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On the Convergence of Minimal Boundaries with Obstacles.

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In this note we continue the analysis of the following problem: given E_h, a set with minimal boundary with respect to the obstacle L_h $(h = 1, 2, ...)$, when will the limit set of the sequence E_h have minimal boundary with respect to the limit obstacle?

(We are assuming that both sequences are convergent in the L^1_{loc}-sense; see Section 1 below for the definition of this and related concepts).

In Tamanini [6], a first result in this direction was established, and some examples and counterexamples were discussed. It is the purpose of the present paper to investigate an alternative condition, still guaranteeing that the answer to the above question is in the affirmative.

Roughly speaking, we will assume that the mean curvature of the obstacles be uniformly bounded from above by a fixed integrable function $H(x)$. We remark that a similar condition has been exploited in Barozzi-Tamanini [2], where it is shown that on the stated assumption, the obstacle problem for minimal boundaries is equivalent to the problem of minimizing an *unconditioned* functional, containing a curvature term depending on $H(x)$. (We refer to [2] for a detailed discussion of this result). The above equivalence is not needed in its fullness in the study of convergence properties of minimal boundaries. Consequently, the basic hypotesis can be slightly relaxed,

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and the argument becomes still more elementary. This will be done in Section 1 below.

Examples showing the independence of the assumptions used here and in [6] are presented in Section 2. Finally, we show in Section 3 that the same result can also be obtained by refining the argument of [6].

1. In the following, Ω will denote a fixed open set of $\mathbf{R}^n$, $n \geq 2$, and A an open bounded subset of Ω, with $A \subset\subset \Omega$ (i.e. $\bar{A} \subset \Omega$). Let $E, L \subset \mathbf{R}^n$ be sets of locally finite perimeter in Ω, i.e.

$$|D\varphi_E|(A) = \sup\left\{\int_E \operatorname{div}\phi(x)dx:\ \phi \in C_0^1(A, \mathbf{R}^n),\ |\phi(x)| \leq 1\ \forall x\right\} < +\infty$$

for every $A \subset\subset \Omega$, and similarly for L.

Here, φ_E denotes the characteristic function of E, and $|D\varphi_E|$ is the total variation of the distributional gradient of φ_E. We recall that when ∂A is locally lipschitz, then φ_E has a *trace* (which we also denote by φ_E) belonging to $L^1(\partial A)$. See Giusti [3] or Massari-Miranda [4].

Assume that

$$E \cap \Omega \supset L \cap \Omega \tag{1.1}$$

$$|D\varphi_E|(A) \leq |D\varphi_F|(A) \tag{1.2}$$

$$\text{for every } A \subset\subset \Omega \text{ and}$$

$$\text{for every } F: F \,\Delta\, E = (F \cup E) - (F \cap E) \subset\subset A$$

$$F \cap A \supset L \cap A$$

where set inclusion is to be intended in the usual measure theoretical sense, so that (1.1) means $|\Omega \cap L - E| = 0$, $|\cdot|$ denoting Lebesgue measure in $\mathbf{R}^n$.

When (1.1), (1.2) hold, E will be called *a set of minimal boundary in Ω, with respect to the obstacle L.* The reader is referred to Massari-Miranda [4], Barozzi-Massari [1], Tamanini [7], for a discussion of the existence and regularity of the solution of (1.1), (1.2) corresponding to a given obstacle L.

Now, assume we are given a sequence $\{L_h\}$ of obstacles and a corresponding sequence $\{E_h\}$ of solutions, satisfying (1.1), (1.2).

Assume that $L_h \to L_0$ and $E_h \to E_0$, locally in Ω, that is:

$$(1.3) \qquad \lim_h \int_A |\varphi_{L_h} - \varphi_{L_0}|\,dx = 0 \quad \text{for every } A \subset\subset \Omega$$

and similarly for E_h, E_0. As a simple example shows (see e.g. Tamanini [6], Section 3, ii)), it may happen that E_0 is *not* a solution with respect to L_0. An additional condition implying that E_0 is a solution with respect to L_0 was introduced in Tamanini [6]. Specifically, setting for E, L

$$(1.4) \quad \psi_A(E, L) = |D\varphi_E|(A) - \inf\{|D\varphi_F|(A) : F \,\Delta\, E \subset\subset A,\, F \cap A \supset L \cap A\}$$

with $E \cap A \supset L \cap A$, and assuming that

$$(1.5) \qquad L_h \to L_0,\; E_h \to E_0, \qquad E_h \cap A \supset L_h \cap A \qquad \forall h$$

and that

$$(1.6) \quad |D\varphi_{L_h}|(A') \to |D\varphi_{L_0}|(A') \quad \text{for every } A' \text{ open } \subset\subset A : |D\varphi_{L_0}|(\partial A') = 0$$

it holds (see Tamanini [6], pag. 155):

$$(1.7) \qquad \psi_A(E_0, L_0) \leq \liminf_h \psi_A(E_h, L_h)\,.$$

From this the above assertion follows at once, by noting that $\psi_A(E, L) = 0$ *iff E is a solution with respect to L.*

In this paper the same result will be proved with (1.6) replaced by the following assumption

$$(H) \quad \begin{cases} \textit{there exists } H \in L^1_{\mathrm{loc}}(\Omega) \textit{ such that for every } h \geq 1 \\ |D\varphi_{L_h}|(A) \leq |D\varphi_G|(A) + \displaystyle\int_{L_h - G} H(x)\,dx, \\ \qquad\qquad\qquad\qquad \forall A \subset\subset \Omega,\; \forall G \subset L_h : G \,\Delta\, L_h \subset\subset A\,. \end{cases}$$

The meaning of condition (H) is illustrated in Barozzi-Massari [1], where it is used in connection with the study of the regularity of minimal boundaries with obstacles. A variant of (H) has been used in Barozzi-Tamanini [2], in connection with a problem of penalization.

Roughly speaking, it implies an upper bound of the mean curvature of ∂L_h in Ω.

Among the simplest geometrical conditions implying (H), we recall the *Internal Sphere Condition of radius R*:

$$(\mathrm{ISC}_R) \qquad \begin{cases} \textit{there exists a sequence } \{x_i\} \subset L_h, \textit{ such that} \\ L_h = \bigcup\limits_{i=1}^{\infty} B_{x_i,R} \end{cases}$$

where, as usual, $B_{x,t}$ denotes the open n-ball of centre x and radius $t > 0$. The fact that (ISC_R) implies (H), with $H(x) = n/R$, follows from (1.10) of Tamanini [5].

We are now in position to prove the following

THEOREM. *Suppose that, for every $h \geqslant 1$, E_h has minimal boundary in Ω with respect to the obstacle L_h; let $L_h \to L_0$ and $E_h \to E_0$ locally in Ω and, moreover, let condition (H) hold; then E_0 has minimal boundary with respect to the obstacle L_0, and in addition*

$$(1.8) \qquad |D\varphi_{E_h}|(A) \to |D\varphi_{E_0}|(A) \quad \textit{for every } A \subset\subset \Omega\colon |D\varphi_{E_0}|(\partial A) = 0$$

PROOF. First, we show that

$$(1.9) \qquad |D\varphi_{E_h}|(A) \leqslant |D\varphi_F|(A) + \int\limits_{A \cap L_h - F} H(x)\,dx \qquad \forall A \subset\subset \Omega,\ \forall F\colon F \,\Delta\, E_h \subset\subset A$$

For, if A and F are as in (1.9), then by (1.2), 2.1.2 (10) of Massari-Miranda [4] and (H), we obtain:

$$|D\varphi_{E_h}|(A) \leqslant |D\varphi_{F \cup L_h}|(A) \leqslant |D\varphi_F|(A) + |D\varphi_{L_h}|(A) - |D\varphi_{L_h \cap F}|(A) \leqslant$$
$$\leqslant |D\varphi_F|(A) + \int\limits_{A \cap L_h - F} H(x)\,dx$$

as required.

Next, fix $A \subset\subset \Omega$ and $F\colon F \,\Delta\, E_0 \subset\subset A$. Then, by passing, if necessary, to subsequences, we can pick an open set B, with lipschitz boundary

∂B, such that:

$$F \,\Delta\, E_0 \subset\subset B \subset\subset A \tag{1.10}$$

$$|D\varphi_{E_h}|(\partial B) = 0 \quad \forall h \geqslant 0, \qquad \lim_{h \to +\infty} \int_{\partial B} |\varphi_{E_h} - \varphi_{E_0}|\, dH_{n-1} = 0 \,. \tag{1.11}$$

Put $F_h = (E_h - B) \cup (F \cap B)$, so that $F_h \,\Delta\, E_h \subset\subset A$ and

$$|D\varphi_{F_h}|(A) = |D\varphi_{E_h}|(A - B) + |D\varphi_F|(B) + \int_{\partial B} |\varphi_{E_h} - \varphi_{E_0}|\, dH_{n-1} \,. \tag{1.12}$$

Writing (1.9) for F_h, using (1.12) and reducing terms, we find for every $h \geqslant 1$:

$$|D\varphi_{E_h}|(B) \leqslant |D\varphi_F|(B) + \int_{\partial B} |\varphi_{E_h} - \varphi_{E_0}|\, dH_{n-1} + \int_{B \cap L_h - F} H(x)\, dx \tag{1.13}$$

(recall that $E_h \cap \Omega \supset L_h \cap \Omega \ \forall h \geqslant 1$).

Letting $h \to +\infty$ in (1.13) we obtain, by virtue of (1.10), (1.11), the lower semicontinuity of the perimeter and Lebesgue dominated convergence theorem:

$$|D\varphi_{E_0}|(A) \leqslant |D\varphi_F|(A) + \int_{A \cap L_0 - F} H(x)\, dx, \qquad \forall A \subset\subset \Omega \,, \tag{1.14}$$

which holds for every F: $F \,\Delta\, E_0 \subset\subset A$. If in addition we assume that $F \cap A \supset L_0 \cap A$, then the integral in (4.14) vanishes, thus showing that E_0 has minimal boundary in Ω with respect to L_0.

Moreover, writing (1.13) for $F = E_0$ and letting $h \to +\infty$ we get

$$\limsup_h |D\varphi_{E_h}|(B) \leqslant |D\varphi_{E_0}|(B)$$

which implies

$$\lim_h |D\varphi_{E_h}|(A') = |D\varphi_{E_0}|(A')$$

for every open $A' \subset\subset B$ such that $|D\varphi_{E_0}|(\partial A') = 0$ (see e.g. Giusti [3], Prop. 1.13).

Assertion (1.8) now follows at once. C.V.D.

2. We point out that our present assumption (H) is independent of (1.6).

To show this, we consider the following examples.

I) Consider, for every $h \geqslant 1$, $L_h = L^1 \cup L^2$ (see figure 1), where

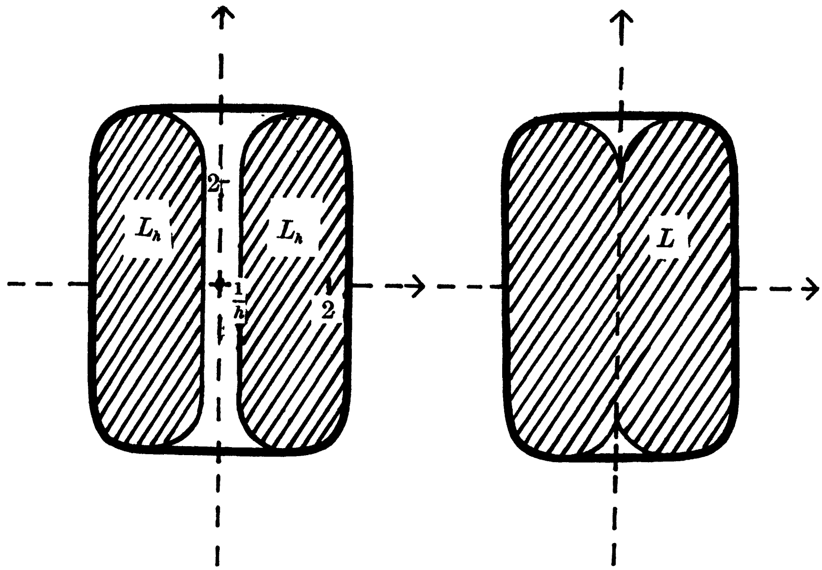

Fig. 1.

$L^i = conv\,(B^i_-, B^i_+)$ and $B^i_\pm$, $i = 1, 2$, denotes the closed disk in $\Omega = \mathbf{R}^2$ of unit radius and centered at

$$\big((-1)^i\,(1 + h^{-1}),\ \pm 2\big)$$

and where *conv* denotes the convex hull. Clearly, (ISC_1) holds for every $h \geqslant 1$, so that (H) holds for every $h \geqslant 1$, while

$$\lim_h |D\varphi_{L_h}|(A) > |D\varphi_{L_0}|(A) \quad \text{for every } A : L_0 \subset\subset A$$

Here, $E_h = conv\, L_h$ for every $h \geqslant 0$

II) Denote by S_i the sector of the circle of radius $2\pi/i$ and angle $2\pi/i$ in figure 2 and, for $j = 1, 2, \ldots, i$, denote by $S_{i,j}$ the sector S_i rotated clockwise by $2\pi(j-1)/i$.

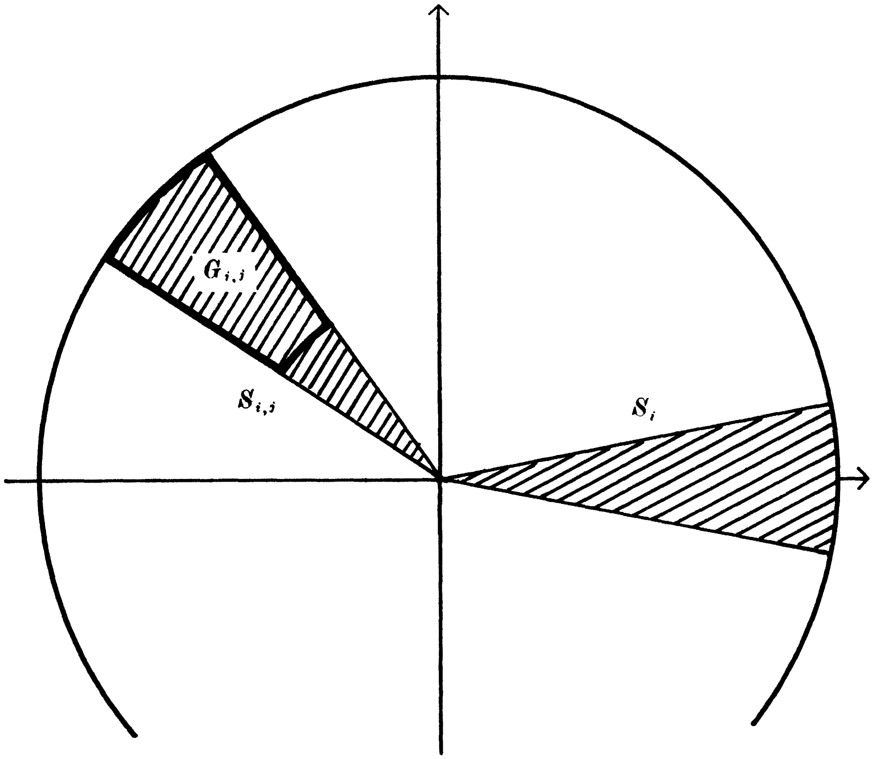

Fig. 2.

Then form the sequence L_h as follows:

$$L_1 = S_{1,1}$$

$$L_2 = S_{2,1} \qquad L_3 = S_{2,2}$$

$$L_4 = S_{3,1} \qquad L_5 = S_{3,2} \qquad L_6 = S_{3,3}$$

$$\cdots\cdots\cdots$$

We have $|D\varphi_{L_h}|(R^2) \to 0$, $|L_h| \to 0$ as $h \to +\infty$, so that (1.6) holds, with $L_0 = \emptyset$.

On the other hand, assume by contradiction that (H) holds. Then, recalling that $L_h = S_{i,j}$ for suitable $i \geqslant 1$ and $j \in \{1, 2, \ldots, i\}$, and setting

$$G = G_{i,j} = S_{i,j} \cap \left\{x \in R^2 \colon |x| > \frac{\pi}{\mathrm{i}}\right\},$$

we would derive from (H):

$$\frac{2\pi}{i} \leqslant \frac{2\pi^2}{i^2} + \int_{S_{i,j}-G_{i,j}} H(x)\,\mathrm{d}x$$

for every $i \geqslant 1$ and $j \in \{1, \ldots, i\}$. By summing over $j = 1, \ldots, i$ we thus get for every i:

$$2\pi \leqslant \frac{2\pi^2}{i} + \int_{B_{0,\pi/i}} H(x)\,\mathrm{d}x$$

which gives the desired contradiction, since $H \in L^1_{\mathrm{loc}}(R^2)$.

3. We notice that the preceding theorem can also be proved by the method developed in Tamanini [6].

In fact, one can show that (1.7) holds in the hypothesis (1.5) and (H). To see this, fix $A \subset\subset \Omega$ and $F \colon F \Delta E_0 \subset\subset A$, $F \cap A \supset L_0 \cap A$. Then, as in the proof of the Theorem, choose B with lipschitz boundary ∂B such that

$$(1.15) \qquad F \Delta E_0 \subset\subset B \subset\subset A, \qquad |D\varphi_{E_h}|(\partial B) = |D\varphi_{L_h}|(\partial B) = 0 \qquad \forall h \geqslant 0\,,$$

$$(1.16) \qquad \lim_h \int_{\partial B} |\varphi_{E_h} - \varphi_{E_0 \cup L_h}|\, dH_{n-1} = \lim_h \int_{\partial B} |\varphi_{L_h} - \varphi_{L_0 \cap L_h}|\, dH_{n-1} = 0$$

Setting $F_h = (E_h - B) \cup \big((F \cup L_h) \cap B\big)$, we have $F_h \Delta E_h \subset\subset A$, $F_h \cap A \supset L_h \cap A$ and

$$|D\varphi_{F_h}|(A) \leqslant |D\varphi_{E_h}|(A - B) + |D\varphi_F|(B) + |D\varphi_{L_h}|(B) - \\ - |D\varphi_{F \cap L_h}|(B) + \int_{\partial B} |\varphi_{E_h} - \varphi_{E_0 \cup L_h}|\, dH_{n-1}$$

Thus:

$$(1.17)\qquad \psi_A(E_h, L_h) \geq |D\varphi_{E_h}|(A) - |D\varphi_{F_h}|(A) \geq |D\varphi_{E_h}|(B) - |D\varphi_F|(B) + \\ + |D\varphi_{F\cap L_h}|(B) - |D\varphi_{L_h}|(B) - \int_{\partial B} |\varphi_{E_h} - \varphi_{E_0\cup L_h}|\, dH_{n-1} .$$

Now we use (H) to show that

$$(1.18)\qquad \liminf_h \{|D\varphi_{F\cap L_h}|(B) - |D\varphi_{L_h}|(B)\} \geq 0 .$$

For, setting $G_h = (L_h - B) \cup (F \cap L_h \cap L_0 \cap B)$ and observing that $G_h \triangle L_h \subset\subset A$ and $G_h \subset L_h$, we get from (H) that

$$|D\varphi_{L_h}|(A) \leq |D\varphi_{L_h}|(A - B) + |D\varphi_{F\cap L_h\cap L_0}|(B) + \\ + \int_{\partial B} |\varphi_{L_h} - \varphi_{L_h\cap L_0}|\, dH_{n-1} + \int_{L_h - G_h} H(x)\, dx$$

Using 2.10.2 (10) of Massari-Miranda [4] and rearranging, we obtain

$$|D\varphi_{F\cap L_h}|(B) - |D\varphi_{L_h}|(B) \geq |D\varphi_{(F\cap L_h)\cup L_0}|(B) - |D\varphi_{L_0}|(B) - \\ - \int_{\partial B} |\varphi_{L_h} - \varphi_{L_h\cap L_0}|\, dH_{n-1} - \int_{B\cap L_h - L_0} H(x)\, dx ,$$

which implies (1.18) (recall (1.16) and that $F \cap A \supset L_0 \cap A$).

From (1.17), (1.18) and (1.16) we get

$$\liminf_h \psi_A(E_h, L_h) \geq |D\varphi_E|(B) - |D\varphi_F|(B) = |D\varphi_E|(A) - |D\varphi_F|(A)$$

and (1.7) follows immediately.

Remark. Assuming a stronger integrability of H, the conclusion in (1.8) can be improved.

For example, if condition (H) holds with $H \in L^p_{\mathrm{loc}}(\Omega)$ and $p > n$l then we derive from known regularity results for almost-minima, boundaries (see Tamanini [7], [8]) that

(i) $x_h \in \partial E_h \ \forall h \geqslant 1, \ x_h \to x_0 \in \Omega \Rightarrow x_0 \in \partial E_0$

(ii) $x_h \in \partial E_h \ \forall h \geqslant 1, \ x_h \to x_0 \in \partial^* E_0 \Rightarrow x_h \in \partial^* E_h$

for every h sufficiently large, and $\nu_{E_h}(x_h) \to \nu_{E_0}(x_0)$, where, as usual (see Giusti [3] or Massari-Miranda [4]), $\partial^* E$ denotes the « reduced boundary » of E and $\nu_E(x)$ is the inner unit normal to ∂E at x.

We refer for details to Tamanini [8].

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