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## On Dedekind Domains in Infinite Algebraic Extensions.

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**SUMMARY** - The aim of this note is to prove the following result: Let  $A$  be a Dedekind domain with quotient field  $F$ , and  $K$  an algebraic extension of  $F$ . Let  $W$  be a set of discrete valuations of rank one on  $K$ , and  $O = \{x: x \in K \text{ such that } w(x) \geq 0 \text{ for all } w \in W\}$ . Then  $O$  is a Dedekind domain whose quotient field is  $K$ , if and only if for every maximal ideal  $P$  of  $A$  the set  $W(P) = \{w|w \in W \text{ such that } w \text{ extend the valuation on } F \text{ defined by } P\}$ , is finite. In such a way we can give various examples of Dedekind domains in infinite algebraic extensions.

1. The theory of Dedekind domains was created as a generalisation of results concerning the rings of integers in finite extensions of the rational field, obtained mainly by Dedekind.

We shall say that a field  $K$  has a classical ideal theory relative to a Dedekind domain  $A$ , if  $A$  is a proper subring of  $K$  and  $K$  is its quotient field. It is clear that not every field has a classical ideal theory (for example the finite fields and algebraically closed fields).

The aim of this work is to make some remarks on the Dedekind domains in infinite algebraic extensions of a field  $F$  which has a classical ideal theory. Since the theorem which we shall prove has a general character, the main applications of this theorem concerne infinite algebraic extensions of rational numbers.

In [3] are defined so-called Stiemke fields. In a Stiemke field,

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there exist a classical ideal theory relative to a smallest Dedekind domain. On the other hand we can define infinite extensions of rational numbers which have a classical ideal theory relative to a smallest Dedekind domain but these fields are not Stiemke fields (Example 3). Moreover (Example 1) we show that there exist infinite extensions of rational numbers which do not contain a smallest Dedekind domain.

Another question is relative to residue class fields of a Dedekind domain. We show (Example 1) that in some infinite extensions of the field of rational number there exist Dedekind domains  $A$  with property that  $A/P$  is a finite field for every maximal ideal  $P$  of  $A$ . On the other hand (Example 2) we may define Dedekind domains  $A$  whose field of quotients is an infinite algebraic extension of rational numbers and such that  $A$  contains infinitely (or finitely) many maximal ideals  $P$  such that  $A/P$  is finite, and in the same time infinitely (or finitely) many maximal ideals  $P$  such that  $A/P$  is an infinite field.

**2.** Let  $A$  be a Dedekind domain and  $F$  its quotient field. Denote  $M$  the set of all maximal ideals of  $A$ . For every  $P \in M$  denote  $v_P$  the valuation on  $F$  associated with  $P$ .

The set  $V = \{v_P\}_{P \in M}$  is called the set of the valuations of  $F$  which define  $A$ .

Let  $K$  be an algebraic extension of  $F$ . Assume that for every maximal ideal  $P$  of  $A$  there exists a non-empty set  $W(P)$  of non-equivalent (rank-one and discrete) valuations on  $K$  which extend the valuation  $v_P$ . Let  $W = \bigcup_{P \in M} W(P)$ . For every  $v \in W$ , denote  $O_v$  the valuation ring of  $v$  in  $K$ , and let  $P_v$  be the maximal ideal of  $O_v$ . Let us denote

$$O = \bigcap_{v \in W} O_v .$$

**THEOREM.** The hypotheses and notations are as above. The following assertions are equivalent:

- a) The ring  $O$  is a Dedekind domain whose quotient field is  $K$ .
- b) For every  $P \in M$ , the set  $W(P)$  is finite.

**PROOF.** a)  $\Rightarrow$  b). Let  $P \in M$ . There exists a non-zero element  $a \in F$  such that  $v_P(a) \neq 0$ . Thus one has:

$$W(P) \subseteq \{v \in W, v(a) \neq 0\} .$$

The hypothesis  $O$  Dedekind domain shows that the set in the right of the above inclusion is finite, i.e.  $W(F)$  is finite.

$b) \Rightarrow a)$ . We shall show that conditions I-IV in [3], p. 95 are accomplished (see also [1]).

(I). Let  $x \in K$  and  $O_x = O \cap F(x)$ . Let  $A_x$  be the integral closure of  $A$  in  $F(x)$ . It is clear that  $A_x \subseteq O_x$ . Since  $F(x)$  is a finite extension of  $F$ , one sees that  $F(x)$  is the quotient field of  $A_x$  hence of  $O_x$ . This means that  $x = ab^{-1}$ , where  $a, b \in O_x$ . Therefore the quotient field of  $O$  is  $K$ .

(II). Let  $x$  be a non-zero element of  $K$ . The inclusion  $A_x \subseteq O_x$  shows that  $O_x$  is also a Dedekind domain. Hence the set of all  $v \in W$  such that  $v(x) \neq 0$  must be finite.

(III). Let  $v_1, v_2 \in W$ ,  $v_1 \neq v_2$ . This means that there exist  $x \in K$  such that  $v_1(x) \neq v_2(x)$ . Since  $O_x = F(x) \cap O$  is a Dedekind domain, there exist  $a \in O_x$  such that  $v_1(a) > 0$  and  $v_2(a) = 0$ .

(IV). We must show that every prime non-zero ideal of  $O$  is maximal. Denote by  $D$  the set of all finite subsets of  $K$ . It is clear that  $K = \bigcup_{S \in D} F(S)$ , where  $F(S)$  is the smallest subfield of  $K$  which contains  $F$  and  $S$ . For every  $S \in D$ , denote  $O_S = O \cap F(S)$ , and  $A_S$  the integral closure of  $A$  in  $F(S)$ . Since  $A_S \subseteq O_S \subseteq F(S)$  and  $A_S$  is obviously a Dedekind domain it follows that  $O_S$  is also a Dedekind domain. One has

$$O = O \cap K = O \cap \left( \bigcup_{S \in D} F(S) \right) = \bigcup_{S \in D} (O \cap F(S)) = \bigcup_{S \in D} O_S.$$

Now let  $I$  be an ideal of  $O$  and  $I_S = I \cap O_S$ . Then  $I_S$  is an ideal of  $O_S$  and as above  $I = \bigcup_{S \in D} I_S$ . Furthermore, let  $P$  be a non-zero prime ideal of  $O$ , and  $I$  an ideal of  $O$  such that  $P \subset I$  and  $P \neq I$ . Since  $P$  is non-zero, there exists an element  $S_1 \in D$  so that  $P_{S_1} \neq 0$ . Also for any  $S \in D$  one has  $P_S \subset I_S$ , and the condition  $P \neq I$ , implies that there exists  $S_2 \in D$ , such that  $P_{S_2} \neq I_{S_2}$ . Let  $S = S_1 \cup S_2$ . Since  $0 \neq P_{S_1} \subset P_S$  one sees that  $P_S$  is a non-zero prime ideal of  $O_S$ . On the other hand, the condition  $P_{S_2} \neq I_{S_2}$ , means that  $P_S \neq I_S$ , hence  $I_S = O_S$ . Therefore the ideal  $I_S$ , hence also  $I$ , contains the identity element of  $O$  i.e.  $I = O$ . This proves that  $P$  is a maximal ideal of  $O$ .

As a consequence of the above Theorem one has the following result.

**COROLLARY** ([3], Ch. IV, Theorem 5). Let  $A$  be a Dedekind domain,  $F$  its quotient field and  $K$  an algebraic extension of  $K$ . Let  $V$  be the set of all valuations of  $F$  which define  $A$  and denote  $W$  the set of all prolongations of all elements of  $V$  to  $K$ . The following assertions are equivalent:

1) The ring  $O = \bigcap_{w \in W} O_w$  is a Dedekind domains.

2) Every  $v \in V$  has a finite number of prolongations to  $K$ , and every element of  $W$  is a discrete valuation.

A field  $K$  which verifies the equivalent condition of the Corollary is called a Stiemke field with respect to the pair  $(A, F)$  (see [3], pag. 110). Also, [3], pag. 111 gives an example of a Stiemke field with respect to the pair  $(\mathbb{Z}, Q)$ , where  $\mathbb{Z}$  is the ring of integers and  $Q$  the field of rational numbers. It is easy to see that every Stiemke field with respect to a pair  $(A, F)$  can be constructed as in [3], Lemma 31, pag. 111.

On the other hand, using the ideas of [3] (Lemma 31, pag. 111) we can define various examples of Dedekind domains.

In what follows let  $p_1, p_2, \dots, p_n \dots$  be the increasing sequence of prime numbers; denote  $v_n$  the valuation on  $Q$  defined by  $p_n$ .

**EXAMPLE 1.** We shall define an infinite algebraic extension  $K$  of  $Q$  such that the prolongation to  $K$  of any valuation  $v$  on  $Q$ , is a discrete valuation.

For every natural number  $m$  we may define an algebraic number field  $K_n$  such that:

i)  $[K_{n+1}:K_n] = 2$ ,  $K_n \subset K_{n+1}$  for all  $n \geq 1$ .

ii) For every  $1 < i \leq n$ , the valuation  $v_i$  has at least  $2^{n-i+1}$  prolongations to  $K_n$ .

Suppose that  $K_n$  is already constructed, and let  $W_1, \dots, W_r$  be the set of all prolongations to  $K_n$  of the first  $n + 1$  valuations  $v_1, \dots, v_{n+1}$ . Let  $B$  be the integral closure of  $\mathbb{Z}$  in  $K_n$ . There exists for every  $j$ ,  $1 \leq j \leq r$ , a monic polynomial  $p_j(X) \in B[X]$  of degree two and such that the image of  $p_j(X)$  in the residue class field associated with  $w_j$ , has two distinct roots. By the Chinese Remainder Theorem there exists then an irreducible polynomial  $p(X)$  of degree two,  $p(X) \in B[X]$  such that  $p(X) \equiv p_j(X) \pmod{P_j}$ , where  $P_j$  is the prime ideal of  $B$  associated with  $w_j$ ,  $1 \leq j \leq r$ . Let  $K_{n+1}$  be generated by a root of  $p(X)$ .

Each valuation  $w_j$  determines in  $K_{n+1}$  precisely two prolongations.

We let  $K = \bigcup_n K_n$ . It is easy to see that every valuation of  $K$  is discrete and its residue class field is finite. Also it is plain that every valuation  $v_n$  of  $Q$  has infinitely many prolongations to  $K$ .

Let  $A$  be the integral closure of  $\mathbb{Z}$  in  $K$ . According to the above theorem one sees that  $A$  is not a Dedekind domain. On the other hand according to the above theorem we can define infinitely many Dedekind domains  $A_n$ ,  $n \in \mathbb{N}$ , such that  $K$  is the quotient field of  $A_n$  for all  $n$ .

**EXAMPLE 2.** We shall define an infinite algebraic extension  $K$  of  $Q$ , such that there exist on  $K$  infinitely many discrete valuations, whose residue class field is infinite, and also infinitely many discrete valuations, whose residue class field is finite. In this way, according to the above theorem we can define examples of Dedekindian domains  $A$  such that  $A$  contains infinitely (or finitely) many maximal ideals  $P$  such that  $A/P$  is an infinite field, and also infinitely (or finitely) many maximal ideals  $P'$  such that  $A/P'$  is a finite field.

We wish to construct  $K$  as the join  $\bigcup_{n \geq 1} K_n$  of finite extensions  $K_n/Q$  such that:

$$1) K_n \subset K_{n+1} \text{ and } [K_{n+1}:K_n] = 2.$$

2) For every  $1 \leq k \leq n$ , the valuation  $v_{2k}$  has at least  $2(n - k + 1)$  prolongations to  $K_n$ .

3) For every  $1 \leq k \leq n$  the valuation  $v_{2^{k-1}}$  has at the most  $2k - 1$  prolongations to  $K_n$ .

Let us assume that  $K_n$  has been defined such that above conditions are accomplished. We shall define  $K_{n+1}$ . By the Chinese Remainder Theorem, there exists a monic polynomial  $f(X)$  in  $B[X]$  ( $B$  being the ring of integers in  $K_n$ ) of degree two such that for every  $1 \leq k \leq n + 1$ , the image of  $f(X)$  is irreducible in the residue class field of all prolongations of  $v_{2^{k-1}}$  and that the image of  $f(X)$  has two distinct roots in the residue class field of all prolongations of  $v_{2^k}$ . Let us define  $K_{n+1} = K_n(a)$  where  $a$  is a root of  $f(X)$ .

**EXAMPLE 3.** Now we shall define an infinite algebraic extension  $K$  of  $Q$  such that every valuation on  $Q$  has finitely many prolongations to  $K$  which are discrete (of course, generally a valuation on  $Q$  has another set, eventually infinite, of prolongations to  $K$  which are not discrete).

For any natural number  $n$  we shall define an extension  $K_n$  of  $Q$  such that

1)  $K_n \subset K_{n+1}$  and  $[K_{n+1}:K_n] = 2$ .

2) The valuation  $v_n$  has a prolongation  $w_n$  to  $K_n$  such that  $w_n$  has only a prolongation to  $K_{n+t}$  for all  $t \geq 1$  and this prolongation is unramified.

3) If  $u_1, \dots, u_s$  are other prolongations of  $v_n$  to  $K_n$ , then these valuations are totally ramified in  $K_{n+t}$ , for all  $t \geq 1$ .

Let us assume that  $K_n$  has been defined such that above conditions are accomplished. We shall define  $K_{n+1}$ . For that let  $w_i$ , be the extension of  $v_i$   $1 \leq i \leq n$ , at  $K_n$  which satisfies the above condition 2) and let us choose  $w_{n+1}$  an prolongation of  $v_{n+1}$  to  $K_n$ . Furthermore let  $u_1, \dots, u_h$  be all the prolongations of  $v_1, \dots, v_{n+1}$  to  $K_n$  which are distinct of  $w_1, \dots, w_{n+1}$ . Denote by  $B$  the ring of algebraic integers of  $K_n$ . According to the Chinese Remainder theorem we can define a monic polynomial, of degree two  $f(X) \in B[X]$  such that the image of  $f(X)$  over the residue field of  $w_i$ ,  $1 \leq i \leq n+1$  is irreducible and its image over the residue field of all  $u_i$ ,  $1 \leq i \leq h$  is the square of an irreducible polynomial. It is clear that  $f(X)$  is irreducible over  $K_n$  and thus we define  $K_{n+1} = K_n(a)$ , where  $a$  is a root of  $f(X)$ . It is plain to show that above conditions 1)-3) are accomplished. Let  $K = \bigcup_{n \geq 1} K_n$ . The field  $K$  is an infinite extension of  $Q$  and every valuation  $v_n$  of  $Q$  has only one prolongation  $\bar{w}_n$  to  $K$  which is discrete. According to the above Theorem the ring  $A = \bigcap_{n \geq 1} Q_{\bar{w}_n}$  is the smallest Dedekind domain of  $K$ , and obviously  $K$  is not Stiemke field.

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