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Some Cardinal Invariants for Valuation Domains.

LUIGI SALCE - PAOLO ZANARDO

Introduction.

The condition on a valuation domain V of being maximal, which goes back to Krull [8], and the very close condition of being almost maximal, due to Kaplansky [7], were extensively investigated by many authors, on account of their importance for the consequences deriving for the ring structure of R and for many classes of R -modules.

On the contrary, the problem of measuring in some way the failure of the maximality did not receive too much attention up to now; the only contributions in this direction known by the authors are by Brandal [1], and by Facchini and Vamos [2].

As any valuation domain R is a subring of a maximal valuation domain S , which is an immediate extension of it, it is natural to try to measure the failure of the maximality for R by looking for cardinal invariants which measure, roughly speaking, how large is S over R .

In this paper, given any ideal I of R , we will introduce two cardinal invariants associated with I : the *completion defect at I* , denoted by $c_R(I)$, and the *total defect at I* , denoted by $d_R(I)$; their definition, which seems very technical, raised up naturally in the investigation of indecomposable finitely generated modules in [13].

The completion defect $c_R(I)$ measures how large is $(R/I)^\wedge$, the com-

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pletion of R/I in the ideal topology, and the total defect $d_R(I)$ measures how large is S/IS over R/I ; recall that both R/I and $(R/I)^\wedge$ are contained in S/IS up to canonical isomorphisms.

It is noteworthy that the invariants that we are introducing do not depend on the ring structure of S , which is not unique up to isomorphism, but only on the R -module structure of S , which is a pure-injective hull of R (see [7] and [12]).

In the first section we introduce the *breadth ideal (of non maximality)* of the valuation domain R , a concept originally due to Brandal [1], and the *breadth ideal of a unit* of S , a concept defined in a slightly different way by Ostrowsky [11], Kaplansky [7] and Nishi [10].

The breadth ideals of the units of S are used in the second section to define the completion defect and the total defect at an ideal I of R . The main result in this section is an inequality which relates the total defect at an ideal I with the completion defects at the ideals containing I . This inequality however is in general strict, as is shown, for a special class of discrete valuation domains, by Facchini and the second author in [3].

In section 3 we compare the total defect at an ideal I with the Goldie dimension $g_R(I)$ of S/IS as an R/I -module; they turn out to be equal if I is a prime ideal, while in the non-prime case the total defect becomes generally larger.

We remark that the invariants of the valuation domain R that we investigate here play a relevant role in the study of many classes of R -modules: besides finitely generated modules (see [13]), the R -modules JS/IS , where $I > J$ are fractional ideals of R (see [4]); indecomposable injective R -modules (see [2]) and torsion-free R -modules of finite rank (see [5]).

1. The breadth.

R will always denote a valuation domain, P its maximal ideal and Q its field of quotients. Recall that R is maximal if it is linearly compact (in the discrete topology); R is almost maximal if every proper factor of it is linearly compact.

A valuation domain S containing R as a subring is an immediate extension of R if

- (i) every ideal of S is of the form IS , where I is an ideal of R , and $IS \cap R = I$;

(ii) S/PS is naturally isomorphic to R/P or, equivalently, $S = PS + R$.

An immediate extension S of R is maximal if, given any valuation domain S' containing S as a proper subring, either (i) or (ii) fails for S' .

It is well known (see [7] or [12]) that every valuation domain R is contained in a maximal immediate extension S , which is a maximal valuation domain. However S is uniquely determined up to R -isomorphism only, and not as a ring, unless R is almost maximal, in which case S is the completion of R (in the valuation topology). R coincides with S if and only if it is maximal.

Brandal considered in [1] the family of ideals of R

$$\mathcal{F} = \{I \leq R: R/I \text{ is not linearly compact}\}$$

and he showed that either $\mathcal{F} = \emptyset$, or there exists a prime ideal L of R such that

$$\mathcal{F} = \{I: I \leq L\} \quad \text{or} \quad \mathcal{F} = \{I: I < L\}.$$

Fixed a maximal immediate extension S of R , we reformulate this result by introducing the following subset of R , called the *breadth of R* (with a more meaningful term we could call it the *breadth of non maximality of R*):

$$B(R) = \{a \in R: S > R + aS\}.$$

Notice that $B(R) = \emptyset$ whenever $S = R + aS$ for all $a \in R$; this happens exactly if $S = R$, i.e. if R is maximal; thus from now on we shall assume that R is a valuation domain not maximal, so $B(R)$ is an ideal of R .

PROPOSITION 1.1. Let R be a valuation domain not maximal. Then its breadth $B(R)$ is a prime ideal of R such that:

$$\begin{aligned} B(R) &= \cap \{I: R/I \text{ is linearly compact}\} \\ &= \cup \{I: R/I \text{ is not linearly compact}\}. \end{aligned}$$

PROOF. Assume that $a, b \in R \setminus B(R)$. Then $S = R + aS = R + bS$, and $bS = b(R + aS)$ implies $S = R + b(R + aS) = R + abS$. Therefore $ab \notin B(R)$, so that $B(R)$ is prime.

If R/I is linearly compact, then $R/I \cong S/IS$ in a natural way, therefore $S = R + IS$; thus $I \geq B(R)$. It follows that $B(R)$ is contained in $\cap \{I: R/I \text{ is linearly compact}\}$. Conversely, if $I > B(R)$, then $S = R + IS$, thus $R/I \cong S/IS$ is linearly compact. Being $B(R)$ prime, either $B(R) = P$, in which case the first equality is trivial, or $B(R)$ is the intersection of the ideals properly containing it, thus the first equality is obvious. The second equality can be proved in a similar way. ///

From Proposition 1.1 it follows that $B(R)$ does not depend on the choice of S , and that it coincides with the ideal L quoted in the Brandal's result. Notice that R is almost maximal (and not maximal) if and only if $B(R) = 0$.

The valuation domain $R/B(R)$ is always almost maximal; Brandal gives examples in [1] showing that $R/B(R)$ can be maximal or not.

Let us denote by $U(S)$ and (UR) respectively the multiplicative groups of the units of S and R . Every $0 \neq x \in S$ can be written in a unique way, up to units of E , in the form $x = \varepsilon r$, with $\varepsilon \in U(S)$ and $r \in R$; by this reason we will confine ourselves to consider units of S in the following discussion.

Given any $\varepsilon \in U(S) \setminus R$, consider the ideal of R , called the *breadth of ε*

$$B(\varepsilon) = \{a \in R: \varepsilon \notin R + aS\}.$$

REMARK. Our definition of breadth of a unit of S is essentially the same as the one given by Nishi [10], which is a slight modification of the original definition of breadth of a pseudoconvergent set of elements of R given by Kaplansky [7], and originally due to Ostrowsky [11]. The definition of breadth (of non maximality) of R is originated by the two above definitions.

From the definitions of $B(R)$ and $B(\varepsilon)$ it trivially follows that $B(\varepsilon) \leq B(R)$. Conversely, let $a \in B(R)$; then $S > R + aS$, thus there exists $\varepsilon \in U(S)$ such that $\varepsilon \notin R + aS$, therefore $a \in B(\varepsilon)$; we have proved

PROPOSITION 1.2. Let R be a valuation domain not maximal. Then $B(R) = \cup \{B(\varepsilon): \varepsilon \in U(S) \setminus R\}$.

The following result will be useful in the next section; it is similar to [10, Prop. 6].

LEMMA 1.3. Let R be a valuation domain not maximal and $\varepsilon \in U(S) \setminus R$. If $u \in U(R)$ and $0 \neq r \in P$, then $B(u + r\varepsilon) = rB(\varepsilon)$.

PROOF. $\varepsilon \notin R + aS (a \in R)$ if and only if $r\varepsilon \notin R + raS$, and this obviously is equivalent to $u + r\varepsilon \notin R + raS$. ///

We introduce the following notation: given $I \leq R$, let $f_I: S \rightarrow S/IS$ be the canonical surjection; then the image $f_I R$ of R is a subring of S/IS isomorphic to R/I ; its completion, whenever R/I is Hausdorff, is denoted by $(f_I R)^\wedge$. Notice that, being $f_I R$ pure in S/IS and S/IS complete, we have the following inclusions:

$$(1) \quad f_I R \leq (f_I R)^\wedge \leq S/IS.$$

The topology considered above, as in the following proposition, on the factor ring R/I is the «ideal topology», which has as a basis of neighborhoods of 0 the ideals $(aR)/I$, $a \in R \setminus I$.

PROPOSITION 1.4. Let R be a valuation domain, and $I \leq R$. Then R/I is Hausdorff and non complete if and only if $I = B(\varepsilon)$ for some $\varepsilon \in U(S) \setminus R$.

PROOF. In order to show that $R/B(\varepsilon)$ is Hausdorff, it is enough to prove that $a \notin B(\varepsilon)$ implies $pa \notin B(\varepsilon)$ for some $p \in P$. So let $\varepsilon \in R + aS$; then $\varepsilon = r + as (r \in R, s \in S)$. But $S = R + PS$ implies that $s = t + ps'$, for some $t \in R$, $p \in P$ and $s' \in S$; therefore we get: $\varepsilon = r + at + aps' \in R + paS$, as we want. Clearly $\varepsilon + B(\varepsilon)S \notin f_{B(\varepsilon)} R$, but it is the limit of a Cauchy net of elements of $f_{B(\varepsilon)} R$: for, given $r \notin B(\varepsilon)$, $\varepsilon \in R + rS$ implies that there exists $u_r \in U(R)$ such that $\varepsilon - u_r \in rS$; thus $\varepsilon + B(\varepsilon)S$ is the limit of the Cauchy net $\{u_r + B(\varepsilon)S : r \notin B(\varepsilon)\}$. So we have proved that $R/B(\varepsilon) \cong f_{B(\varepsilon)} R$ is not complete.

Conversely, assuming that R/I is Hausdorff and not complete, from the inclusions (1) we get an element $\varepsilon \in U(S) \setminus R$ such that $\varepsilon + IS$ is the limit of a Cauchy net $\{u_r + IS : r \notin I\}$ in $f_I R$. So $\varepsilon \in R + rS$ if and only if $r \notin I$, therefore $I = B(\varepsilon)$. ///

From the proof of the preceding proposition we deduce the following

COROLLARY 1.5. Let $\varepsilon \in U(S) \setminus R$, and $I \leq R$. Then $\varepsilon + IS \in f_I R$ if and only if $B(\varepsilon) < I$; $\varepsilon + IS \in (f_I R)^\wedge \setminus (f_I R)$ if and only if $B(\varepsilon) = I$. ///

A particular case is when $I = 0$; then the elements of the completions $\hat{R}$ of R are exactly those $x = r\varepsilon \in S$ ($0 \neq r \in R$, $\varepsilon \in U(S)$) such that $B(\varepsilon) = 0$. It was shown by Nishi [10] that $\hat{R}$ is the center $Z(A)$ of the ring $A = \text{End}_R E(R/P)$, which is isomorphic to $\text{End}_R(S)$; so we have the inclusions:

$$R \leq \hat{R} = Z(A) \leq S \leq A = \text{End}_R E(R/P) \cong \text{End}_R S.$$

2. The completion defect and the total defect.

We introduce now a new concept, which first appeared in [13]. Let R be a valuation domain not maximal, and S a maximal immediate extension of R ; let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in U(S)$ and $I \leq P$; we say that the ε_i 's are *u-independent over I* if

$$(2) \quad a_0 + \sum_{i=1}^n a_i \varepsilon_i \in IS \quad (a_i \in R, 0 \leq i \leq n)$$

implies $a_i \in P$ for all i . Conversely, if (2) holds for some $a_i \in U(R)$ the ε_i 's are said *u-dependent over I* .

LEMMA 2.1. (i) If $\varepsilon \in U(S) \setminus R$, then ε is *u-independent over $B(\varepsilon)$* .
(ii) If $\varepsilon_1, \dots, \varepsilon_n \in U(S)$ are *u-independent over $I \leq P$* , then

$$\varepsilon_i \notin R \text{ and } B(\varepsilon_i) \geq I \text{ for all } i.$$

PROOF. (i) If $a_0 + a_1 \varepsilon \in B(\varepsilon)S$, then $a_0 \notin P$ if and only if $a_1 \notin P$ and, in this case, $\varepsilon \in B(\varepsilon)S + R$, which is absurd.

(ii) If, for some j , $\varepsilon_j \in R$, then (2) holds with $a_0 = \varepsilon_j$, $a_j = -1$ and $a_i = 0$ for $0 \neq i \neq j$. Assume now that $B(\varepsilon_j) < I$ for some j . Then $\varepsilon_j + IS = u + IS$ for some $u \in U(R)$, so (2) holds with $a_0 = u$, $a_j = -1$ and $a_i = 0$ for $0 \neq i \neq j$. ///

We say that a family $\{\varepsilon_\lambda: \lambda \in \Lambda\}$ of units of S not in R is *u-independent over an ideal $I \leq P$* , if any finite subset of it is *u-independent over I* ; so the *u-independence* is a property of finite character, and maximal families of units with this property do exist.

Having fixed the ideal $I \leq P$, we consider all the families $\{\varepsilon_\lambda\}_{\lambda \in \Lambda}$ of units of S which are *u-independent over I* , such that $B(\varepsilon_\lambda) = I$ for all $\lambda \in \Lambda$. Let $c_R(I)$ be the minimal cardinal such that $c_R(I) \geq |\Lambda| + I$ for all these families.

We call $c_R(I)$ the *completion defect of R at I* ; clearly $c_R(I)$ is an invariant of R not depending on the choice of S , being the u -independence defined by linearity.

Obviously R is almost maximal if and only if $c_R(I) = 1$ for all non-zero ideals I .

The following result compares the completion defects at isomorphic ideals.

PROPOSITION 2.2. Let $I \cong J$ be isomorphic ideals of R contained in P . Then $c_R(I) = c_R(J)$.

PROOF. It is enough to show that, given a family $\{\varepsilon_\lambda: \lambda \in \Lambda\} \subseteq S$ which is u -independent over I , where $I = B(\varepsilon_\lambda)$ for all $\lambda \in \Lambda$, there exists a family $\{\eta_\lambda: \lambda \in \Lambda\}$ which is u -independent over J , where $J = B(\eta_\lambda)$ for all $\lambda \in \Lambda$. Being $I \cong J$, there exists $a \in R$ such that either $J = aI$ or $aJ = I$; we can assume $a \in P$, otherwise $J = I$ and the claim is trivial. If $J = aI$, let $\eta_\lambda = 1 + a\varepsilon_\lambda$ for all $\lambda \in \Lambda$. Then $B(\eta_\lambda) = J$ for all $\lambda \in \Lambda$ follows from Lemma 1.3. Assume now that

$$a_0 + \sum_1^n a_i \eta_{\lambda_i} \in JS \quad (a_i \in R, \ 0 \leq i \leq n);$$

then $a_0 + \sum_1^n a_i(1 + a\varepsilon_{\lambda_i}) \in aIS$ implies that

$$a^{-1}\left(a_0 + \sum_1^n a_i\right) + \sum_1^n a_i \varepsilon_{\lambda_i} \in IS$$

thus, by the u -independence of the ε_λ 's, we deduce that $a_1, \dots, a_n \in P$, and $a^{-1}\left(a_0 + \sum_1^n a_i\right) \in P$; it follows that $a_0 \in P$ too.

Conversely, assume that $aJ = I$ ($a \in P$). Notice that $a \notin I$, because $J \leq P$. Being $B(\varepsilon_\lambda) = I$, there exists an $u_\lambda^a \in U(R)$ such that $\varepsilon_\lambda = u_\lambda^a + a\eta_\lambda$ for some $\eta_\lambda \in S$. Without loss of generality, we can assume that $\eta_\lambda \in U(S)$: for, if $\eta_\lambda \in PS$, substitute u_λ^a and η_λ respectively by $u_\lambda^a - a \in U(R)$ and $1 + \eta_\lambda \in U(S)$. From $aJ = I = B(\varepsilon_\lambda)$ and from Lemma 1.3, we deduce that $aJ = aB(\eta_\lambda)$, so $B(\eta_\lambda) = J$. Assume now that

$$a0 + \sum_1^n a_i \eta_{\lambda_i} \in JS \quad (a_i \in R, \ 0 \leq i \leq n).$$

Then

$$\left(aa_0 - \sum_1^n a_i u_{\lambda_i}^a\right) + \sum_1^n a_i(u_{\lambda_i} + a\eta_{\lambda_i}) \in aJS = IS;$$

recalling that $u_{\lambda_i}^a + a\eta_{\lambda_i} = \varepsilon_{\lambda_i}$ ($1 \leq i \leq n$), and that the ε_{λ_i} 's are u -independent over I , it follows that $a_1, \dots, a_n \in P$; then $a_0 + \sum_1^n a_i \eta_{\lambda_i} \in JS$ implies $a_0 \in P$ too. ///

Given an ideal $I \leq P$, we introduce now another invariant; we consider all the families $\{\varepsilon_\lambda: \lambda \in \Lambda\}$ of units of S as in the definition of $c_R(I)$, but we assume only that the ε_λ 's are u -independent over I , without assuming that $B(\varepsilon_\lambda) = I$ for all $\lambda \in \Lambda$; thus, by Lemma 2.1, we only know that $B(\varepsilon_\lambda) \geq I$ for all $\lambda \in \Lambda$. Let $d_R(I)$ be the minimal cardinal such that $d_R(I) \geq |\Lambda| + 1$ for all these families. We call $d_R(I)$ the *total defect of R at I* ; here too we notice that $d_R(I)$ is an invariant of R not depending on the choice of S .

The following result is an immediate consequence of the definition.

LEMMA 2.3. (i) If $I \leq J \leq P$, then $d_R(I) \geq d_R(J)$.

(ii) $d_R(I) = 1$ if and only if $I > B(R)$ or $I = B(R)$ and $R/B(R)$ is complete.

(iii) R is almost maximal if and only if $d_R(I) = 1$ for every $I \neq 0$. ///

Given an R -module M and an ideal $I \leq P$, we say that the elements $x_1, \dots, x_n \in M$ are *linearly independent over I* if $x_1 + IM, \dots, x_n + IM$ are linearly independent elements of the R/I -module M/IM , i.e. if $\sum_1^n a_i x_i \in IM$ ($a_i \in R$) implies that $a_i \in I$ for all i . Obviously one can extend this definition to a family of elements of M .

Recall that, if M is torsion-free, then the rank $rk_R M$ of M is the dimension of the Q -vector space $M \otimes_R Q$, where Q is the field of quotients of R , or, equivalently, the cardinality of a maximal system of linearly independent elements of M .

PROPOSITION 2.4. Let R be a valuation domain and I a prime ideal of R . Then $c_R(I) = rk_{R/I}(R/I)^\wedge$ and $d_R(I) = rk_{R/I} S/IS$.

PROOF. Given a family of elements $\{x_\lambda: \lambda \in A\}$ of $(R/I)^\wedge$ which are linearly independent over I , one can assume, without loss of generality, that $x_\lambda \in U(S)$ for all $\lambda \in A$, and that one of them, say $x_{\bar{\lambda}}$, is 1. It follows trivially from the definition that $\{x_\lambda: \lambda \neq \bar{\lambda}\}$ is a family of elements u -independent over I , and $B(x_\lambda) = I$ by Corollary 1.5; therefore $rk_{R/I}(R/I)^\wedge \leq c_R(I)$. In a similar way one can see that $rk_{R/I}S/IS \leq d_R(I)$.

Conversely, to prove that $c_R(I) \leq rk_{R/I}(R/I)^\wedge$ (respectively $d_R(I) \leq rk_{R/I}S/IS$) it is enough to show that, given a family $\{\varepsilon_\lambda: \lambda \in A\}$ of units of S with $B(\varepsilon_\lambda) = I$ for all $\lambda \in A$ (resp. with $B(\varepsilon_\lambda) \geq I$), which are u -independent over I , then $\{1, \varepsilon_\lambda: \lambda \in A\}$ are linearly independent over I . Assume that

$$a_0 + \sum_1^n a_i \varepsilon_{\lambda_i} \in IS \quad (a_i \in R);$$

if some $a_i \notin I$, let a_j be one of the coefficients not in I with minimal value. By multiplying by a_j^{-1} , we get

$$a_j^{-1} \left(a_0 + \sum_1^n a_i \varepsilon_{\lambda_i} \right) \in a_j^{-1} IS = IS,$$

because $a_j I = I$; the last relation is absurd, because the coefficient of ε_{λ_i} is equal to 1, which contradicts the u -independence of $\{\varepsilon_\lambda: \lambda \in A\}$ over I . ///

We wish to compare now the total defect $d_R(I)$ at the ideal I with the completion defects $c_R(J)$ at the ideals $J \geq I$.

LEMMA 2.5. For every $i = 1, \dots, n$, let E_i be a family of units of S u -independent over the ideals J_i , such that $J_i = B(\varepsilon)$ for all $\varepsilon \in E_i$. If $J_1 > J_2 > \dots > J_n$, and the J_i 's are pairwise non isomorphic, then $\cup \{E_i: 1 \leq i \leq n\}$ is u -independent over J_n .

PROOF. We induct on n , the claim being trivial for $n = 1$. So, assume that $n > 1$ and that $\cup \{E_i: 1 \leq i \leq t\}$ is u -independent over J_t , for $1 \leq t \leq n-1$. Let

$$(3) \quad a_0 + \sum_1^k a_j \varepsilon_j + \sum_1^m b_h \eta_h \in J_n S$$

where $a_j, b_h \in R \setminus \{0\}$ ($0 \leq j \leq k$), $1 \leq h \leq m$; $\varepsilon_j \in \cup \{E_i: 1 \leq i \leq n-1\}$ for $1 \leq j \leq k$ and $\eta_h \in E_n$ for $1 \leq h \leq m$. First, notice that $a_1, \dots, a_k \in P$: for, let $r \in J_n \setminus J_{n-1}$; then, for all $h \leq m$ there exists $v_h^r \in U(R)$ such that $\eta_h - v_h^r \in rS$. It follows that

$$a_0 + \sum_1^k a_j \varepsilon_j + \sum_1^m b_h v_h^r \in J_{n-1}S$$

and the u -independence of the ε_j 's implies that $a_1, \dots, a_k \in P$. Recall now that $B(\varepsilon_j)$ is one of the J_i 's, for $1 \leq i \leq n-1$, for all j ; we will show that

$$a_j B(\varepsilon_j) = B(1 + a_j \varepsilon_j) < J_n \text{ for all } j.$$

Being J_n not isomorphic to $J_1, \dots, J_{n-1}$, $a_j B(\varepsilon_j) \neq J_n$ for all j ; let

$$(4) \quad A_1 = \{j: a_j B(\varepsilon_j) < J_n\}; \quad A_2 = \{j: a_j B(\varepsilon_j) > J_n\}.$$

Let $t \in J_n \setminus \cup \{a_j B(\varepsilon_j): j \in A_1\}$; for all $j \in A_1$ we can choose an element $w_j^t \in U(R)$ such that

$$(1 + a_j \varepsilon_j) - w_j^t \in tS < J_n S;$$

substituting in (3), we get:

$$(5) \quad a_0 - k1 + \sum_{j \in A_1} w_j^t + \sum_{j \in A_2} (1 + a_j \varepsilon_j) + \sum_1^m b_h \eta_h \in J_n S$$

where the sums with indexes in A_1, A_2 are intended to be 0 whenever A_1 or A_2 is void.

Assume now that $A_2 \neq \emptyset$. Let $c \in \bigcap_{j \in A_2} a_j B(\varepsilon_j) \setminus J_n$, and choose for all $h \leq m$, $v_h^c \in U(R)$ such that $\eta_h - v_h^c \in cS$; substituting in (5) we get:

$$(6) \quad a_0 - k1 + \sum_{j \in A_1} w_j^t + \sum_{j \in A_2} (1 + a_j \varepsilon_j) + \sum_1^m b_h v_h^c \in \bigcap_{j \in A_2} a_j B(\varepsilon_j) S;$$

we will show that (6) is absurd. Let $j_0 \in A_2$ be such that a_{j_0} has minimal

value among the a_j 's with $j \in A_2$; then

$$(7) \quad a_{j_0}^{-1}(a_0 - k1 + \sum_{j \in A_1} w_j^t + \sum_1^m b_h v_h^c + |A_2| \cdot 1) + \\ + \sum_{j \in A_2} a_{j_0}^{-1} a_j \varepsilon_j \in \bigcap_{j \in A_2} a_{j_0}^{-1} a_j B(\varepsilon_j) S;$$

notice that in (7) $a_{j_0}^{-1} a_j \in R$ for all $j \in A_2$, and that

$$B(\varepsilon_{j_0}) \geq a_{j_0}^{-1} \bigcap_{j \in A_2} a_j B(\varepsilon_j)$$

therefore also the first summand in (7) is in R . But then (7) is absurd, because the coefficient of ε_{j_0} is 1 and by the inductive hypothesis. Thus we have proved that $A_2 = \emptyset$, therefore (5) becomes simply:

$$(8) \quad (a_0 - k1 + \sum_{j \in A_1} w_j^t) + \sum_1^m b_h \eta_h \in J_n S;$$

by the u -independence of the η_h 's over J_n , (8) gives that $b_1, \dots, b_m \in P$; being $a_1, \dots, a_k \in P$, from (3) it follows that $a_0 \in P$. ///

Given an ideal $J \leq R$, let $[J]$ denote the isomorphism class of J ; if $I \leq R$ is another ideal, then $[J] \geq I$ means that there exists $J' \in [J]$ such that $J' \geq I$. By Proposition 2.2, we can define $c_R[J]$ as the common value $c_R(J')$, where J' ranges over $[J]$. We can easily obtain from the preceding Lemma 2.5 the following

$$\text{PROPOSITION 2.6. } d_R(I) \geq \sum \{c_R[J] - 1 : [J] \geq I\} + 1. \quad ///$$

The inequality in Proposition 2.6 is in general strict, as is shown by Facchini and the second author in [3]; actually, they prove a multiplicative formula relating $d_R(I)$ and the $c_R[J]$'s, $[J] \geq I$, for I a prime ideal of a discrete valuation domain R with $\text{Spec } R$ well ordered by the opposite inclusion; they also give a realization theorem for these domains with preassigned completion defects, using an idea of Nagata [9].

3. - Total defect and Goldie dimension.

Let I be an ideal of the valuation domain R , and let $g_R(I)$ denote the Goldie dimension of S/IS as an R/I -module. If I is a prime ideal,

then $g_R(I) = rk_{R/I}S/IS$. It follows from the definitions that, for an arbitrary ideal I , $g_R(I) \leq d_R(I)$, and Proposition 2.4 shows that this inequality becomes an equality if I is a prime ideal.

Recall that, if $0 \neq I \leq P$, then the subset of R

$$I^\# = \{r \in R: rI < I\}$$

is a prime ideal, which is the union of all the ideals $< R$ isomorphic to I (see [10] and [4]). If $I = 0$, we set $I^\# = 0$.

LEMMA 3.1. Given any $I < R$, $g_R(I) = g_R(I^\#)$.

PROOF. It is enough to show that, given $\varepsilon_1, \dots, \varepsilon_n \in U(S)$, they are linearly independent over $I^\#$ if and only if they are linearly independent over I . So, assume that they are linearly independent over $I^\#$ and let $\sum_1^n a_i \varepsilon_i \in IS$. If some $a_i \notin I$, let $a \in R$ be such that $v(a) = \min \{v(a_i): 1 \leq i \leq n\}$. Then

$$a^{-1} \sum_1^n a_i \varepsilon_i \in a^{-1}IS \leq I^\#S$$

because $a^{-1}I$ is an ideal isomorphic to I , hence $a^{-1}I \leq I^\#$. But this is a contradiction, because some $a^{-1}a_i$ is a unit.

Conversely, let $\varepsilon_1, \dots, \varepsilon_n$ be linearly independent over I and let $x = \sum_1^n a_i \varepsilon_i \in I^\#S$. If $x \in IS$, then $a_i \in I$ for all i , hence $a_i \in I^\#$ for all i . If $x \notin IS$, then $x = r\eta$, where $\eta \in U(S)$ and $r \in I^\# \setminus I$. Being $I^\#$ the union of the ideals isomorphic to I , there exists an ideal $J \leq I^\#$ such that $tJ = I$ for some $t \in P$ and $r \in J$. Then $tr \in I$, therefore $t \sum_1^n a_i \varepsilon_i \in IS$; the independence of the ε_i 's over I implies that $ta_i \in I = tJ$, hence $a_i \in J \leq I^\#$ for all i . ///

Recall that an ideal $I < R$ is archimedean if $I^\# = P$. As an immediate consequence of the preceding lemma we get

COROLLARY 3.2. Given two ideals $I \cong J$, then $g_R(I) = g_R(J)$; moreover $g_R(I) = 1$ if I is archimedean.

PROOF. The first claim follows from the equality $I^\# = J^\#$; the second equality follows from the isomorphism $S/PS \cong R/P$. ///

Lemma 3.1 and Proposition 2.4 give the following

COROLLARY 3.4. Given any ideal $I < R$, $g_R(I) = d_R(I^\#)$. ///

COROLLARY 3.5. Given any ideal $I < R$, then $g_R(I) = 1$ if and only if either $I^\# > B(R)$, or $I^\# = B(R)$ and $R/B(R)$ is complete. ///

REFERENCES

- [1] W. BRANDAL, *Almost maximal integral domains and finitely generated modules*, Trans. Amer. Math. Soc., **183** (1973), pp. 203-222.
- [2] A. FACCHINI - P. VAMOS, *Almost reflexive rings* (to appear).
- [3] A. FACCHINI - P. ZANARDO (to appear).
- [4] L. FUCHS - L. SALCE, *Modules over valuation rings*, Lecture Notes in Pure and Applied Math., Marcel Dekker, New York (in prints).
- [5] L. FUCHS - G. VILJOEN, *On finite rank torsion-free modules over almost maximal valuation domains*, Comm. Alg., **12**, 3 (1984), pp. 245-258.
- [6] R. GILMER, *Multiplicative ideal theory*, Marcel Dekker, New York, 1972.
- [7] I. KAPLANSKY, *Maximal fields with valuations*, Duke Math. J., **9** (1942), pp. 303-321.
- [8] W. KRULL, *Allgemeine Bewertungstheorie*, J. für Math., **177** (1937), pp. 215-237.
- [9] M. NAGATA, *Local rings*, Interscience Tracts Pure Appl. Math., J. Wiley and S., New York, London, 1962.
- [10] M. NISHI, *On the ring of endomorphisms of an indecomposable injective module over a Prüfer ring*, Hiroshima Math. J., **2** (1972), pp. 271-283.
- [11] A. OSTROWSKY, *Untersuchungen zur arithmetischen Theorie der Körper*, Math. Z., **39** (1935), pp. 269-404.
- [12] R. B. WARFIELD JR., *Purity and algebraic compactness for modules*, Pacific J. Math., **28** (1969), pp. 699-719.
- [13] P. ZANARDO, *Indecomposable finitely generated modules over valuation domains* (to appear).

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