own.

For completing the review on the results about representation formulas, we want to recall also that Bogovskiĭ [4] has obtained an explicit formula for a solution of

$$
\begin{cases} \operatorname{div} v = g \quad \text{in } \Omega\,, & \int_\Omega g(y)\, dy = 0\,, \\ v|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega\,, \end{cases} \tag{1.17}
$$

in terms of the datum g. However, he doesn't utilize Green's func-

functions for a ball, pp. 25-28, 97-106; see also [26]); Modjtahédi [23] (Ω a ball; see also Villat [32], pp. 257-267). For other informations about these classical results, see Berker [2], pp. 262-276.

In a much more generale context, Colautti [6] proved the existence of the solution and obtained a representation formula in the case $g = 0$, $\varphi = 0$. The methods employed by this last author seem to be the most fruitful with regard to numerical approximations.

tions, and among the family of solutions that he finds in general none satisfies $\Delta v - \nabla p = 0$.

At the end of this introduction we remark that in the sequel we shall assume that Ω is a bounded connected open subset of $\mathbf{R}^3$, and its boundary $\partial\Omega$ is a regular manifold, say $\partial\Omega \in C^3$. Consequently, the functions $g_k^i(x, y)$ and $g_k(x, y)$ defined in § 3, (3.3) satisfy $g_k^i(x, \cdot) \in C^{2+\lambda}(\bar{\Omega}_y)$, $g_k(x, \cdot) \in C^{1+\lambda}(\bar{\Omega}_y)$ for each $x \in \Omega$, $0 < \lambda < 1$. (We denote by $C^{k+\lambda}(\bar{\Omega})$, $k \in \mathbf{N}$, $0 < \lambda < 1$ the usual Hölder's spaces, and $W^{s,p}(\Omega)$ or $W^{s,p}(\partial\Omega)$, $s \geqslant -1$, $1 < p < +\infty$, the usual Sobolev's spaces).

Finally, we shall use freely the properties of the Dirac delta «function» $\delta(x - y)$: however all the calculations can be developed in a classical way by deleting from Ω a small ball $B_\varepsilon(x)$ of center x and radius ε, and by taking the limit as $\varepsilon \to 0^+$.

2. Green's formulas.

By setting

$$T_{ij}(\boldsymbol{v}, p) \equiv -p\delta_{ij} + \nabla_i v^j + \nabla_j v^i ,$$

$$T'_{ij}(\boldsymbol{u}, q) \equiv q\delta_{ij} + \nabla_i u^j + \nabla_j u^i ,$$

one obtains at once (see also Odqvist [25]; Ladyzhenskaya [15], pag. 53)

$$\begin{aligned}(2.1)\qquad &\int_\Omega [(\Delta v - \nabla p + \nabla \operatorname{div} v) \cdot u - p \operatorname{div} u - \\ &\qquad - (\Delta u + \nabla q + \nabla \operatorname{div} u) \cdot v - q \operatorname{div} v]\, dy = \\ &\qquad\qquad = \int_{\partial\Omega} [T_{ij}(\boldsymbol{v}, p) u^i n^j - T'_{ij}(\boldsymbol{u}, q) v^i n^j]\, d\sigma_y ,\end{aligned}$$

where n is the unit outward normal vector to $\partial\Omega$, and (v, p), (u, q) are smooth functions.

By setting

$$(2.2)\qquad \begin{cases} \Delta v - \nabla p \equiv f , \\ \operatorname{div} v \equiv g , \\ v|_{\partial\Omega} \equiv \varphi , \end{cases}$$

and choosing in (2.1) $(u^i, q) \equiv (u^i_k, q_k)$ for each $k = 1, 2, 3$, one gets easily by (1.10) that for $x \in \Omega$

$$
(2.3) \qquad v^k(x) = \int_\Omega u^i_k(x, y)[f^i(y) + \nabla_i g(y)]\, dy - \int_\Omega q_k(x, y) g(y)\, dy + \\
+ \int_{\partial\Omega} [T'_{ij}(\boldsymbol{u}_k, q_k)_y (x, y) \varphi^i(y) n^j(y) - T_{ij}(\boldsymbol{v}, p)(y) u^i_k(x, y) n^j(y)]\, d\sigma_y .
$$

By proceeding as in [25], [15] and observing that

$$
(2.4) \qquad -\Delta\Big[\int_\Omega q_k(x, y) g(y)\, dy\Big] = -\nabla_k \Delta\Big[\int_\Omega E(x, y) g(y)\, dy\Big] = \nabla_k g(x) ,
$$

one also gets for $x \in \Omega$

$$
(2.5) \qquad p(x) = 2g(x) + \int_\Omega q_i(x, y)[f^i(y) + \nabla_i g(y)]\, dy + \\
+ \int_{\partial\Omega} [-2\nabla_{x_j} q_i(x, y) \varphi^i(y) n^j(y) - T_{ij}(\boldsymbol{v}, p)(y) q_i(x, y) n^j(y)]\, d\sigma_y ,
$$

up to an additive constant.

Moreover, by (1.10) one has

$$
(2.6) \qquad \int_\Omega u^i_k(x, y)\, \nabla_i g(y)\, dy = \int_{\partial\Omega} u^i_k(x, y) n^i(y) g(y)\, d\sigma_y ,
$$

and by $\nabla_{y_i} q_i(x, y) = -\Delta_y E(x, y) = \delta(x - y)$ one gets

$$
(2.7) \qquad \int_\Omega q_i(x, y)\, \nabla_i g(y)\, dy = -g(x) + \int_{\partial\Omega} q_i(x, y) n^i(y) g(y)\, d\sigma_y .
$$

Hence we can write the Green's formulas in this way:

$$
(2.8) \qquad v^k(x) = \int_\Omega u^i_k(x, y) f^i(y)\, dy - \int_\Omega q_k(x, y) g(y)\, dy + \\
+ \int_{\partial\Omega} [T'_{ij}(\boldsymbol{u}_k, q_k)_y (x, y) \varphi^i(y) n^j(y) - T_{ij}(\boldsymbol{v}, p)(y) u^i_k(x, y) n^j(y) + \\
+ u^i_k(x, y) n^i(y) g(y)]\, d\sigma_y ,
$$

$$p(x) = \int_{\Omega} q_i(x, y) f^i(y)\, dy + g(x) + \int_{\partial\Omega} [-2\nabla_{x_j} q_i(x, y) \varphi^i(y) n^j(y) - \\ - T_{ij}(v, p)(y) q_i(x, y) n^j(y) + q_i(x, y) n^i(y) g(y)]\, d\sigma_y \tag{2.9}$$

(of course, (2.9) is satisfied up to an additive constant).

REMARK 2.1. Formula (2.9) for $p(x)$ can be obtained also by choosing in (2.1) $(u^i, q) \equiv (q_i, 0)$ (see Villat [32], pag. 134, 160).

Furthermore, we can choose also $(u^i, q) \equiv (q_i, -\delta)$, that is the solution of

$$\begin{cases} \Delta_y u^i(x, y) + \nabla_{y_i} q(x, y) = 0\,, \\ \nabla_{y_i} u^i(x, y) = \delta(x - y)\,. \end{cases}$$

The matrix

$$U \equiv \begin{bmatrix} u_1^1 & u_2^1 & u_3^1 & q_1 \\ u_1^2 & u_2^2 & u_3^2 & q_2 \\ u_1^3 & u_2^3 & u_3^3 & q_3 \\ q_1 & q_2 & q_3 & -\delta \end{bmatrix}$$

is the fundamental solution for both the operators L_y^* and L_x.

3. Green's tensor.

Construct now the Green's tensor by setting

$$G_k^i(x, y) \equiv u_k^i(x, y) - g_k^i(x, y)\,, \tag{3.1}$$

$$G_k(x, y) \equiv q_k(x, y) - g_k(x, y)\,, \tag{3.2}$$

where g_k^i and g_k are the solution of

$$\begin{cases} \Delta_y g_k^i(x, y) + \nabla_{y_i} g_k(x, y) = 0\,, & y \in \Omega\,, \\ \nabla_{y_i} g_k^i(x, y) = 0\,, & y \in \Omega\,, \\ g_k^i(x, y)|_{y\in\partial\Omega} = u_k^i(x, y)|_{y\in\partial\Omega}\,, & y \in \partial\Omega\,, \end{cases} \tag{3.3}$$

for $x \in \Omega$. As we already said, the existence of g^i_k, g_k follows from well known results (see for instance Ladyzhenskaya [15], pag. 60), since

$$\int_{\partial\Omega} u^i_k(x, y) n^i(y)\, d\sigma_y = \int_{\Omega} \nabla_{y_i} u^i_k(x, y)\, dy = 0\,.$$

Moreover, $g_k(x, y)$ is defined up to an additive function of x. It is easily verified that G^i_k and G_k satisfy $L^*_y(\boldsymbol{G}_k, G_k)(x, y) = (\delta(x-y)\boldsymbol{e}_k, 0)$. Hence we can repeat the same calculations performed in § 2 by choosing in (2.1) $(u^i, q) \equiv (G^i_k, G_k)$, and we obtain

$$v^k(x) = \int_{\Omega} G^i_k(x, y) f^i(y)\, dy - \int_{\Omega} G_k(x, y) g(y)\, dy + \\ + \int_{\partial\Omega} T'_{ij}(\boldsymbol{G}_k, G_k)_y\, (x, y)\varphi^i(y) n^j(y)\, d\sigma_y\,,$$

since $G^i_k(x, y)|_{y\in\partial\Omega} = 0$. Moreover

$$\int_{\partial\Omega} \nabla_{y_i} G^j_k(x, y)\varphi^i(y) n^j(y)\, d\sigma_y = \int_{\Omega} \nabla_{y_j}[\nabla_{y_i} G^j_k(x, y) v^i(y)]\, dy = \\ = \int_{\Omega} \nabla_{y_i} G^j_k(x, y) \nabla_j v^i(y)\, dy = -\int_{\Omega} G^j_k(x, y) \nabla_j g(y)\, dy = \\ = \int_{\Omega} \nabla_{y_j} G^j_k(x, y) g(y)\, dy = 0\,.$$

Hence

$$v^k(x) = \int_{\Omega} G^i_k(x, y) f^i(y)\, dy - \int_{\Omega} G_k(x, y) g(y)\, dy + \\ + \int_{\partial\Omega} [G_k(x, y)\, \delta_{ij} + \nabla_{y_j} G^i_k(x, y)]\varphi^i(y) n^j(y)\, d\sigma_y\,. \tag{3.4}$$

This is Green's representation formula for the solution of (1.1).

We can obtain an analogous formula for $p(x)$. First of all we need to show that

$$G^i_k(x, y) = G^k_i(y, x) \text{ for each } x, y \in \bar{\Omega},\ x \neq y\,. \tag{3.5}$$

This is easily proved for $x, y \in \Omega$, $x \neq y$ by choosing in (2.1)

$$v^i(z) \equiv G^i_j(x, z) , \qquad p(z) \equiv - G_j(x, z) ,$$

$$u^i(z) \equiv G^i_k(y, z) , \qquad q(z) \equiv G_k(y, z) ,$$

moreover for $x \in \partial\Omega$, $y \in \bar{\Omega}$ we set $G^i_k(x, y) = 0$. Obviously, one has also

$$g^i_k(x, y) = g^k_i(y, x) . \tag{3.6}$$

Now calculate $\Delta v^k(x)$ from (3.4): by using the results that we have already proved in § 2, we get

$$\begin{aligned}\Delta v^k(x) = f^k(x) - \int_\Omega \Delta_x g^i_k(x, y) f^i(y)\, dy + \nabla_k\Big[\int_\Omega q_i(x, y) f^i(y)\, dy\Big] + \\ + \nabla_k\, g(x) + \int_\Omega \Delta_x\, g_k(x, y) g(y)\, dy + \\ + \int_{\partial\Omega} [- \Delta_x\, g_k(x, y)\, \delta_{ij} + \nabla_{y_j} \Delta_x G^i_k(x, y)] \varphi^i(y) n^j(y)\, d\sigma_y ,\end{aligned}$$

where we have choosen a suitable $g_k(x, y)$ in such a way that it is regular in x, for instance by requiring that

$$\int_{\partial\Omega} G_k(x, y)\, d\sigma_y = 0 \qquad \forall x \in \Omega . \tag{3.7}$$

Moreover, by $G^j_k(x, y) = (G^k_j \circ S)(x, y)$ (where we have defined $S(x, y) \equiv \equiv (y, x)\big)$, one has

$$\begin{aligned}\Delta_x G^i_k = \Delta_x(G^k_i \circ S) = (\Delta_y G^k_i) \circ S = \\ = - (\nabla_{y_k} G_i) \circ S = - \nabla_{x_k}(G_i \circ S) \quad \text{for each } i = 1, 2, 3,\end{aligned}$$

and

$$\Delta_x g^i_k = - \nabla_{x_k}(g_i \circ S) .$$

Hence

$$\nabla_k p(x) = \Delta v^k(x) - f^k(x) = \nabla_k\Big\{\int_\Omega (q_i + g_i \circ S)(x, y) f^i(y)\,dy + \tag{3.8}$$

$$+ g(x) - \int_{\partial\Omega} \nabla_{y_j}(G_i \circ S)(x, y)\varphi^i(y) n^j(y)\,d\sigma_y\Big\} +$$

$$+ \int_\Omega \Delta_x g_k(x, y) g(y)\,dy - \int_{\partial\Omega} \Delta_x g_k(x, y)\varphi(y)\cdot n(y)\,d\sigma_y\,.$$

Finally

$$\nabla_{y_i}\Delta_x g_k = -\Delta_x\Delta_y g_k^i = -\Delta_x\Delta_y(g_i^k \circ S) = \tag{3.9}$$

$$= -\Delta_y[(\Delta_y g_i^k)\circ S] = \Delta_y[(\nabla_{y_k} g_i)\circ S] = \nabla_{x_k}\Delta_y(g_i \circ S)\,,$$

hence

$$\int_\Omega \Delta_x g_k(x, y) g(y)\,dy - \int_{\partial\Omega} \Delta_x g_k(x, y)\varphi(y)\cdot n(y)\,d\sigma_y = \tag{3.10}$$

$$= \int_\Omega \Delta_x g_k(x, y)\,\mathrm{div}\, v(y)\,dy - \int_{\partial\Omega} \Delta_x g_k(x, y) v(y)\cdot n(y)\,d\sigma_y =$$

$$= -\int_\Omega \nabla_{y_i}\Delta_x g_k(x, y) v^i(y)\,dy = -\nabla_k \int_\Omega \Delta_y(g_i \circ S)(x, y) v^i(y)\,dy\,.$$

Remark that from (3.9) we have that $[\nabla_{x_k}\Delta_y(g_i \circ S)](x, y)$ is continuous in $K X \bar{\Omega}$, K any compact set contained in Ω.

From (3.4) and (3.10) we can write

$$\int_\Omega \Delta_x g_k(x, y) g(y)\,dy - \int_\Omega \Delta_x g_k(x, y)\varphi(y)\cdot n(y)\,d\sigma_y =$$

$$= -\nabla_k\Big\{\int_\Omega \Delta_y(g_i \circ S)(x, y)\Big[\int_\Omega G_i^j(y, \eta) f^j(\eta)\,d\eta - \int_\Omega G_i(y, \eta) g(\eta)\,d\eta +$$

$$+ \int_{\partial\Omega} (G_i(y, \eta)\,\delta_{js} + \nabla_{\eta_j} G_i^s(y, \eta))\varphi^s(\eta) n^j(\eta)\,d\sigma_\eta\Big]\,dy\Big\}\,.$$

Moreover from (3.9) one has

$$\nabla_k\Big\{\int_\Omega \Delta_y(g_i\circ S)(x,y)\Big[\int_\Omega G_i^j(y,\eta)f^j(\eta)\,d\eta\Big]dy\Big\} = \tag{3.11}$$

$$= \Delta\Big\{\int_\Omega \nabla_{y_i} g_k(x,y)\Big[\int_\Omega G_j^i(\eta,y)f^j(\eta)\,d\eta\Big]dy\Big\} =$$

$$= \Delta\Big\{\int_\Omega - g_k(x,y)\Big[\int_\Omega \nabla_{y_i} G_j^i(\eta,y)f^j(\eta)\,d\eta\Big]dy +$$

$$+ \int_{\partial\Omega} g_k(x,y)n^i(y)\Big[\int_\Omega G_j^i(\eta,y)f^j(\eta)\,d\eta\Big]d\sigma_y\Big\} = 0\,,$$

since $\nabla_{y_i} G_j^i(\eta, y) = 0$ and $G_j^i(\eta, y)|_{y\in\partial\Omega} = 0$.
Hence

$$p(x) = \int_\Omega (q_i + g_i\circ S)(x,y)f^i(y)\,dy + g(x) + \tag{3.12}$$

$$+ \int_\Omega \Delta_y(g_i\circ S)(x,y)\Big[\int_\Omega G_i(y,\eta)g(\eta)\,d\eta\Big]dy -$$

$$- \int_{\partial\Omega} \nabla_{y_j}(G_i\circ S)(x,y)\varphi^i(y)n^j(y)\,d\sigma_y -$$

$$- \int_\Omega \Delta_y(g_i\circ S)(x,y)\Big\{\int_{\partial\Omega}[G_i(y,\eta)\,\delta_{js} +$$

$$+ \nabla_{\eta_j} G_i^s(y,\eta)]\varphi^s(\eta)n^j(\eta)\,d\sigma_\eta\Big\}dy\,,$$

up to an additive constant.
Define now

$$h_k^i(x,y) \equiv (g_k^i\circ S)(x,y) = g_i^k(x,y)\,, \tag{3.13}$$

$$h_k(x,y) \equiv -(g_k\circ S)(x,y)\,, \tag{3.14}$$

and

$$H_k^i(x,y) \equiv u_k^i(x,y) - h_k^i(x,y) = (G_k^i\circ S)(x,y) = G_i^k(x,y)\,, \tag{3.15}$$

$$H_k(x,y) \equiv q_k(x,y) - h_k(x,y) = -(G_k\circ S)(x,y)\,. \tag{3.16}$$

It is verified at once that for $y \in \Omega$

$$(3.17)\qquad \begin{cases} \Delta_x H^i_k(x,y) - \nabla_{x_i} H_k(x,y) = \delta_{ki}\delta(x-y)\,, & x \in \Omega\,,\\ \nabla_{x_i} H^i_k(x,y) = 0\,, & x \in \Omega\,,\\ H^i_k(x,y)|_{x\in\partial\Omega} = 0\,, & x \in \partial\Omega\,; \end{cases}$$

hence these functions correspond to

$$H^i_k(x,y) = G_{ik}(x,y)\,, \qquad H_k(x,y) = r_k(x,y)$$

(see Ladyzhenskaya [15], pag. 65), or to

$$H^i_k(x,y) = -G_{ik}(x,y)\,, \qquad H_k(x,y) = -g_k(x,y)$$

(see Odqvist [25]).
We can rewrite (3.4) and (3.12) in this form (see also (3.28), (3.24)):

$$(3.18)\qquad v^k(x) = \int_\Omega H^k_i(x,y) f^i(y)\,dy + \int_\Omega H_k(y,x) g(y)\,dy + \int_{\partial\Omega} [\nabla_{x_j} H^i_k(y,x) - H_k(y,x)\,\delta_{ij}]\varphi^i(y) n^j(y)\,d\sigma_y$$

$$(3.19)\qquad p(x) = \int_\Omega H_i(x,y) f^i(y)\,dy + g(x) + \int_\Omega \Delta_y h_i(x,y)\Big[\int_\Omega H_i(\eta,y) g(\eta)\,d\eta\Big]dy + \int_{\partial\Omega} \nabla_{y_j} H_i(x,y)\varphi^i(y) n^j(y)\,d\sigma_y + \int_\Omega \Delta_y h_i(x,y)\Big\{\int_{\partial\Omega}[\nabla_{x_j} H^s_i(\eta,y) - H_i(\eta,y)\,\delta_{sj}]\varphi^s(\eta) n^j(\eta)\,d\sigma_\eta\Big\}dy\,.$$

For $g = 0$, $\varphi = 0$ (3.18) and (3.19) gives (45) of [15] and (5.01) of [25].

REMARK 3.1. One can verify that in general the third term in (3.19) cannot be deleted. In fact, take $\Omega = \{x \in \mathbb{R}^3 \colon |x| < 1\}$, $f = 0$,

$\varphi = 0$, and $g(x) = \sum_{i=1}^{3} x^i$. It is easily seen that the solution is given by

$$v^i(x) = \frac{|x|^2 - 1}{2} \quad \text{for each } i = 1, 2, 3 ,$$

$$p(x) = 3 \sum_{i=1}^{3} x^i = 3g(x) ,$$

and indeed

$$\nabla_k \left\{ \int_\Omega \Delta_y h_i(x, y) \left[\int_\Omega H_i(\eta, y) g(\eta)\, d\eta \right] dy \right\} = 2 \nabla_k g(x) ,$$

as one can verify directly by using the relations

$$v^i(y) = \int_\Omega H_i(\eta, y)\, g(\eta)\, d\eta = \frac{|y|^2 - 1}{2} \quad \text{for each } i = 1, 2, 3,$$

$$\nabla_{x_k} \Delta_y h_i(x, y) = - \nabla_{y_i} \Delta_x g_k(x, y) .$$

On the other hand, if $f = 0$, $\varphi = 0$, $g = \Delta \psi$, $\psi \in C_0^\infty(\Omega)$ one gets at once that

$$v^i(x) = \nabla_i \psi(x) ,$$

$$p(x) = \Delta \psi(x) = g(x) ,$$

and in this case the third term in (3.19) does not give a contribution.

REMARK 3.2. Observe that the relation

$$g_k(x, y) = - g_k(y, x) \tag{3.20}$$

in general does not hold. In fact, it follows from (3.20) and $(3.3)_1$ that

$$\Delta_x g_k(x, y) = - (\Delta_y g_k)(y, x) = (\nabla_{y_i} \Delta_y g_k^i)(y, x) = 0$$

for each $x \in \Omega$, $y \in \bar{\Omega}$. Hence in (3.8) the last two terms disappear, and consequently the same happens for the third and the fifth term in (3.19). This contradicts Remark 3.2.

REMARK 3.3. As we observed in Remark 2.1, we can obtain (2.9) in a direct way by choosing $(u^i, q) \equiv (q_i, -\delta)$, where $L_y^*(q_i, -\delta) = (0, \delta)$. Hence, if we find the solution (l^i, l) of

$$
(3.21) \qquad \begin{cases} \Delta l^i(x, y) + \nabla_{y_i} l(x, y) = 0 \,, & y \in \Omega \,, \\ \nabla_{y_i} l^i(x, y) = 0 \,, & y \in \Omega \,, \\ l^i(x, y)|_{y\in\partial\Omega} = q_i(x, y)|_{y\in\partial\Omega} - \dfrac{1}{|\partial\Omega|}\, n^i(y) \,, & y \in \partial\Omega \,, \end{cases}
$$

($|\partial\Omega| \equiv$ surface measure of $\partial\Omega$), and we put

$$
(3.22) \qquad L^i(x, y) \equiv q_i(x, y) - l^i(x, y) \,,
$$

$$
(3.23) \qquad L(x, y) \equiv -\delta(x - y) - l(x, y) \,,
$$

by choosing in (2.1) $(u^i, q) \equiv (L^i, L)$ we can repeat the same calculations, and we get easily (up to an additive constant)

$$
(3.24) \qquad p(x) = \int_\Omega L^i(x, y) f^i(y)\, dy - \int_\Omega L(x, y) g(y)\, dy + \\ + \int_{\partial\Omega} [L(x, y)\, \delta_{ij} + \nabla_{y_j} L^i(x, y)] \varphi^i(y) n^j(y)\, d\sigma_y \,.
$$

Observe that (3.24) is formally quite similar to (3.4).

Remark also that the solution of (3.21) exists since

$$
\int_{\partial\Omega} \left[q_i(x, y) - \frac{1}{|\partial\Omega|}\, n^i(y) \right] n^i(y)\, d\sigma_y = 0 \,.
$$

Though formula (3.24) looks simpler than (3.19), we prefer to use this last in the following arguments, since in the classical paper of Odqvist [25] the author studies in great detail the behaviour of the Green's functions H_k^i and H_k, while L^i and L are not considered. (If Ω is a ball, see however Oseen [27], pag. 103-106).

REMARK 3.4. We want to precise some properties of L^i and L which are useful to clarify the relations between (3.19) and (3.24). By choosing in (2.1) $v^i(z) \equiv G_k^i(x, z)$, $p(z) \equiv -G_k(x, z)$, $u^i(z) \equiv L^i(y, z)$, $q(z) \equiv L(y, z)$ one gets

$$G_k(x, y) - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} G_k(x, z)\, d\sigma_z = -L^k(y, x)\,; \tag{3.25}$$

by choosing u^i and q as before and $v^i(z) \equiv -L^i(x, z)$, $p(z) \equiv L(x, z)$ one obtains

$$L(y, x) - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} L(y, z)\, d\sigma_z = L(x, y) - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} L(x, z)\, d\sigma_z\,. \tag{3.26}$$

Here one utilizes (4.14) and the relation

$$\int_{\partial\Omega} \nabla_{z_j} L^i(y, z)\, n^i(z)\, n^j(z)\, d\sigma_z = \text{const.} \qquad \forall y \in \Omega\,,$$

which can be proved by extending $n(z)$ in $\bar{\Omega}$ and by using the divergence theorem.

Relations (3.25) and (3.26) make it possible to simplify formula (3.19). In fact, by (3.25) and by assuming that (3.7) holds, we have

$$h_k(x, y) = l^k(x, y)\,. \tag{3.27}$$

Consequently, by using the divergence theorem and (2.2)

$$\int_{\Omega} \Delta_y h_i(x, y)\left[\int_{\partial\Omega} \nabla_{x_j} H_i^s(\eta, y)\, \varphi^s(\eta)\, n^j(\eta)\, d\sigma_\eta\right] dy =$$

$$= \int_{\Omega} \Delta_y h_i(x, y)\left[\int_{\Omega} \nabla_{x_j} H_i^s(\eta, y)\, \nabla_j v\, (\eta)\, d\eta\right] dy -$$

$$- \int_{\Omega} \nabla_{y_i} l(x, y)\left[\int_{\Omega} \Delta_x H_i^s(\eta, y)\, v^s(\eta)\, d\eta\right] dy\,.$$

On the other hand

$$\int_{\Omega} \nabla_{x_j} H_i^s(\eta, y)\, \nabla_j v^s(\eta)\, d\eta = -\int_{\Omega} H_i^s(\eta, y)\, \Delta v^s(\eta)\, d\eta = -\int_{\Omega} G_i^s(y, \eta)\, f^s(\eta)\, d\eta\,;$$

and

$$\int_{\Omega} \nabla_{y_i} l(x, y) \Delta_x H_i^s(\eta, y)\, dy = \Delta_\eta \left[\int_{\Omega} \nabla_{y_i} l(x, y) H_i^s(\eta, y)\, dy \right] =$$

$$= \Delta_\eta \left[\int_{\Omega} \nabla_{y_i} l(x, y) H_s^i(y, \eta)\, dy \right] = 0 .$$

Hence by (3.11) and by Fubini's theorem we get that

$$\nabla_k \left\{ \int_{\Omega} \Delta_y h_i(x, y) \left[\int_{\partial\Omega} \nabla_{x_j} H_i^s(\eta, y) \varphi^s(\eta) n^j(\eta)\, d\sigma_\eta \right] dy \right\} = 0 .$$

We can thus rewrite the formula for $p(x)$ in this way

$$\begin{aligned} p(x) = {} & \int_{\Omega} H_i(x, y) f^i(y)\, dy + g(x) + \\ & + \int_{\Omega} \Delta_y h_i(x, y) \left[\int_{\Omega} H_i(\eta, y) g(\eta)\, d\eta \right] dy + \\ & + \int_{\partial\Omega} \nabla_{y_j} H_i(x, y) \varphi^i(y) n^j(y)\, d\sigma_y - \\ & - \int_{\Omega} \Delta_y h_i(x, y) \left[\int_{\partial\Omega} H_i(\eta, y) \varphi(\eta) \cdot n(\eta)\, d\sigma_\eta \right] dy . \end{aligned} \tag{3.28}$$

Moreover, by assuming that

$$\int_{\partial\Omega} L(x, y)\, d\sigma_y = -\int_{\partial\Omega} l(x, y)\, d\sigma_y = 0 \qquad \forall x \in \Omega , \tag{3.29}$$

we get

$$\int_{\Omega} \Delta_y h_i(x, y) H_i(\eta, y)\, dy = -\int_{\Omega} \nabla_{y_i} l(x, y) H_i(\eta, y)\, dy = \tag{3.30}$$

$$= \int_{\Omega} l(x, y) \nabla_{y_i} L^i(\eta, y)\, dy - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} l(x, y)\, d\sigma_y = l(x, \eta) \qquad \forall x \in \Omega , \ \eta \in \bar{\Omega} .$$

Hence, if we assume that (3.7) and (3.29) hold, Fubini's theorem gives at once that the representation formulas (3.28) and (3.24) are exactly the same.

4. The synthesis of the solution to the Stokes system.

We want to prove now that, if we assigne f, g and φ, smooth functions satisfying

$$\int_{\Omega} g(y)\,dy = \int_{\partial\Omega} \varphi(y)\cdot n(y)\,d\sigma_y\,, \tag{4.1}$$

then v and p given by (3.18) and (3.28) are the solution of

$$\begin{cases} \Delta v - \nabla p = f & \text{in } \Omega\,,\\ \operatorname{div} v = g & \text{in } \Omega\,,\\ v|_{\partial\Omega} = \varphi & \text{on } \partial\Omega\,. \end{cases} \tag{4.2}$$

From $(4.2)_2$ and $(4.2)_3$ one sees that condition (4.1) is obviously necessary for the existence of the solution.
We begin by verifying $(4.2)_2$.

LEMMA 4.1. *One has that*

(i) *for any* $y \in \Omega$, $\nabla_{y_i} h_i(x, y)$ *is constant in* $x \in \bar{\Omega}$;

(ii) *for any* $x \in \Omega$, $\nabla_{x_i} g_i(x, y)$ *is constant in* $y \in \bar{\Omega}$.

Moreover, by (3.7) *one gets that these constants are equal to zero.*

PROOF. One has only to recall that for $x \in \Omega$, $y \in \Omega$

$$\Delta_x g_k^i(x, y) = \Delta_x h_i^k(x, y) = \nabla_{x_k} h_i(x, y)\,,$$

and moreover by $(3.3)_2$

$$0 = \nabla_{y_i} \Delta_x g_k^i(x, y) = \nabla_{x_k} \nabla_{y_i} h_i(x, y)\,.$$

Since, for a fixed $y \in \Omega$, $\nabla_{y_i} h_i(x, y)$ is regular in $x \in \bar{\Omega}$, one obtains (i).

Finally

$$\nabla_{y_i} h_i(x, y) = -\nabla_{y_i}(g_i \circ S)(x, y) = -(\nabla_{x_i} g_i)(y, x) .$$

Hence for any $x \in \Omega$, $y \in \bar{\Omega}$ we can write

$$\nabla_{x_i} g_i(x, y) = A(x) ,$$

for a certain function $A(x)$; on the other hand by (3.7)

$$0 = \nabla_{x_i}\Big[\int_{\partial\Omega} q_i(x, y)\, d\sigma_y\Big] = \nabla_{x_i}\Big[\int_{\partial\Omega} g_i(x, y)\, d\sigma_y\Big] = \\ = \int_{\partial\Omega} A(x)\, d\sigma_y = A(x)|\partial\Omega| . \quad \square$$

LEMMA 4.2. *The function* $v(x)$ *given by* (3.18) *satisfies*

$$\operatorname{div} v(x) = g(x) , \tag{4.3}$$

for any $x \in \Omega$.

PROOF. By direct calculation. One has by $(3.17)_2$

$$\nabla_k\Big[\int_{\Omega} H_i^k(x, y) f^i(y)\, dy\Big] = 0 .$$

Moreover

$$\nabla_k\Big[\int_{\Omega} H_k(y, x) g(y)\, dy\Big] = -\Delta\Big[\int_{\Omega} E(x, y) g(y)\, dy\Big] - \\ -\nabla_k \int_{\Omega} (h_k \circ S)(x, y) g(y)\, dy = g(x) . \tag{4.4}$$

Finally, from $(3.17)_2$

$$\nabla_k\Big\{\int_{\partial\Omega} [\nabla_{x_j} H_k^i(y, x) - H_k(y, x)\, \delta_{ij}] \varphi^i(y) n^j(y)\, d\sigma_y\Big\} = \\ = -\nabla_k\Big[\int_{\partial\Omega} (H_k \circ S)(x, y) \varphi(y) \cdot n(y)\, d\sigma_y\Big] + \tag{4.5}$$

$$+\int_{\partial\Omega} \nabla_{x_k}\nabla_{y_j} H_i^k(x, y)\varphi^i(y) n^j(y)\, d\sigma_y = \int_{\partial\Omega} \Delta_x E(x, y)\varphi(y)\cdot n(y)\, d\sigma_y +$$
$$+ \nabla_k \int_{\Omega} (h_k \circ S)(x, y)\varphi(y)\cdot n(y)\, d\sigma_y = 0 \,. \qquad \square$$

Now we want to verify that v assume the boundary confition, that is

$$\lim_{\substack{x\to x_0\\ x\in\Omega}} v(x) = \varphi(x_0) \qquad \text{for any } x_0 \in \partial\Omega \,.$$

One sees easily that

$$\int_{\Omega} H_i^k(x, y) f^i(y)\, dy \qquad \text{and} \qquad \int_{\Omega} H_k(y, x) g(y)\, dy$$

are continuous functions on $\bar{\Omega}$, since for $x, y \in \bar{\Omega}$, $x \neq y$ one has

$$\text{(4.6)} \qquad |H_i^k(x, y)| \leq \frac{c}{|x-y|}\,,$$
$$|\nabla_x H_i^k(x, y)| + |\nabla_y H_i^k(x, y)| + |H_k(y, x)| \leq \frac{c}{|x-y|^2}\,,$$

as it is proved in Odqvist [25] (see also Ladyzhenskaya [15], pag. 68; Miranda [22], pag. 25).

On the other hand

LEMMA 4.3. *For any $y_0 \in \partial\Omega$ we have*

$$\lim_{\substack{y\to y_0\\ y\in\Omega}} H_k(x, y) = \frac{1}{|\partial\Omega|}\, n^k(y_0)\,.$$

PROOF. By (3.7) and (3.25) we have for $x \in \Omega$, $y \in \Omega$

$$L^k(x, y) = -G_k(y, x) = H_k(x, y)\,.$$

Hence, by the properties of L^k we get for each $x \in \Omega$, $y_0 \in \partial\Omega$

$$\lim_{\substack{y\to y_0\\ y\in\Omega}} H_k(x, y) = L^k(x, y_0) = \frac{1}{|\partial\Omega|}\, n^k(y_0)\,. \qquad \square$$

Hence for $x \in \Omega$ we can extend $H_k(x, y)$ up to $y \in \partial\Omega$ in a continuous way, by setting

$$H_k(x, y) = \frac{1}{|\partial\Omega|}\, n^k(y)\,, \qquad x \in \Omega\,,\ y \in \partial\Omega\,.$$

LEMMA 4.4. *The function $v(x)$ given by* (3.18) *satisfies*

$$\lim_{\substack{x\to x_0\\ x\in\Omega}} v^k(x) = \varphi^k(x_0) + \frac{1}{|\partial\Omega|}\, n^k(x_0)\left[\int_\Omega g(y)\,dy - \int_{\partial\Omega}\varphi(y)\cdot n(y)\,d\sigma_y\right], \tag{4.7}$$

for any $x_0 \in \partial\Omega$.

PROOF. One has, from $(3.17)_3$ and Lemma 4.3:

$$\lim_{\substack{x\to x_0\\ x\in\Omega}} \int_\Omega H_i^k(x, y) f^i(y)\,dy = \int_\Omega H_i^k(x_0, y) f^i(y)\,dy = 0\,, \tag{4.8}$$

$$\lim_{\substack{x\to x_0\\ x\in\Omega}} \int_\Omega H_k(y, x)\, g(y)\,dy = \int_\Omega H_k(y, x_0)\, g(y)\,dy = \frac{1}{|\partial\Omega|}\, n^k(x_0)\int_\Omega g(y)\,dy\,. \tag{4.9}$$

The third term in (3.18) requires some more calculations.
First of all

$$\lim_{\substack{x\to x_0\\ x\in\Omega}} \int_{\partial\Omega} [\nabla_{x_j} H_k^i(y, x) - H_k(y, x)\,\delta_{ij}]\varphi^i(y) n^j(y)\,d\sigma_y = \tag{4.10}$$

$$= -\lim_{\substack{x\to x_0\\ x\in\Omega}} \int_{\partial\Omega} H_k(y, x)\varphi(y)\cdot n(y)\,d\sigma_y + \lim_{\substack{x\to x_0\\ x\in\Omega}} \int_{\partial\Omega} \nabla_{y_j} G_k^i(x, y)\varphi^i(y) n^j(y)\,d\sigma_y\,.$$

Take the second term into account. We can find two regular functions ϕ and π (say $\phi \in W^{2,2}(\Omega)$, $\pi \in W^{1,2}(\Omega)$) such that

$$\begin{cases} \Delta\phi - \nabla\pi = 0 & \text{in } \Omega\,,\\ \operatorname{div}\phi = 0 & \text{in } \Omega\,,\\ \phi|_{\partial\Omega} = \varphi - \dfrac{1}{|\partial\Omega|}\left[\displaystyle\int_{\partial\Omega}\varphi(y)\cdot n(y)\,d\sigma_y\right] n & \text{on } \partial\Omega\,, \end{cases} \tag{4.11}$$

(see for instance Odqvist [25], Ladyzhenskaya [15], pag. 79; remark however that for proving (4.15) it is enough to extend φ in $\bar{\Omega}$ as a $C^2(\bar{\Omega})$-function, and then use the properties of H^i_k and H_k as in (4.8), (4.9)).

Hence we can write

$$\int_{\partial\Omega} \nabla_{y_j} G^i_k(x, y)\varphi^i(y)n^j(y)\,d\sigma_y = \int_{\Omega} \nabla_{y_j}[\nabla_{y_j} G^i_k(x, y)\phi^i(y)]\,dy + \\ + \frac{1}{|\partial\Omega|}\left[\int_{\partial\Omega}\varphi(y)\cdot n(y)\,d\sigma_y\right]\int_{\partial\Omega}\nabla_{y_j}G^i_k(x, y)n^i(y)n^j(y)\,d\sigma_y\,.$$

The first integral can be written in this way, by integrating by parts and by using (4.11), (3.7):

$$(4.12)\qquad \int_{\Omega}[\Delta_y G^i_k(x, y)\phi^i(y) + \nabla_{y_j}G^i_k(x, y)\nabla_j\phi^i(y)]\,dy = \\ = \phi^k(x) - \int_{\Omega}[\nabla_{y_i}G_k(x, y)\phi^i(y) + G^i_k(x, y)\nabla_i\pi(y)]\,dy = \\ = \phi^k(x) - \int_{\partial\Omega} G_k(x, y)\phi(y)\cdot n(y)\,d\sigma_y = \\ = \phi^k(x) + \int_{\partial\Omega} H_k(y, x)\varphi(y)\cdot n(y)\,d\sigma_y\,.$$

Hence we have

$$(4.13)\qquad \lim_{\substack{x\to x_0\\ x\in\Omega}}\int_{\partial\Omega}[\nabla_{x_j}H^i_k(y, x) - H_k(y, x)\,\delta_{ij}]\varphi^i(y)n^j(y)\,d\sigma_y = \varphi^k(x_0) - \\ - \frac{1}{|\partial\Omega|}\left[\int_{\partial\Omega}\varphi(y)\cdot n(y)\,d\sigma_y\right]\left\{n^k(x_0) - \lim_{\substack{x\to x_0\\ x\in\Omega}}\int_{\partial\Omega}\nabla_{y_j}G^i_k(x, y)n^i(y)n^j(y)\,d\sigma_y\right\}.$$

On the other hand for $x\in\Omega$, $y\in\partial\Omega$

$$(4.14)\qquad \nabla_{y_j}G^i_k(x, y)n^i(y)n^j(y) = 0\,,\qquad \text{for each } k = 1, 2, 3\,.$$

In fact for $x\in\Omega$, $y\in\partial\Omega$

$$\nabla_{y_j}G^i_k(x, y) = C^i_k(x, y)n^j(y)\,,$$

and moreover, as for $x \in \Omega$ $\nabla_{y_i} G^i_k(x, y)$ is regular in y near and up to $\partial\Omega$,

$$0 = \nabla_{y_i} G^i_k(x, y) = C^i_k(x, y)\, n^i(y) = \nabla_{y_j} G^i_k(x, y)\, n^i(y)\, n^j(y) \,.$$

The thesis follows from (4.8), (4.9), (4.13) and (4.14).

Observe that in particular we have proved

$$\text{(4.15)} \qquad \lim_{\substack{x \to x_0 \\ x \in \Omega}} \int_{\partial\Omega} [\nabla_{x_j} H^i_k(y, x) - H_k(y, x)\, \delta_{ij}] \varphi^i(y) n^j(y)\, d\sigma_y = \\ = \varphi^k(x_0) - \frac{1}{|\partial\Omega|} \left[\int_{\partial\Omega} \varphi(y) \cdot n(y)\, d\sigma_y \right] n^k(x_0) \,. \qquad \square$$

We are now in a position to prove that v and p given by (3.18), (3.28) satisfy $(4.2)_1$:

LEMMA 4.5. *The functions $v(x)$ and $p(x)$ given by* (3.18) *and* (3.28) *satisfy*

$$\text{(4.16)} \qquad \Delta v(x) - \nabla p(x) = f(x) \,,$$

for any $x \in \Omega$.

PROOF. We have already proved that (see (3.8))

$$\Delta v^k(x) = f^k(x) + \nabla_k \Big\{ \int_\Omega H_i(x, y) f^i(y)\, dy + g(x) + \\ + \int_{\partial\Omega} \nabla_{y_j} H_i(x, y) \varphi^i(y) n^j(y)\, d\sigma_y \Big\} + \\ + \int_\Omega \Delta_x g_k(x, y)\, g(y)\, dy - \int_{\partial\Omega} \Delta_x g_k(x, y)\, \varphi(y) \cdot n(y)\, d\sigma_y \,.$$

On the other hand, from (3.9), (3.14), (4.4) and (4.9)

$$\nabla_k \Big\{ \int_\Omega \Delta_y h_i(x, y) \Big[\int_\Omega H_i(\eta, y)\, g(\eta)\, d\eta \Big] dy \Big\} = \\ = -\int_\Omega \nabla_{y_i} \Delta_x g_k(x, y) \Big[\int_\Omega H_i(\eta, y)\, g(\eta)\, d\eta \Big] dy =$$

$$
= \int_{\Omega} \Delta_x g_k(x, y) \nabla_{y_i} \left[\int_{\Omega} H_i(\eta, y) g(\eta)\, d\eta \right] dy -
$$

$$
- \int_{\partial\Omega} \Delta_x g_k(x, y) n^i(y) \left[\int_{\Omega} H_i(\eta, y) g(\eta)\, d\eta \right] d\sigma_y =
$$

$$
= \int_{\Omega} \Delta_x g_k(x, y) g(y)\, dy - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \Delta_x g_k(x, y)\, d\sigma_y\,.
$$

Furthermore, by (3.7)

$$
\int_{\partial\Omega} \Delta_x g_k(x, y)\, d\sigma_y = \Delta_x \left[\int_{\partial\Omega} q_k(x, y)\, d\sigma_y \right] = \nabla_{x_k} \int_{\partial\Omega} \Delta_x E(x, y)\, d\sigma_y = 0\,.
$$

Finally, from Fubini's theorem, (3.30) and (3.26), by extending φ in $\bar{\Omega}$ as a $C^2(\bar{\Omega})$-function $\bar{\varphi}$, we get

$$
\nabla_k \int_{\Omega} \Delta_y h_i(x, y) \left[\int_{\partial\Omega} H_i(\eta, y) \varphi(\eta) \cdot n(\eta)\, d\sigma_\eta \right] dy =
$$

$$
= \int_{\Omega} \nabla_{\eta_s} \left\{ \bar{\varphi}^s(\eta) \nabla_{x_k} \left[\int_{\Omega} \Delta_y h_i(x, y) H_i(\eta, y)\, dy \right] \right\} d\eta =
$$

$$
= \int_{\Omega} \nabla_{\eta_s} \left\{ \bar{\varphi}^s(\eta) \nabla_{x_k} \left[l(\eta, x) - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} l(\eta, z)\, d\sigma_z \right] \right\} d\eta =
$$

$$
= - \int_{\Omega} \nabla_{\eta_s} \{ \bar{\varphi}^s(\eta) \Delta_y l^k(\eta, x) \}\, d\eta = \int_{\partial\Omega} \Delta_x g_k(x, \eta) \varphi(\eta) \cdot n(\eta)\, d\sigma_\eta\,. \quad \square
$$

From (4.3), (4.7) and (4.16) it is clear now that condition (4.1) is sufficient for having that v and p given by (3.18), (3.28) are the solution of (4.2) (p unique up to an additive constant).

REMARK 4.6. By a variational approach (which was used for the first time by Leray [17]; see for instance Temam [30], pag. 23, 31) one can prove an existence and uniqueness theorem for (4.2). On the other hand, by means of the a priori estimates of Agmon-Douglis-Nirenberg [1] for general elliptic systems, one gets regularity results in Sobolev's and Hölder's spaces (see also [30], pag. 33). More precise a priori estimates for the Stokes system were proved by Cattabriga [5], and these estimates, combined with the existence theorem of variational type that we have recalled before, give the optimal result for $f \in W^{k,s}(\Omega)$, $v \in W^{k+1,s}(\Omega)$, $\varphi \in W^{k+2-1/s,s}(\partial\Omega)$, $\partial\Omega \in C^{m+2}$, $m = \max(k, 0)$, $k \geqslant -1$, $1 < s < +\infty$. A theorem which gives existence, uniqueness and regularity in Sobolev's spaces with $s = 2$ and in Hölder's spaces is proved (among many other results) in Giaquinta-Modica [10], without using potential theory.

REMARK 4.7. The representation formulas (3.18) and (3.28) hold also for $f \in L^s(\Omega)$, $g \in W^{1,s}(\Omega)$, $\varphi \in W^{2-1/s,s}(\partial\Omega)$ (g and φ satisfying (4.1)). In fact, set (v, p) to be the solution of (4.2) with these data (by choosig p in some way, for instance $\int_\Omega p(x)\,dx = 0$). By the results of Cattabriga [5] we have $v \in W^{2,s}(\Omega)$, $p \in W^{1,s}(\Omega)$, and we can approximate v, p in these spaces by $v_n \in C^\infty(\bar{\Omega})$, $p_n \in C^\infty(\bar{\Omega})$. Define

$$f_n \equiv \Delta v_n - \nabla p_n\,, \qquad g_n \equiv \operatorname{div} v_n\,, \qquad \varphi_n \equiv v_n|_{\partial\Omega}\,.$$

We can write v_n and p_n in term of (3.18), (3.28) (p_n up to an additive constant c_n). By classical result on singular kernels (see for instance Miranda [22], pag. 27), estimates (4.6) give that each term in (3.18) (evaluated for f_n, g_n, φ_n), converges in $L^s(\Omega)$ to the corresponding term in f, g, φ. Hence (3.18) hold almost everywhere in Ω. The same calculations can be performed for the first, the second and the fourth term in (3.28). The third and the fifth term converges pointwise in Ω, since for any fixed $x \in \Omega$ $\Delta_y h_i(x, y)$ is in $L^r(\Omega_y)$ for any $r \in [1, +\infty[$ and the terms integrated in η converge in $L^s(\Omega_y)$. Hence also (3.28) holds almost everywhere in Ω (up to an additive constant).

REMARK 4.8. If we choose $G_k(x, y)$ and $L(x, y)$ (see (3.23)) in such a way that

$$\int_{\partial\Omega} G_k(x, y)\,d\sigma_y = \int_{\partial\Omega} L(x, y) = 0 \qquad \forall x \in \Omega\,, \tag{4.17}$$

then by (3.25) and (3.26) we have

$$L^k(x, y) = -G_k(y, x) = H_k(x, y)\,, \tag{4.18}$$

and

$$L(x, y) = L(y, x)\,. \tag{4.19}$$

Define now the Green's tensor for L_y^* by

$$G \equiv \begin{bmatrix} G_1^1 & G_2^1 & G_3^1 & L^1 \\ G_1^2 & G_2^2 & G_3^2 & L^2 \\ G_1^3 & G_2^3 & G_3^3 & L^3 \\ G_1 & G_2 & G_3 & L \end{bmatrix};$$

consequently the Green's tensor for L_y is given by

$$\tilde{G} \equiv \begin{bmatrix} G_1^1 & G_2^1 & G_3^1 & -L^1 \\ G_1^2 & G_2^2 & G_3^2 & -L^2 \\ G_1^3 & G_2^3 & G_3^3 & -L^3 \\ -G_1 & -G_2 & -G_3 & L \end{bmatrix}.$$

These tensors satisfy respectively $L_y^* G(x, y) = \delta(x-y)I$ and $L_y\tilde{G}(x, y) = \delta(x-y)I$, $I \equiv$ identity matrix.

By (3.5), (4.18) and (4.19) one has moreover

$${}^tG = \tilde{G}\circ S \qquad ({}^tA \equiv \text{transpose matrix of } A) \tag{4.20}$$

and consequently $L_x({}^tG) = (L_y\tilde{G})\circ S = \delta(x-y)I$. Hence one observes at once that the functions

$$(V(x), P(x)) \equiv \int_\Omega {}^tG(x, y)\cdot(f(y), -g(y))\,dy\,, \tag{4.21}$$

$$(\cdot \equiv \text{matrix product})$$

formally satisfy $L(V, P) = (f, -g)$.

If one observes that (4.21) gives the first two terms of (3.4) and (3.24) (or (3.28)), it is clear that we have already proved this result in Lemma 4.2 and Lemma 4.5. The compatibility condition (4.1) does not play any rôle at this level, while it is crucial, as we have already seen, to verify the boundary condition for v (see (4.7)). Finally, remark that if one chooses G_k and L in a way which is different from (4.17), then in general condition (4.1) is necessary also for proving that $\Delta v - \nabla p = f$, $\operatorname{div} v = g$.

REFERENCES

[1] S. Agmon - A. Douglis - L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II*, Comm. Pure Appl. Math., **17** (1964), pp. 35-92.

[2] R. Berker, *Intégration des équations du mouvement d'un fluide visqueux incompressible*, Handbuch der Physik, VIII/2, Springer-Verlag, Berlin Göttingen Heidelberg, 1963, pp. 1-384.

[3] T. Boggio, *Sul moto stazionario lento di una sfera in un liquido viscoso*, Rend. Circ. Mat. Palermo, **30** (1910), pp. 65-81.

[4] M. E. Bogovskiĭ, *Solution of the first boundary value problem for the equation of continuity of an incompressible medium*, Soviet Math. Dokl., **20** (1979), pp. 1094-1098 (previously in Dokl. Akad. Nauk SSSR, **248** (1979), pp. 1037-1040 (russian)).

[5] L. Cattabriga, *Su un problema al contorno relativo al sistema di equazioni di Stokes*, Rend. Sem. Mat. Univ. Padova, **31** (1961), pp. 308-340.

[6] M. P. Colautti, *Sui problemi variazionali di un corpo elastico incompressibile*, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I*a*, (8) **8** (1967), pp. 291-343.

[7] U. Crudeli, *Metodo di risoluzione di un problema fondamentale nella teoria del moto lento stazionario dei liquidi viscosi*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., (6) **2** (1925), pp. 247-251.

[8] H. Faxén, *Fredholmsche Integralgleichungen zu der Hydrodynamik zäher Flüssigkeiten I*, Ark. Mat. Astronom. Fys., **21***A* (no. 14) (1928-29), pp. 1-40.

[9] G. B. Folland, *Introduction to partial differential equations*, Princeton University Press, Princeton, 1976.

[10] M. Giaquinta - G. Modica, *Nonlinear systems of the type of the stationary Navier-Stokes system*, J. Reine Angew. Math., **330** (1982), pp. 173-214.

[11] A. Korn, *Allgemeine Lösung des Problems kleiner, stationärer Bewegungen in reibenden Flüssigkeiten*, Rend. Circ. Mat. Palermo, **25** (1908), pp. 259-271.

[12] A. Korn, *Sur les mouvements stationnaires d'un liquide doué de frottement*, C. R. Acad. Sci. Paris, **151** (1910), pp. 50-53, 352.

[13] M. Krzyżański, *Partial differential equations of second order, I*, PWN-Polish Scientific Publishers, Warszawa, 1971 (translated from polish).

[14] O. A. LADYŽENSKAJA, *Investigation of the Navier-Stokes equation for a stationary flow of an incompressible fluid,* Amer. Math. Soc. Transl., (2) **25** (1963), pp. 173-197 (previously in Uspehi Mat. Nauk, **14** (1959), pp. 75-97 (russian)).

[15] O. A. LADYZHENSKAYA, *The mathematical theory of viscous incompressible flow,* 2nd english ed., Gordon and Breach, New York London Paris, 1969 (translated from russian).

[16] O. A. LADYZHENSKAYA - V. A. SOLONNIKOV, *Some problems of vector analysis and generalized formulations of boundary-value problems for the Navier-Stokes equations,* J. Soviet Math., **10** (1978), pp. 257-286 (previously in Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), **59** (1976), pp. 81-116 (russian)).

[17] J. LERAY, *Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'Hydrodynamique,* J. Math. Pures Appl., (9) **12** (1933), pp. 1-82.

[18] L. LICHTENSTEIN, *Über einige Existenzprobleme der Hydrodynamik. Dritte Abhandlung. Permanente Bewegungen einer homogenen inkompressiblen, zähen Flüssigkeit,* Math. Z., **28** (1928), pp. 387-415, 752.

[19] H. A. LORENTZ, *Ein allgemeiner Satz, die Bewegung einer reibenden Flüssigkeit betreffend, nebst einigen Anwendungen desselben,* Zittingsverlag Akad. Wet. Amster., **5** (1896), pp. 168-175.

[20] H. A. LORENTZ, *Abhandlungen über theoretische Physik, I,* Teubner, Leipzig-Berlin, 1906.

[21] A. MATSUMURA - T. NISHIDA, *The initial boundary value problem for the equations of motion of compressible viscous and heat-conductive fluid,* preprint University of Wisconsin, MRC Technical Summary Report n. 2237 (1981).

[22] C. MIRANDA, *Partial differential equations of elliptic type,* 2nd revised ed., Springer-Verlag, New York Heidelberg Berlin, 1970.

[23] M. A. MODJTAHÉDI, *Quelques problèmes concernant le mouvement des fluides visqueux,* Thèse, Paris, 1938.

[24] F. K. G. ODQVIST, *Die Randwertaufgaben der Hydrodynamik zäher Flüssigkeiten,* Dissertation, Stockholm, 1928.

[25] F. K. G. ODQVIST, *Über die Randwertaufgaben der Hydrodynamik zäher Flüssigkeiten,* Math. Z., **32** (1930), pp. 329-375.

[26] C. W. OSEEN, *Über ein Randwertproblem in der Hydrodynamik,* Ark. Mat. Astronom. Fys., **18** (no. 17) (1924), pp. 1-5.

[27] C. W. OSEEN, *Neuere Methoden und Ergebnisse in der Hydrodynamik,* Akademische Verlagsgesellschaft M.B.H., Leipzig, 1927.

[28] P. E. SOBOLEVSKIĬ, *On the smoothness of generalized solutions of the Navier-Stokes equations,* Soviet Math. Dokl., **1** (1967), pp. 341-343 (previously in Dokl. Akad. Nauk SSSR, **131** (1960), pp. 758-760 (russian)).

[29] V. A. SOLONNIKOV, *On estimates of Green's tensors for certain boundary problems,* Soviet Math. Dokl., **1** (1960), pp. 128-131 (previously in Dokl. Akad. Nauk SSSR, **130** (1960), pp. 988-991 (russian)).

[30] R. Temam, *Navier-Stokes equations. Theory and numerical analysis*, North-Holland Publishing Company, Amsterdam New York Oxford, 1977.

[31] A. Valli, *Periodic and stationary solutions for compressible Navier-Stokes equations via a stability method*, Ann. Scuola Norm. Sup. Pisa, (4) **10** (1983), pp. 607-647.

[32] H. Villat, *Leçons sur les fluides visqueux*, Gauthier-Villars, Paris, 1943.

Manoscritto pervenuto in redazione il 22 giugno 1984.