RENDICONTI
*del*
SEMINARIO MATEMATICO
*della*
UNIVERSITÀ DI PADOVA

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**Blocking sets of 16 points in projective planes of order 10 - III**

*Rendiconti del Seminario Matematico della Università di Padova*, tome 74 (1985), p. 163-174

<http://www.numdam.org/item?id=RSMUP_1985__74__163_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 74 (1985)

# Blocking Sets of 16 Points in Projective Planes of Order 10 - III.

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## 1. Introduction.

Let $\Pi = (\mathfrak{P}, \mathfrak{L})$ be a finite projective plane with point-set $\mathfrak{P}$ and line-set $\mathfrak{L}$, and $\mathfrak{B}$ a blocking set of $\Pi$, i.e. $\mathfrak{B} \subset \mathfrak{P}$ and

$$g \cap \mathfrak{B} \neq 0 \neq g \cap (\mathfrak{P} - \mathfrak{B}) \quad \text{for all } g \in \mathfrak{L}\,.$$

It is convenient to introduce some notation:

$$\mathfrak{L}_i = \{g\colon g \in \mathfrak{L},\ |g \cap \mathfrak{B}| = i\},\ a_i = |\mathfrak{L}_i|,\ a = |\mathfrak{L}| - a_1,$$
$$\mathfrak{L}_i(P) = \{g\colon P \in g \in \mathfrak{L}_i\},\ a_i(P) = |\mathfrak{L}_i(P)|,\ a(P) = \sum_{j>1} a_j(P)$$

for every $i > 0$, $P \in \mathfrak{P}$;

$g^* = \{X\colon X \in g,\ X \notin \mathfrak{B}\}$. Elements of $\mathfrak{L}_i$ are called $i$-lines, elements of $\mathfrak{L}_1$ are tangents, elements of $\mathfrak{L} - \mathfrak{L}_1$ are « lines of $\mathfrak{B}$ ». The « strength » of $g \in \mathfrak{L}$ is defined as $st(g) = |g \cap \mathfrak{B}|$. If $g \in \mathfrak{L} - \mathfrak{L}_1$, $g \cap \mathfrak{B} = \{A, B, C, ...\}$, we also write $g = [A, B, C, ...]$.

Let now $\Pi$ have order 10, $g \in \mathfrak{L}$, $X \in g^*$. As $X$ is on 11 lines and each of these lines contains elements of $\mathfrak{B}$, we get $|\mathfrak{B}| \geqslant 11$ and $|g \cap \mathfrak{B}| \leqslant |\mathfrak{B}| - 10$. If further $X \in h \in \mathfrak{L}$, $h \neq g$, then the same counting argument shows $st(g) + st(h) \leqslant |\mathfrak{B}| - 9$.

(*) Indirizzo dell'A.: Mathematisches Institut der Universität, Im Neuenheimer Feld 288, 69 Heidelberg, Rep. Fed. Tedesca.

By counting (1) all the lines, (2) the flags $(P, g)$, $P \in \mathfrak{B}$, $g \in \mathfrak{L}$, $P \in g$, (3) pairs of points of $\mathfrak{B}$, we get the following equations:

$$\sum_{i \geqslant 1} a_i = 111 \tag{1}$$

$$\sum_{i \geqslant 1} i a_i = 11 |\mathfrak{B}|, \tag{2}$$

$$\sum_{i \geqslant 2} \binom{i}{2} a_i = \binom{|\mathfrak{B}|}{2}. \tag{3}$$

Consideration of the left sides shows $(1/2)(|\mathfrak{B}| - 10)((2) - (1)) \geqslant (3)$. The right sides show then $|\mathfrak{B}| \geqslant 15$. It might be noted that the same kind of argument suffices to prove Bruen's fundamental Theorem [4]:

THEOREM (Bruen). If $\mathfrak{B}$ is a blocking set of a projective plane of order $n$, then $|\mathfrak{B}| \geqslant n + \sqrt{n} + 1$. Equality holds exactly for Baer subplanes. □

The case $|\mathfrak{B}| = 15$ $(n = 10)$ has been ruled out by Denniston [6] with the help of a computer-program (see also [5]).

From now on let $|\mathfrak{B}| = 16$. We restate the result of the above counting arguments:

LEMMA (*I*). (i) $a_i = 0$ for $i > 6$.

(ii) If $\{g, h\} \subset \mathfrak{L}$, $g \neq h$, $st(g) + st(h) > 7$, then $g \cap h \in \mathfrak{B}$.

The case $a_6 \neq 0$ was ruled out in [1, 2, 3] with the help of a computer-program. Here we study the case $a_6 = 0$.

THEOREM. Let $\mathfrak{B}$ be a blocking set of 16 points in a projective plane $\Pi = (\mathfrak{P}, \mathfrak{L})$ of order 10. Then one of the following holds:

(i) $a_5 = 6$, $a_4 = 4$, $a_3 = 7$, $a_2 = 15$.

Consider the linear space $(\mathfrak{B}, \mathfrak{L}')$ with parameters $a_5' = 6$, $a_4' = 4$, $a_3' = 11$, $a_2' = 3$, where $\mathfrak{L}' = \bigcup_{i=2}^{5} \mathfrak{L}_i'$, $\mathfrak{L}_5' = \{f_1, \dots, f_6\}$, $\mathfrak{L}_4' = \{v_1, \dots, v_4\}$,

$\mathfrak{L}_3' = \{d_1', d_2', \ldots, d_{11}'\}$, $\mathfrak{L}_2' = \{z_1, z_2, z_3\}$,

$$\mathcal{B} = \{P_0\} \cup \{A_i, B_i, C_i, D_i, R_i \colon i = 1, 2, 3\},$$

$$\begin{array}{ll} f_1 = [R_1, A_1, B_1, C_1, D_1] & v_1 = [P_0, A_1, A_2, A_3] \\ f_2 = [R_1, A_2, B_2, C_2, D_2] & v_2 = [P_0, B_1, B_2, B_3] \\ f_3 = [R_2, A_1, B_2, C_3, D_3] & v_3 = [P_0, C_1, C_2, C_3] \\ f_4 = [R_2, A_3, B_3, C_1, D_2] & v_4 = [P_0, D_1, D_2, D_3] \\ f_5 = [R_3, A_2, B_3, C_3, D_1] & \\ f_6 = [R_3, A_3, B_1, C_2, D_3] & \end{array}$$

$$\begin{array}{lll} d_1' = [R_1, R_2, R_3] & d_2' = [R_1, A_3, C_3] & d_3' = [R_1, B_3, D_3] \\ d_4' = [R_2, A_2, B_1] & d_5' = [R_2, C_2, D_1] & d_6' = [R_3, A_1, D_2] \\ d_7' = [R_3, B_2, C_1] & d_8' = [A_1, B_3, C_2] & d_9' = [A_2, C_1, D_3] \\ d_{10}' = [A_3, B_2, D_1] & d_{11}' = [B_1, C_3, D_2] & \end{array}$$

$$z_1 = [P_0, R_1] \quad z_2 = [P_0, R_2] \quad z_3 = [P_0, R_3].$$

Then $\mathfrak{L}_5 = \mathfrak{L}_5'$, $\mathfrak{L}_4 = \mathfrak{L}_4'$, $\mathfrak{L}_2 \supset \mathfrak{L}_2'$, $\mathfrak{L}_3 \subset \mathfrak{L}_3'$, and $\mathfrak{L}_2 - \mathfrak{L}_2'$ arises by replacing each of the four 3-lines in $\mathfrak{L}_3' - \mathfrak{L}_3$ by three 2-lines. The symmetry-group of $(\mathcal{B}, \mathfrak{L}')$ is $G' = \langle\sigma_1, \sigma_2\rangle\langle\varrho, \tau\rangle \cong \Sigma_4$, where $\sigma_1 = \prod_{i=1}^{3}(A_i, C_i)(B_i, D_i)$,

$$\sigma_2 = (A_1, B_2)(A_2, B_1)(A_3, B_3)(C_1, D_2)(C_2, D_1)(C_3, D_3),$$
$$\varrho = (R_1, R_2, R_3)(A_1, C_3, D_1)(A_2, C_1, D_3)(A_3, C_2, D_2)(B_1, B_2, B_3),$$
$$\tau = (R_2, R_3)(A_1, C_2)(A_2, C_1)(A_3, C_3)(B_1, B_2)(D_1, D_2).$$

(ii) $a_5 = 6$, $a_4 = 5$, $a_3 = 4$, $a_2 = 18$.

Consider the complement $(\mathcal{B}, \mathfrak{L}')$ of an oval in $PG(2, 4)$. Then $(\mathcal{B}, \mathfrak{L}')$ has parameters $a_5' = 6$, $a_4' = 5$, $a_3' = 10$, $a_2' = 0$ and is uniquely determined:

$$\mathfrak{L}' = \bigcup_{i=3}^{5} \mathfrak{L}_i', \quad \mathfrak{L}_5' = \{f_1, \ldots, f_6\}, \quad \mathfrak{L}_4' = \{v_1, \ldots, v_5\},$$

$$\mathfrak{L}_3' = \{d_1', \ldots, d_{10}'\}, \quad \mathcal{B} = \{P_0\} \cup \{A_i, B_i, C_i, D_i, E_i \colon i = 1, 2, 3\},$$

$$f_1 = [A_1, B_1, C_1, D_1, E_1] \quad v_1 = [P_0, A_1, A_2, A_3]$$
$$f_2 = [A_1, B_2, C_2, D_2, E_2] \quad v_2 = [P_0, B_1, B_2, B_3]$$
$$f_3 = [A_2, B_1, C_2, D_3, E_3] \quad v_3 = [P_0, C_1, C_2, C_3]$$
$$f_4 = [A_2, B_3, C_3, D_1, E_2] \quad v_4 = [P_0, D_1, D_2, D_3]$$
$$f_5 = [A_3, B_2, C_3, D_3, E_1] \quad v_5 = [P_0, E_1, E_2, E_3]$$
$$f_6 = [A_3, B_3, C_1, D_2, E_3]$$

$$d'_1 = [A_1, B_3, D_3] \quad d'_2 = [A_1, C_3, E_3] \quad d'_3 = [A_2, B_2, C_1]$$
$$d'_4 = [A_2, D_2, E_1] \quad d'_5 = [A_3, B_1, E_2] \quad d'_6 = [A_3, C_2, D_1]$$
$$d'_7 = [B_1, C_3, D_2] \quad d'_8 = [B_2, D_1, E_3] \quad d'_9 = [B_3, C_2, E_1]$$
$$d'_{10} = [C_1, D_3, E_2] .$$

Then $\mathfrak{L}_5 = \mathfrak{L}'_5$, $\mathfrak{L}_4 = \mathfrak{L}'_4$, $\mathfrak{L}_3 \subset \mathfrak{L}'_3$, and $\mathfrak{L}_2$ arises by replacing each of the six 3-lines in $\mathfrak{L}'_3 - \mathfrak{L}_3$ by three 2-lines.

The symmetry group of $(\mathfrak{B}, \mathfrak{L}')$ is $G' = \langle \varphi, \alpha \rangle \cong \Sigma_5$, where

$$\varphi = (A_1, B_3, C_2, D_1, E_3)(A_2, B_1, C_1, D_2, E_2)(A_3, B_2, C_3, D_3, E_1) ,$$
$$\alpha = (A_1, A_2)(B_2, B_3)(C_1, D_3)(C_2, D_1)(C_3, D_2)(E_1, E_3) .$$

As the lines of $\mathfrak{L}'_3$ correspond to the secants of the oval in $PG(2, 4)$ and as $G'$ has six orbits on the 4-sets of the set of secants, we get six isomorphism-types for $(\mathfrak{B}, \mathfrak{L})$, with respective symmetry-groups $Z_2 \times \Sigma_3$, $D_8$, $\Sigma_4$, $Z_2$, $Z_2$, $Z_2$.

In the statement of the Theorem we have extended our notation for the parameters of a blocking set to the linear spaces $(\mathfrak{B}, \mathfrak{L}')$ in an obvious way. The following paragraph is dedicated to the proof of the Theorem.

## 2. Proof of the Theorem.

We use the notation of the introduction. Further every point $X \in \mathfrak{P} - \mathfrak{B}$ will be called « of type $(i, j, k, ...)$ » if the lines of $\mathfrak{B}$ through $X$ are an $i$-line, a $j$-line, a $k$-line, .... The only types which can occur are $(5, 2)$, $(4, 3)$, $(4, 2, 2)$, $(3, 3, 2)$, $(3, 2, 2, 2)$, and $(2, 2, 2, 2, 2)$. We have $|\mathfrak{B}| = 16$, $a_i = 0$ for $i > 5$. Consider the equations (1), (2), (3) as given in the introduction. We shall use equations $(A)$ and $(B)$, where $(A) = (3) - (2) + (1)$, $(B) = 2(2) - 2(1) - (3)$, precisely $(A)$ $a_3 + 3a_4 + 6a_5 = 55$ $(B)$ $a_2 + a_3 = 10 + 2a_5$.

By Lemma $(I)$, $g \cap h \in \mathcal{B}$ if either $g \in \mathfrak{L}_5$, $st(h) \geqslant 3$ or $\{g, h\} \subseteq \mathfrak{L}_4$.

Assume $a_5 = 0$. Then $a_4 \geqslant 15$ by $(A)$ and $(B)$. Counting along a line $v \in \mathfrak{L}_4$, we see that there is a point $P \in \mathcal{B}$ such that $a_4(P) = 5$. By $(I)$ then $a_4 = a_4(P) = 5$, a contradiction.

As every point of $f^*$, $f \in \mathfrak{L}_5$, has type $(5, 2)$, we see that $a_2 \geqslant 6$. If $a_5 = 1$, then $a_4 \geqslant 15$ because of $a_2 \geqslant 6$, and we get the same contradiction as before.

Assume $a_5 = 2$. The equations show $a_4 \geqslant 12$, hence $a_4 + a_5 \geqslant 14$. Let $\{v_1, v_2\} \subseteq \mathfrak{L}_4$, $P = v_1 \cap v_2$. Then $(I)$ yields $14 \leqslant a_4 + a_5 \leqslant 9 + a_4(P) + a_5(P)$. It follows $a_4(P) = 5$, hence $a_4 = a_4(P)$ by $(I)$, contradiction.

Assume $a_5 = 3$. Our equations read $a_2 + a_3 = 16$, $a_3 + 3a_4 = 37$. Because of $a_2 \geqslant 6$ we get $9 \leqslant a_4 \leqslant 12$. Clearly then $a_4(P) + a_5(P) < 5$ for every $P \in \mathcal{B}$. Choose $P \in \mathcal{B}$ such that $a_4(P) > 1$. Then $a_4 + a_5 \leqslant 9 + a_4(P) + a_5(P) \leqslant 13$, hence $a_4 \in \{9, 10\}$.

Assume there is $P \in \mathcal{B}$ such that $a_4(P) = 4$. If $a_3(P) = a_2(P) = 1$, let $R \in \mathcal{B}$ such that $\overline{RP} \in \mathfrak{L}_2$. Clearly $a_4(R) = a_5(R) = 0$ by $(I)$. As every 3-line through $R$ must contain a point $Q \in \mathcal{B}$ with $a_5(Q) < 1$, we get $a_3(R) \leqslant 3$. Thus $a_2(R) \geqslant 9$, a contradiction.

Thus we have $a_3(P) = 0$, $a_2(P) = 3$. Let $\{R_i : i = 1, 2, 3\} = \{R : P \neq R \in \mathcal{B}, \overline{RP} \in \mathfrak{L}_2\}$. Then $a_4(R_i) = 0$ like above, $i = 1, 2, 3$. If $a_5(R_i) = 0$, we get the contradiction $a(R_i) > 11$. As $\overline{R_i R_j} \notin \mathfrak{L}_5$ $(i \neq j)$, it follows $a_5(R_i) = 1$, $i = 1, 2, 3$ and consequently $a_2(R_i) \geqslant 5$. As $a_5(R_i) \neq 0$, we get $a_2 \geqslant 5 + 6 = 11$, thus $a_4 > 10$, contradiction. We have $a_4(P) \leqslant 3$ for every $P \in \mathcal{B}$ under the above assumption. By counting along $v \in \mathfrak{L}_4$, we get $a_4 = 9$, hence $a_3 = 10$, $a_2 = 6$. It follows $a_4(P) \in \{0, 3\}$ for every $P \in \mathcal{B}$. If $P \in \mathcal{B}$, $a_5(P) \neq 0$, then $a_2(P) = 0$ (as $a_2 = 6$). Let $\mathcal{N} = \{N : N \in \mathcal{B}, a_4(N) = 0\}$, $N \in \mathcal{N} \neq \emptyset$.

If $a_5(N) \neq 0$, then $a_2(N) = 0$, hence $15 = 2a_3(N) + 4a_5(N)$, contradiction. Thus $a_5(N) = 0$. As $a_3(N) \leqslant 3$, we get $a(N) > 11$, contradiction as before. We have proved $a_5 \geqslant 4$. Assume $a_5 < 6$. Equation $(A)$ shows $a_3 \equiv 1 (\text{mod. } 3)$ especially $a_3 \neq 0$. Let $d \in \mathfrak{L}_3$. Because of $(I)$ there is $P \in d \cap \mathcal{B}$ such that $a_5(P) = 3$. It follows $a_3(P) = a_2(P) = 1$, consequently $a_5 - a_5(P) \leqslant 2$, $a_5 \leqslant 5$, contradiction.

We have $a_5 \in \{4, 5, 6\}$.

HYPOTHESIS 1. $a_5 = 4$.

Then $a_3 + 3a_4 = 31$, $a_2 + a_3 = 18$. As $a_2 \geqslant 6$, we get $a_3 \leqslant 12$, by

$(A)$ $a_3 \leqslant 10$. Hence $a_4 \geqslant 7$. It follows $a_4(P) + a_5(P) \leqslant 4$ for every $P \in \mathcal{B}$.

Let $a_4(Q) > 1$. Then $a_4 + a_5 \leqslant 9 + a_4(Q) + a_5(Q) \leqslant 13$, thus $a_4 \leqslant 9$.

Assume $a_4(P) = 4$, $a_3(P) = a_2(P) = 1$, let $\overline{PR} \in \mathfrak{L}_2$, $R \in \mathcal{B}$. Then $a_4(R) = a_5(R) = 0$. If $d = [R, Q_1, Q_2] \in \mathfrak{L}_3$, then $a_5(Q_1) + a_5(Q_2) = 4$. Thus $a_3(R) \leqslant 3$. It follows $a(R) > 11$, a contradiction.

Assume $a_4(P) = 4$, $a_2(P) = 3$, let $\{R_i: i = 1, 2, 3\} = \{R: P \neq R \in \in \mathcal{B}, \overline{PR} \in \mathfrak{L}_2\}$. We have $a_4(R_i) = 0$, $\overline{R_iR_j} \in \mathfrak{L}_2 \cup \mathfrak{L}_3$, $i \neq j$. If $a_5(R_i) = 0$, then we get a contradiction like before because of $a_3(R_i) \leqslant 3$. Thus we have without restriction $a_5(R_1) = 2$, $a_5(R_2) = a_5(R_3) = 1$. Let $i \in \{2, 3\}$. Clearly $a_3(R_i) \leqslant 3$, thus $a_2(R_i) \geqslant 5$.

Assume $z = \overline{R_2R_3} \in \mathfrak{L}_2$. If $X \in z^*$, $a_4(X) \neq 0$, then $X$ has type $(4, 2, 2)$. As $a_4(R_2) = a_4(R_3) = 0$, we get $a_2 \geqslant 1 + 2 \times 4 + a_4 \geqslant 16$, thus $a_4 \geqslant 10$, contradiction. We have $\overline{R_2R_3} \in \mathfrak{L}_3$. Because of $(I)$ we get $d = [R_1, R_2, R_3] \in \mathfrak{L}_3$. As $a_2(R_i) \geqslant 5$, $a_5(R_i) \neq 0$, we get $a_2 \geqslant 5 + 6 = = 11$, thus $a_4 \geqslant 8$. On the other hand $a_4 \leqslant |d^*| = 8$. Thus $a_4 = 8$, $a_3 = 7$, $a_2 = 11$ and further $a_2(R_i) = a_3(R_i) = 3$. Let $\{f\} = \mathfrak{L}_5(R_2)$, $Q \in f \cap \mathcal{B}$, $Q \neq R_2$. As $a_5 \neq a_5(Q)$, clearly $a(Q) \geqslant 5$. Counting along $f$, we get $a \geqslant |f^*| + 4 \times 4 + a(R_2) = 31$, contradiction as $a = 30$.

We have $a_4(P) \leqslant 3$ for every $P \in \mathcal{B}$ under Hypothesis 1.

Assume $a_4 = 9$. Counting along $v \in \mathfrak{L}_4$ shows $a_4(P) \in \{0, 3\}$ for every $P \in \mathcal{B}$. Set $\mathcal{M} = \{M: M \in \mathcal{B}, a_4(M) = 3\}$, $\mathcal{N} = \mathcal{B} - \mathcal{M}$. Clearly $|\mathcal{M}| = 12$.

Let $\{f_1, f_2\} \subset \mathfrak{L}_5$. Then $f_1 \cap f_2 \in \mathcal{N}$. As $a_5 = 4 = |\mathcal{N}|$, there is $N \in \mathcal{N}$ such that $a_5(N) = 3$. Let $f \in \mathfrak{L}_5 - \mathfrak{L}_5(N)$. Then $|f \cap \mathcal{N}| \geqslant 3$, thus $a_4 \leqslant 3|f \cap \mathcal{M}| \leqslant 6$, contradiction.

We have $a_4 \in \{7, 8\}$.

Assume $a_4 = 8$. Then $a_3 = 7$, $a_2 = 11$, hence $a = 30$.

*If* $a_5(P) = 1$, $a_4(P) = 3$, let $\{R_1, R_2\} = \{R: R \in \mathcal{B}, \overline{PR} \in \mathfrak{L}_2 \cap \mathfrak{L}_3\}$. Then $a_4(R_i) = 0$, $\overline{R_1R_2} \in \mathfrak{L}_2 \cup \mathfrak{L}_3$. Thus $a_5(R_1) + a_5(R_2) = 3$. If $a_5(R_1) = = 0$, then $a_5(R_2) = 3$ and by $(I)$ $a_3(R_1) \leqslant 1$. It follows $a_2(R_1) \geqslant 11$. Thus $\mathcal{B} - \{R_1\}$ is a blocking set, contradiction.

Without restriction we have $a_5(R_1) = 1$, $a_5(R_2) = 2$. Let $\{f\} = = \mathfrak{L}_5(R_1)$. Counting along $f$ and observing that $\mathfrak{L}_5(Q) \neq \mathfrak{L}_5$ for every $Q \in f \cap \mathcal{B}$, we get $30 = a \geqslant |f^*| + 4 \times 4 + a(R_1) = 22 + a(R_1)$. Thus $a(R_1) \leqslant 8$. However $a_3(R_1) \leqslant 3$, thus $a_2(R_1) \geqslant 5$ and clearly then $a(R_1) \geqslant 9$, contradiction. We have proved the following: if $P \in \mathcal{B}$, $a_5(P) + a_4(P) > > 3$, then $a_5(P) \geqslant 2$ and $a_5(P) + a_4(P) = 4$.

Set $b_i = |\{P: P \in \mathcal{B}, a_5(P) + a_4(P) = i\}|$, $i \leqslant 4$. As $a_5 + a_4 = 12$, we have $6b_4 + 3b_3 + b_2 = 66$. As $b_4 \leqslant 6$, it follows $b_4 = 6$, $b_3 = 10$. Counting incidences we get six points $P$ with $a_5(P) = a_4(P) = 2$, eight points $P$ with $a_5(P) = 1$, $a_4(P) = 2$, and consequently two points $P$ with $a_4(P) = 3$. This conflicts with $\binom{a_4}{2} = 28$ and Lemma $(I)$.

We have $a_4 = 7$, $a_3 = 10$, $a_2 = 8$ under Hypothesis 1.

As $a_4 + a_5 = 11$ and $\binom{11}{2} = 55 > 16\times 3 = 48$, there is $P \in \mathcal{B}$ such that $a_4(P) + a_5(P) = 4$. Further $a(P) \geqslant 5$. Let $R \in \mathcal{B}$ such that $\overline{PR} \in \mathcal{L}_2 \cup \mathcal{L}_3$. Clearly $a_4(R) = 0$. If $a_5(R) = 1$, then $a_3(R) \leqslant 3$ by $(I)$, hence $a_2(R) \geqslant 5$. This yields a contradiction by counting $a_2$ along $f \in \mathcal{L}_5(R)$. We already know $a_4(P) \leqslant 3$. If $a_5(P) = 1$, $a_4(P) = 3$, we have $\{R_1, R_2\} = \{R\colon R \in \mathcal{B},\ \overline{PR} \in \mathcal{L}_2 \cup \mathcal{L}_3\}$. As $\overline{R_1R_2} \in \mathcal{L}_2 \cup \mathcal{L}_3$, we have $a_5(R_1) + a_5(R_2) = 3$, by the above without restriction $a_5(R_1) = 3$, $a_5(R_2) = 0$. It follows from $(I)$, that $a_3(R_2) \leqslant 1$, hence $a(R_2) \geqslant 14$, contradiction.

Let $a_5(P) = a_4(P) = 2$. Clearly $a_5(R) = 2$, $a_4(R) = 0$, $a_3(R) \leqslant 2$. It follows $a_2(R) \geqslant 3$. By counting along $f \in \mathcal{L}_5(R)$, we get $a_2 \geqslant 3 + 6 = 9$, contradiction. It is clearly impossible that $a_5(P) < 2$. Thus we have excluded Hypothesis 1.

HYPOTHESIS 2. $a_5 = 5$.

We have $a_3 + 3a_4 = 25$, $a_2 + a_3 = 20$.

Assume $a_4(P) + a_5(P) = 5$, $P \in \mathcal{B}$. Then $a_4(P) = 5 = a_4$, $a_3 = a_2 = 10$.

Clearly $a_5(Q) \leqslant 2$ for every $Q \in \mathcal{B} - \{P\}$. Set $\mathcal{B}_i = \{Q\colon Q \in \mathcal{B}, a_5(Q) = i\}$, $b_i = |\mathcal{B}_i|$, $i \leqslant 2$. By counting along $v \in \mathcal{L}_4$, we get $|\mathcal{B}_2 \cap v| = 2$, $|\mathcal{B}_1 \cap v| = 1$, hence $b_2 = 10$, $b_1 = 5$. Let $f \in \mathcal{L}_5$. Then $|f \cap \mathcal{B}_2| = 4$, $|f \cap \mathcal{B}_1| = 1$. It follows $\mathcal{L}_2 = \{[Q_1, Q_2]\colon Q_i \in \mathcal{B}_1, Q_1 \neq Q_2\}$.

The set $\mathcal{A} = \{P\} \cap \mathcal{B}_1$ is a 6-arc. The secants of $\mathcal{A}$ are the lines in $\mathcal{L}_4 \cap \mathcal{L}_2$, and these form a dual blocking set of cardinality 15, which is impossible.

We have $a_4(P) + a_5(P) \leqslant 4$ for every $P \in \mathcal{B}$.

Assume $a_4(P) = 4$. If $a_3(P) = a_2(P) = 1$, let $\mathcal{L}_2(P) = \{[P, R]\}$, $\mathcal{L}_3(P) = \{[P, S_1, S_2]\}$. Clearly $a_4(R) = a_5(R) = 0$. As $a_5(S_1) + a_5(S_2) = 5$, further $a_3(R) \leqslant 1$, thus $a_2(R) \geqslant 13$, contradiction.

Thus $a_2(P) = 3$, $\mathcal{L}_2(P) = \{[P, R_i]\colon i = 1, 2, 3\}$, $a_4(R_i) = 0$, $\overline{R_iR_j} \in \mathcal{L}_2 \cup \mathcal{L}_3$, $i \neq j$. If $a_5(R_1) = 3$, we have without restriction $a_5(R_2) \leqslant 1$, by $(I)$ $a_3(R_2) \leqslant 1$, thus $a_2(R_2) \geqslant 9$ and $\mathcal{B} - \{R_2\}$ is a blocking set, contradiction. We have without restriction $a_5(R_1) = a_5(R_2) = 2$, $a_5(R_3) = 1$. Counting along $f \in \mathcal{L}_5(R_3)$, we get $a \geqslant |f^*| + 4\times 4 + a(R_3) = 22 + a(R_3)$. As $a_3(R_3) \leqslant 3$, we get $a(R_3) \geqslant 9$, hence $a \geqslant 31$.

Let $\mathcal{N} = \{Q\colon Q \in \mathcal{B} - \{P\},\ \overline{QP} \in \mathcal{L}_4,\ a_5(Q) = 1\}$. As $a_5(Q) \leqslant 2$ for every $Q \in \mathcal{B}$, we have $|v \cap \mathcal{N}| = 1$ for every $v \in \mathfrak{L}_4(P)$, thus $|\mathcal{N}| = 4$. As $a \geqslant 31$, we have $a_4 \geqslant 6$. Let $v' \in \mathfrak{L}_4 - \mathfrak{L}_4(P)$. Then $|v' \cap \mathcal{N}| = 3$ because of $(I)$ and $a_5 = 5$. Thus $a_4 - a_4(P) \leqslant 1$, $a_4 \leqslant 5$, contradiction. We have proved: $a_4(P) \leqslant 3$ for every $P \in \mathcal{B}$ under Hypothesis 2. Assume $a_5(P) = a_4(P) = 2$ (hence $a_2(P) = 1$). Let $\{[P, R]\} = \mathfrak{L}_2(P)$. Then $a_4(R) = 0$, $a_5(R) = 3$. Let $\{f_1, f_2\} = \mathfrak{L}_5(P)$, $\{f_3, f_4, f_5\} = \mathfrak{L}_5(R)$, $\{v_1, v_2\} = \mathfrak{L}_4(P)$, set $\{S_i: i = 1, 2\} = \{S: S \in f_i \cap \mathcal{B}, a_5(S) = 1, i = 1, 2\}$. Clearly $\mathfrak{L}_4 = \mathfrak{L}_4(P) \cup \mathfrak{L}_4(S_1) \cap \mathfrak{L}_4(S_2)$, $\overline{S_1 S_2} \in \mathfrak{L}_3 \cup \mathfrak{L}_2$.

The basic equations show $a_4 \leqslant 8$. By $(I)$ we have

$$\sum_{Q \in \mathcal{B}} \binom{a_5(Q) + a_4(Q)}{2} = \binom{a_5 + a_4}{2}.$$

Set

$$c(\mathcal{M}) = \sum_{M \in \mathcal{M}} \binom{a_5(M) + a_4(M)}{2}, \qquad \text{for every } \mathcal{M} \subseteq \mathcal{B}.$$

Assume first $a_4 = 8$. Then $a_4(S_1) = a_4(S_2) = 3$, $c(f_i - \{P\}) \leqslant 15$, $c(v_i - \{P\}) \leqslant 18$. Thus

$$78 = \binom{a_5 + a_4}{2} \leqslant 6 + 3 + 2 \times 15 + 2 \times 18 = 75\,,$$

contradiction. Assume $a_4 = 7$. Without restriction $a_4(S_1) = 3$, $a_4(S_2) = 2$. Then $c(f_1 - \{P\}) \leqslant 13$, $c(f_2 - \{P\}) \leqslant 12$, $c(v_i - \{P\}) \leqslant 15$, $i = 1, 2$. Thus $66 \leqslant 6 + 3 + 13 + 12 + 2 \times 15 = 64$, contradiction.

Thus $a_4 \leqslant 6$. Let $d \in \mathfrak{L}_3$. If $d \cap \{R, S_1, S_2\} \neq \emptyset$, then $d = [R, S_1, S_2]$. Consideration of $f_1$ and $f_2$ shows because of $(I)$ that $a_3 - a_3(R) \leqslant 6$. Thus $a_3 \leqslant 7$. It follows $a_3 = 7$, $a_4 = 6$, $a_2 = 13$.

Assume first $a_4(S_1) = 3$, $a_4(S_2) = 1$. Then $c(f_1 - \{P\}) \leqslant 11$, $c(f_2 - \{P\}) \leqslant 10$, $c(v_i - \{P\}) \leqslant 12$, $i = 1, 2$, thus $55 \leqslant 54$, contradiction. We have $a_4(S_1) = a_4(S_2) = 2$, $c(f_i - \{P\}) \leqslant 10$, $c(v_i - \{P\}) \leqslant 13$, $55 \leqslant 55$. Thus we have equality all the way. Set $\mathcal{N} = \{N: N \in \mathcal{B} - \{P\}, \overline{PN} \in \mathfrak{L}_4,\ a_4(N) = 3\}$. We have $|v_i \cap \mathcal{N}| = 2$, $i = 1, 2$, and $a_4(Q) = 1$ for every $Q \in \mathcal{B}$, $\overline{QP} \in \mathfrak{L}_4$, $Q \notin \mathcal{N}$. This is impossible as $|v \cap \mathcal{N}| = 2$ for every $v \in \mathfrak{L}_4 \setminus \mathfrak{L}_4(P)$, hence $a_4 - a_4(P) \leqslant 2$.

Let $a_5(P) = 1$, $a_4(P) = 3$, $P \in \mathcal{B}$, $\{R_1, R_2\} = \{R: P \neq R \in \mathcal{B}, \overline{PR} \in \mathfrak{L}_2 \cup \mathfrak{L}_3\}$, $\{f\} = \mathfrak{L}_5(P)$, $\{v_1, v_2, v_3\} = \mathfrak{L}_4(P)$. Then $a_4(R_i) = 0$, $\overline{R_1 R_2} \notin \mathfrak{L}_5$, hence $a_5(R_1) + a_5(R_2) = 4$. If $a_5(R_1) = 3$, then $a_5(R_2) = 1$, by $(I)$ $a_3(R_2) \leqslant 1$, hence $a(R_2) = 11$ and $\mathcal{B} - \{R_2\}$ is a blocking set, contradiction. Thus $a_5(R_1) = a_5(R_2) = 2$. Let $\mathcal{B}_i = \{Q : Q \in \mathcal{B},\ a_5(Q) + a_4(Q) = i\}$.

If $Q \in \mathcal{B}_4$, then $a_5(Q) = 1$. Thus $|\mathcal{B}_4| \leqslant \binom{a_4}{2}\Big/3$.

If $a_4 = 8$, we get the contradiction

$$78 = \binom{a_5 + a_4}{2} \leqslant 9\times 6 + 7\times 3 = 75\,.$$

Assume $a_4 = 7$. We have $|v \cap \mathcal{B}_4| = 3$ for every $v \in \mathfrak{L}_4$. This shows $|\mathcal{B}_4| = 7$, $\mathcal{B}_4 \subset v_1 \cup v_2 \cup v_3$. Hence $c(f - \{P\}) = 12$, $c(v_i - \{P\}) = 15$, $i = 1, 2, 3$, thus

$$66 = \binom{a_5 + a_4}{2} = 6 + 1 + 1 + 12 + 3\times 15 = 65\,, \quad \text{contradiction}\,.$$

Assume $a_4 = 6$. If $v \in \mathfrak{L}_4$, then $|v \cap \mathcal{B}_4| = 2$ because of $a_4 = 6$ and $a_4 + a_5 = 11$. It follows $|v \cap \mathcal{B}_3| = 2$. Thus $c(v_i - \{P\}) = 12$, $i = 1, 2, 3$. As $a_4 + a_5 = 11$, we get $c(f - \{P\}) = 11$. This implies $|f \cap \mathcal{B}_3| = 3$, $|f \cap \mathcal{B}_2| = 1$. Especially $|\mathcal{B}_4| = 4$ and $\mathfrak{L}_4 = \{\overline{RS} : \{R, S\} \subset \mathcal{B}_4\}$. We have to be more precise. Let $\mathcal{B}(i, j) = \{Q : Q \in \mathcal{B}, a_5(Q) = i, a_4(Q) = j\}$. Then $|v \cap \mathcal{B}(1, 3)| = 2$, $|v \cap \mathcal{B}(1, 2)| = |v \cap \mathcal{B}(2, 1)| = 1$ for every $v \in \mathfrak{L}_4$. It follows

$$\sum_{Q \in v_1 \cup v_2 \cup v_3} \binom{a_4(Q)}{2} = 15 = \binom{6}{2}, \quad \text{thus } a_4(F) \leqslant 1 \quad \text{for every } F \in f - \{P\}\,.$$

Further

$$\sum_{Q \notin f - \{P\}} \binom{a_5(Q)}{2} = 5\,, \quad \text{thus} \sum_{F \in f - \{P\}} \binom{a_5(F)}{2} = 5\,.$$

This yields $|f \cap \mathcal{B}(3, 0)| = 1$, $|f \cap \mathcal{B}(2, 1)| = 2$, $|f \cap \mathcal{B}(1, 1)| = 1$.

This is impossible as, by the above, there is no $v \in \mathfrak{L}_4$ such that $v \cap \mathcal{B}(1, 1) \neq \emptyset$.

Assume $a_4 = 5$. If $|\mathcal{B}_4| > 1$, $\{P, P'\} \subseteq \mathcal{B}_4$, then clearly $\overline{PP'} \in \mathfrak{L}_4$, without restriction $\overline{PP'} = v_1$. Further $\mathfrak{L}_4 = \mathfrak{L}_4(P) \cup \mathfrak{L}_4(P')$, hence $a_4(Q) \leqslant 2$ for every $Q \in \mathcal{B} - \{P, P'\}$. Especially $|\mathcal{B}_4| = 2$.

We have $c(v_1 - \{P\}) = 10$, $c(v_i - \{P\}) = 9$, $i = 2, 3$, hence $c(f - \{P\}) = 9$. As $Q \in \mathcal{B}_3$, but $a_4(Q) \leqslant 2$ for every $Q \in v_i - \{P\}$, $i = 2, 3$, we get $a_5(Q) \neq 0$. It follows

$$\sum_{Q \notin f - \{P\}} \binom{a_5(Q)}{2} = 5 , \quad \text{thus} \sum_{F \in f - \{P\}} \binom{a_5(F)}{2} = 5 .$$

More precisely we have $|f \cap \mathfrak{B}(3, 0)| = |f \cap \mathfrak{B}(1, 0)| = 1$, $|f \cap \mathfrak{B}(2, 1)| = 2$, $|v_1 \cap \mathfrak{B}(2, 1)| = |v_1 \cap \mathfrak{B}(1, 1)| = 1$, $|v_i \cap \mathfrak{B}(1, 2)| = 2$, $|v_i \cap \mathfrak{B}(2, 1)| = 1, i = 2, 3$. Let $v \in \mathfrak{L}_4(P')$, $v \neq v_1$. Then $v \cap f \in \mathfrak{B}(2, 1)$, $v \cap v_i \in \mathfrak{B}(1, 2)$, $i = 2, 3$. Let now $\{g\} = \mathfrak{L}_5(P')$. Then $g \cap f \in \mathfrak{B}(3, 0)$, $g \cap v_i \in \mathfrak{B}(2, 1)$, $i = 2, 3$. Counting along $g$, we get $a_5 = 6$, contradiction.

We have $\mathfrak{B}_4 = \{P\}$. It follows $c(v_i - \{P\}) = 9$, $i = 1, 2, 3$. Thus $c(f - \{P\}) = 10$, which is impossible.

We have $a_4 < 5$, thus $a_5 + a_4 + a_3 \geqslant 22$. However, this is impossible because there is a triangle of 5-lines, implying $a_5 + a_4 + a_3 \leqslant 21$ by $(I)$. We have proved $a_5(Q) + a_4(Q) \leqslant 3$ for every $Q \in \mathfrak{B}$ under Hypothesis 2. Counting along $v \in \mathfrak{L}_4$, we get $a_5 + a_4 \leqslant 9$, thus $a_4 \leqslant 4$, a contradiction like before. Hypothesis 2 has been ruled out.

Thus $a_5 = 6$.

We have $a_5 = 6$, $a_3 + 3a_4 = 19$, $a_2 + a_3 = 22$.

Hypothesis 3. $a_5(Q) + a_4(Q) \leqslant 4$ for every $Q \in \mathfrak{B}$.

Lemma. Under Hypothesis 3, the following hold for every $P \in \mathfrak{B}$:

(i) If $a_5(P) + a_4(P) = 4$, then $a_4(P) \geqslant 3$.

(ii) If $a_3(P) \neq 0$, then $a_5(P) = 2$.

Proof. (i) Clearly $a_4(P) \geqslant 2$. If $a_4(P) = a_5(P) = 2$, then $a_5(R) = 4$, where $\{[P, R]\} = \mathfrak{L}_2(P)$, contradiction.

(ii) Let $d \in \mathfrak{L}_3$, $P \in d \cap \mathfrak{B}$. If $a_5(P) = 3$, then $a_3(P) = a_2(P) = 1$, thus $a_5 - a_5(P) \leqslant 2$, contradiction. Assertion (ii) follows now from $(I)$. □

We continue under Hypothesis 3. Let $\mathfrak{B}_i = \{Q : Q \in \mathfrak{B},\ a_5(Q) + a_4(Q) = i\}$.

If $a_4 = 6$, then for every $v \in \mathfrak{L}_4$ we have $|v \cap \mathfrak{B}_4| \geqslant 3$. This shows $a_4 \geqslant 7$ by part (i) of the Lemma, contradiction.

Assume $a_4 = 5$. The same argument shows $|v \cap \mathfrak{B}_4| = 2$ for every $v \in \mathfrak{L}_4$. Thus $|\mathfrak{B}_4| \geqslant 4$. However $|\mathfrak{B}_4| \leqslant \binom{a_4}{2} \Big/ 3 = 10/3$ by (i), contradiction. We have $a_4 \leqslant 4$. Assume there is $P \in \mathcal{B}$ such that $a_5(P) = 1$, $a_4(P) = 3$. If $a_4 = 4$, then $\mathcal{B}_4 = \{P\}$ by (i). There is a line $v \in \mathfrak{L}_4 - \mathfrak{L}_4(P)$. This yields the contradiction $a_4 + a_5 \leqslant 9$.

Thus $a_4 = 3$ and $\mathfrak{L}_4 = \mathfrak{L}_4(P)$. Clearly $|v_i \cap \mathcal{B}_3| = 2$, $|v_i \cap \mathcal{B}_2| = 1$, $i = 1, 2, 3$ where $\mathfrak{L}_4 = \{v_1, v_2, v_3\}$. Let $\mathfrak{L}_5(P) = \{f\}$, $Q \in f \cap \mathcal{B}$, $Q \neq P$. Assume $a_5(Q) = 1$. By (ii) we have $a_3(Q) = 0$. As $a_4(Q) = 0$, we get $a_2(Q) = 11$, a contradiction. Thus $|f \cap \mathcal{B}_3| = 1$, $|f \cap \mathcal{B}_2| = 3$, and we can count:

$$36 = \binom{a_5 + a_4}{2} = 6 + 7 \times 3 + 6 + 3 + 1 = 37\,, \qquad \text{contradiction}\,.$$

We have proved: if $P \in \mathcal{B}_4$, then $a_4(P) = 4$, under Hypothesis 3. Assume $\mathcal{B}_4 = \emptyset$. Counting along $v \in \mathfrak{L}_4$ shows $a_4 \leqslant 3$. Let first $a_4 = 3$. If $a_4(P) \neq 0$, then $P \in \mathcal{B}_3$. Assume $a_4(P) = 3$. Then $a_5(P) = 0$, by (ii) $a_3(P) = 0$, hence $a_2(P) = 6$. Let $t \in \mathfrak{L}_1(P)$. Then $a_2(X) \neq 0$ for every $X \in t^*$, hence $a_2 \geqslant 6 + |t^*| = 16$, contradiction. Thus there is $P \in \mathcal{B}$ such that $a_4(P) = 2$, $a_5(P) = 1$. By (ii) we have $a_3(P) = 0$, $a_2(P) = 5$. If $t \in \mathfrak{L}_1(P)$, there is at most one point $X \in t^*$ such that $a_2(X) = 0$. Hence $12 = a_2 \geqslant 5 + |t^*| - 1 = 14$, contradiction.

Thus $a_4 \leqslant 2$. If $a_4 < 2$, then $a_5 + a_4 + a_3 \geqslant 23$, which is impossible by (*I*) as there is a triangle of 5-lines.

We have $a_4 = 2$, $a_3 = 13$, $a_2 = 9$. Let $\mathfrak{L}_4 = \{v_1, v_2\}$, $P = v_1 \cap v_2$. As $a_5(P) \neq 2$, we have $a_3(P) = 0$ by (ii). If $a_5(P) = 1$, then $a_2(P) = 5$ and consequently $a_2 \geqslant 11$, contradiction. Thus $a_5(P) = 0$, $a_2(P) = 9$. We get a contradiction like above by considering $t \in \mathfrak{L}_1(P)$.

We have $a_4 \leqslant 4$ and $\mathcal{B}_4 \neq \emptyset$ under Hypothesis 3. As every point $P \in \mathcal{B}_4$ satisfies $a_4(P) = 4$, necessarily $\mathcal{B}_4 = \{P_0\}$, $a_4 = 4$, $a_3 = 7$, $a_2 = 15$. Then $(\mathcal{B}, \mathfrak{L}_4)$ is like in case (i) of the Theorem. As $a_5(Q) < 3$ for every $Q \in \mathcal{B} - \{P_0\}$, we get $a_5(Q) = 2$ for every $Q \in \mathcal{B} - \{P_0\}$.

Further $a_3(P_0) = 0$ by (ii) of the Lemma, hence $a_2(P_0) = 3$. Clearly then $(\mathcal{B}, \mathfrak{L}_5 \cup \mathfrak{L}_4 \cup \mathfrak{L}_2(P_0))$ is like in (i) of the Theorem and it is easily seen, that we have case (i) of the Theorem.

We can henceforth assume, that Hypothesis 3 is not satisfied. Let $P_0 \in \mathcal{B}$ such that $a_4(P_0) = 5$. Then clearly $a_4 = 5$, $a_3 = 4$, $a_2 = 18$. As $a_5(Q) \leqslant 2$ for every $Q \in \mathcal{B} - \{P_0\}$, we get $a_5(Q) = 2$ for every $Q \in \mathcal{B} - \{P_0\}$. It is easily seen, that $(\mathcal{B}, \mathfrak{L}_5 \cup \mathfrak{L}_4)$ is uniquely determined and can be chosen like in case (ii) of the Theorem. Further it is easy to check, that $(\mathcal{B}, \mathfrak{L})$ arises in the way descrived in the Theorem out of a uniquely determined linear space $(\mathcal{B}, \mathfrak{L}')$ with 16 points and 21 lines as given in the statement of the Theorem. Again it is easy to see, that $(\mathcal{B}, \mathfrak{L}')$ can be completed in exactly one way to yield $PG(2, 4)$. The five « new » points form an oval in $PG(2, 4)$, together with $P_0$ they form a hyperoval. The proof of the Theorem is complete.

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Manoscritto pervenuto in redazione il 1° agosto 1984.