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**\mathcal{F} -constraint of the automorphism group
of a finite group**

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\mathcal{F} -Constraint of the Automorphism Group of a Finite Group.

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If \mathcal{F} is a homomorph and we denote $\mathcal{F}' = \{G | S^{\mathcal{F}} = S \forall S \trianglelefteq G\}$, we say that a group G is \mathcal{F} -constrained when there is a maximal normal \mathcal{F} -subgroup \bar{M} of $\bar{G} = G/G_{\mathcal{F}'}$, such that $C_{\bar{G}}(\bar{M}) \trianglelefteq \bar{M}$.

In ([7]), R. Laue proves that if G has no direct abelian factors and $C_G(F(G)) \trianglelefteq F(G)$ (i.e. G is Nilpotent-constrained) then $C_{\text{Aut } G}(F(\text{Aut } G)) \trianglelefteq F(\text{Aut } G)$.

In ([10]) it is proved for a saturated Fitting formation and a group G verifying that:

- i) $G_{\mathcal{F}'} \trianglelefteq \Phi(G)$,
- ii) $G/G_{\mathcal{F}'}$ has no direct abelian factors,
- iii) G is \mathcal{F} -constrained,

that $\text{Aut } G$ is \mathcal{F} -constrained.

The purpose of this paper is mainly to prove the above result when \mathcal{F} is a homomorph closed for direct products (D_0 -closed) and normal subgroups that is: i) saturated or ii) closed for central extensions.

All group considered are finite.

If \mathcal{F} is a homomorph then the class \mathcal{F}' is a s -closed (i.e. closed for subgroups) extensible Fitting formation and a group G is said \mathcal{F} -sepa-

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rable if it possesses a normal series:

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \dots \trianglerighteq G_r = 1$$

whose factor groups G_i/G_{i+1} are either \mathcal{F} -groups or \mathcal{F}' -groups.

If we do not state the contrary here on we shall suppose that \mathcal{F} is a n -closed (i.e. closed for normal subgroups) homomorph.

The class of \mathcal{F} -separable groups is an extensible Fitting formation that contains the solvable groups.

The product of all normal semisimple subgroups of a group G is again a semisimple normal subgroup of G , and it is denoted by $L(G)$. It is called the semisimple radical of G ([6]).

We use the concepts of semisimple and perfect-quasisimple groups given by Gorenstein and Walter [6].

LEMMA 1 (s. [5] p. 127 or [9] p. III-32). *For every group G , we have:*

$$C_G(F(G)L(G)) \trianglelefteq F(G).$$

From the definition and properties of class \mathcal{F}' it follows that G is \mathcal{F} -constrained if and only if G/N is \mathcal{F} -constrained, when $N \trianglelefteq G_{\mathcal{F}'}$.

LEMMA 2. *If \mathcal{F} is a saturated Fitting formation, the following are equivalent. 1) G is \mathcal{F} -constrained, 2) $L(\bar{G}) \in \mathcal{F}$ ([8]).*

Our remainder notation is standard and it is based on Huppert's book ([4]).

In the following we shall say that \mathcal{F} verifies:

- A) If $\{Q, E_\Phi, D_0, S_n\}\mathcal{F} = \mathcal{F}$;
- B) If $\{Q, E_z, D_0, S_n\}\mathcal{F} = \mathcal{F}$, where $E_z\mathcal{F} = \{G|N; N \trianglelefteq Z(G), G/N \in \mathcal{F}\}$.

As a consequence of the following proposition we obtain that the \mathcal{F} -constraint is equivalent to the constraint with respect to a suitable saturated Fitting formation.

PROPOSITION 3. *If $\bar{G} = G/G_{\mathcal{F}'}$ and if \mathcal{F} verifies A or B, the following are equivalent:*

- 1) G is \mathcal{F} -constrained,
- 2) \bar{G} is (\mathcal{F} -separable)-constrained,
- 3) $L(\bar{G}) \in \mathcal{F}$.

PROOF. 1) \Rightarrow 2) Obvious.

2) \Rightarrow 3). By lemma 2, $L(\bar{G})$ is \mathcal{F} -separable, hence $L(\bar{G})/Z(L(\bar{G}))$ is \mathcal{F} -separable and direct product of non-abelian simple groups ([6]). Moreover, since \mathcal{F}' is saturated $(L(\bar{G})/Z(L(\bar{G})))_{\mathcal{F}'} = 1$, thus $L(\bar{G})/Z(L(\bar{G})) \in \mathcal{F}$, but since $Z(L(\bar{G})) = \Phi(L(\bar{G}))$, $L(\bar{G}) \in \mathcal{F}$. 3) \Rightarrow 1). Since $\bar{G}_{\mathcal{F}'} = 1$, both $L(\bar{G})$ and $F(\bar{G})$ are normal \mathcal{F} -subgroups of \bar{G} and since $[L(\bar{G}), F(\bar{G})] = 1$, then $L(\bar{G})F(\bar{G})$ is a normal \mathcal{F} -subgroup of \bar{G} . Let \bar{M} be a maximal normal \mathcal{F} -subgroup of \bar{G} containing $L(\bar{G})F(\bar{G})$, by lemma 1 it follows:

$$C_{\bar{G}}(\bar{M}) \leq C_{\bar{G}}(L(\bar{G})F(\bar{G})) \leq F(\bar{G}) \leq \bar{M}$$

COROLLARY. If G is \mathcal{F} -constrained group then $C_{\bar{G}}(\bar{M}) \leq \bar{M}$ for all maximal normal \mathcal{F} -subgroups of \bar{G} .

PROOF. By proposition 3, $L(\bar{G}) \in \mathcal{F}$. Since $L(C_{\bar{G}}(\bar{M})) \trianglelefteq L(\bar{G})$ then $L(C_{\bar{G}}(\bar{M})) \in \mathcal{F}$ hence $L(C_{\bar{G}}(\bar{M})) \leq \bar{M}$ by the maximality of \bar{M} and so $L(C_{\bar{G}}(\bar{M})) = 1$, thus $C_{\bar{G}}(\bar{M})$ is a \mathcal{N} -constrained group. On the other hand $F(C_{\bar{G}}(\bar{M})) \in \mathcal{F}$ hence $F(C_{\bar{G}}(\bar{M})) \leq \bar{M}$.

Clearly:

$$C_{\bar{G}}(\bar{M}) \leq C_{\bar{G}}(F(C_{\bar{G}}(\bar{M}))) \cap C_{\bar{G}}(\bar{M}) = C_{C_{\bar{G}}(\bar{M})}(F(C_{\bar{G}}(\bar{M}))) \leq F(C_{\bar{G}}(\bar{M})) \leq \bar{M}$$

REMARKS. The following conditions, are not equivalent: i) \bar{G} is (\mathcal{F} -separable)-constrained and ii) G is (\mathcal{F} -separable)-constrained. In fact, in [6] it is proved that $G = C_x(\tau)$, where $X = SL(4, 2^a)$, $a > 1$, and τ is the central involution $I_4 + xE_{14}$, $x \neq 0$, is 2-constrained, hence G is (2-separable)-constrained or equivalently (2'-separable)-constrained, however G is not 2'-constrained. Thus G is (2'-separable)-constrained but $G/O_2(G)$ is not (2'-separable)-constrained.

If \mathcal{F} is a saturated Fitting formation it is known that:

- i) The class of the \mathcal{F} -constrained groups is a Fitting class that contains the solvable groups ([8])
- ii) G is \mathcal{F} -constrained if and only if $\text{Inn } G$ is \mathcal{F} -constrained ([10]),
- iii) The class of the \mathcal{F} -constrained groups is extensible ([10]). As a consequence of the above proposition, the properties i), ii), iii) are still valid when \mathcal{F} verifies A or B.

LEMMA 4. *Let \mathcal{F} be a homomorph that verifies A or B, then if G is a group without direct abelian factors and $G_{\mathcal{F}'} = 1$, it follows $(\text{Aut } G)_{\mathcal{F}'} = 1$.*

PROOF. Let H be a normal \mathcal{F}' -subgroup of $\text{Aut}(G)$, then $H \cap \text{Inn}(G)$ is a normal \mathcal{F}' -subgroup of $\text{Inn}(G)$; as $(G/Z(G))_{\mathcal{F}'} = 1$, consequently $H \cap \text{Inn } G = 1$, thus $[H, \text{Inn } G] = 1$ and so $H < C_{\text{Aut } G}(\text{Inn } G) \leqslant F(\text{Aut } G)$ because G has no direct abelian factors ([7]). Set $\pi = \text{car } \mathcal{F} = \{p \mid C_p \in \mathcal{F}\}$, since $F(G)$ is nilpotent and $G_{\mathcal{F}'} = 1$ it follows that $F(G)$ is a π -group and thus $F(\text{Aut } G)$ is a π -group ([7]). Since $H \in \mathcal{F}'$, H is a π' -group and so $H = 1$.

COROLLARY 1. *Let \mathcal{F} be a homomorph that verifies A or B, and G a group such that $G/G_{\mathcal{F}'}$ has no direct abelian factors, then if G is \mathcal{F} -constrained, $\text{Aut}(G/G_{\mathcal{F}'})$ is \mathcal{F} -constrained.*

PROOF. As G is \mathcal{F} -constrained group, so $G/G_{\mathcal{F}'}$ is \mathcal{F} -constrained. By proposition 3, $G/G_{\mathcal{F}'}$ is (\mathcal{F} -separable)-constrained. We can apply (2.6) of [10] to obtain that $\text{Aut}(G/G_{\mathcal{F}'})$ is (\mathcal{F} -separable)-constrained. By lemma 4 $(\text{Aut}(G/G_{\mathcal{F}'}))_{\mathcal{F}'} = 1$, hence by proposition 3, $\text{Aut}(G/G_{\mathcal{F}'})$ is \mathcal{F} -constrained.

COROLLARY 2. *Let \mathcal{F} be a homomorph verifying B, and G a group whithout direct abelian factors. If G is \mathcal{F} -constrained, then $\text{Aut } G$ is \mathcal{F} -constrained.*

PROOF. If \mathcal{F} verifies B), then $\mathcal{F}' = 1$ now we can apply corollary 1, to obtain the result.

However, in the case of saturated homomorph, it does not appear that the constraint of $\text{Aut } G$ can be obtained as an easy consequence of the constraint of $\text{Aut}(G/G_{\mathcal{F}'})$.

THEOREM 5. *Let \mathcal{F} be a homomorph verifying A, and G a group that verifies:*

- i) $G_{\mathcal{F}'} \leqslant \Phi(G)$
- ii) $G/G_{\mathcal{F}'}$ has no direct abelian factors
- iii) G is \mathcal{F} -constrained.

Then $\text{Aut } G$ is \mathcal{F} -constrained.

PROOF. We denote $\overline{\text{Aut } G} = \text{Aut } G / (\text{Aut } G)_{\mathcal{F}'}$. We have:

$$\begin{aligned}\overline{\text{Inn } G} &= \overline{\text{Inn } G(\text{Aut } G)_{\mathcal{F}'}} / (\text{Aut } G)_{\mathcal{F}'} \\ &\simeq \overline{\text{Inn } G} / \overline{\text{Inn } G} \cap (\text{Aut } G)_{\mathcal{F}'} = \overline{\text{Inn } G} / (\text{Inn } G)_{\mathcal{F}'} \simeq \\ &\simeq (G/Z(G)) / (G/Z(G))_{\mathcal{F}'} = (G/Z(G)) / (G_{\mathcal{F}'} Z(G)) / Z(G) \simeq \\ &\simeq G/G_{\mathcal{F}'} Z(G).\end{aligned}$$

Since G is \mathcal{F} -constrained we know that $\text{Inn}(G)$ is \mathcal{F} -constrained and so $\overline{\text{Inn } G}$ is \mathcal{F} -constrained too. By proposition 3, $L(\overline{\text{Inn } G})$ is a characteristic \mathcal{F} -subgroup of $\overline{\text{Inn } G}$. Since $(\overline{\text{Inn } G})_{\mathcal{F}'} = 1$, consequently $F(\overline{\text{Inn } G})$ is also a characteristic \mathcal{F} -subgroup of $\overline{\text{Inn } G}$. Since $[L(\overline{\text{Inn } G}), F(\overline{\text{Inn } G})] = 1$ and \mathcal{F} is a D_0 -closed homomorph, hence $L(\overline{\text{Inn } G}) F(\overline{\text{Inn } G})$ is a normal \mathcal{F} -subgroup of $\overline{\text{Aut } G}$. Let \bar{M} be a maximal normal \mathcal{F} -subgroup of $\overline{\text{Aut } G}$ that contains $L(\overline{\text{Inn } G}) F(\overline{\text{Inn } G})$. Set $\bar{L} = C_{\overline{\text{Aut } G}}(\bar{M})$, since \bar{L} centralizes \bar{M} , \bar{L} will centralize $L(\overline{\text{Inn } G}) F(\overline{\text{Inn } G})$. Let L be the subgroup of $\text{Aut } G$ such that $\bar{L} = L / (\text{Aut } G)_{\mathcal{F}'}$, and let H be the normal subgroup of G such that $L(\overline{\text{Inn } G}) F(\overline{\text{Inn } G}) \simeq H/G_{\mathcal{F}'} Z(G)$. Then L centralizes $H/G_{\mathcal{F}'} Z(G)$ and hence L centralizes $G/C_G(H/G_{\mathcal{F}'} Z(G))$, too. Now since $H/G_{\mathcal{F}'} Z(G) = L(G/G_{\mathcal{F}'} Z(G)) / F(G/G_{\mathcal{F}'} Z(G))$, by lemma 1 consequently:

$$C_{G/G_{\mathcal{F}'} Z(G)}(H/G_{\mathcal{F}'} Z(G)) \leq H/G_{\mathcal{F}'} Z(G),$$

hence $C_G(H/G_{\mathcal{F}'} Z(G)) \leq H$, and so L centralizes G/H , therefore L stabilizes the series:

$$G/G_{\mathcal{F}'} Z(G) > H/G_{\mathcal{F}'} Z(G) > 1,$$

thus $L/C_L(G/G_{\mathcal{F}'} Z(G))$ is nilpotent by a well-known P. Hall's result. We consider now $C_{\text{Aut } G}(G/G_{\mathcal{F}'} Z(G))$. This group induces an automorphism group of $G/G_{\mathcal{F}'}$ which is isomorphic to:

$$(1) \quad C_{\text{Aut } G}(G/G_{\mathcal{F}'} Z(G)) / C_{\text{Aut } G}(G/G_{\mathcal{F}'})$$

By ii) $G/G_{\mathcal{F}'}$ has no direct abelian factors, hence we know ([7]) that:

$$C_{\text{Aut}(G/G_{\mathcal{F}'})}(G/G_{\mathcal{F}'}/Z(G/G_{\mathcal{F}'}))$$

is nilpotent and so:

$$(2) \quad C_{\text{Aut}(G/G_{\mathcal{F}'})}(G/G_{\mathcal{F}'}/Z(G)G_{\mathcal{F}'}/G_{\mathcal{F}'})$$

is nilpotent, too.

Since the subgroup of $\text{Aut}(G/G_{\mathcal{F}'})$ isomorphic to (1) is contained in (2), we deduce that (1) is nilpotent. Now, since $C_{\text{Aut} G}(G/\Phi(G))$ is nilpotent ([11]) and by i) $C_{\text{Aut} G}(G/G_{\mathcal{F}'})$ is nilpotent too. Consequently $C_{\text{Aut} G}(G/G_{\mathcal{F}'}Z(G))$ is solvable and so L and hence \bar{L} are solvable. Now $F(\bar{L}) \in \mathcal{F}$, hence $F(\bar{L})\bar{M} \in \mathcal{F}$, as \bar{M} is a maximal normal \mathcal{F} -subgroup of $\overline{\text{Aut } G}$, then $F(\bar{L}) \leqslant \bar{M}$, therefore:

$$\bar{L} = C_{\overline{\text{Aut } G}}(\bar{M}) \leqslant C_{\overline{\text{Aut } G}}(F(\bar{L})) \cap \bar{L} = C_{\bar{L}}(F(\bar{L})) \leqslant F(\bar{L}) \leqslant \bar{M}.$$

Next, we give some counterexamples which prove that the conditions imposed in theorem 5, are not superfluous.

1) The assumption $G_{\mathcal{F}'} \leqslant \Phi(G)$ is necessary.

It is enough to take $G = C_3 \times C_3 \times C_3 \times Q$ where Q is the quaternion group of order 8. The $\text{Aut } G = GL(3, 3) \times S_4$. If \mathcal{F} is the class of 2-groups, then $G_{\mathcal{F}'} = C_3 \times C_3 \times C_3$ is not a subgroup of $\Phi(G) = Z(Q)$. Make note that G is 2-constrained because it is solvable, now, if $\text{Aut } G$ is 2-constrained, $GL(3, 3)$ would be also 2-constrained and hence his normal subgroup $SL(3, 3)$ would be also 2-constrained, but $SL(3, 3)$ is a simple group that is neither a 2-group nor a 2'-group and therefore it is not 2-constrained group.

2) The condition that $G/G_{\mathcal{F}'}$ has no direct abelian factors is equally necessary. We can consider $G = C_3 \times C_3 \times C_3$ and as \mathcal{F} the class of nilpotent groups, then G is \mathcal{N} -constrained but $\text{Aut } G$ is not \mathcal{N} -constrained, because $GL(3, 3)$ is not \mathcal{N} -constrained by a similar argument, to the above one.

Finally, we give some examples of classes verifying A or B that are not saturated Fitting formations

a) The class of the supersolvable groups is a formation that verifies A and B but it is not a Fitting class.

b) Let $\mathcal{F} = \mathcal{N}\mathcal{A} \cap \mathcal{S}_\pi$ be, where \mathcal{A} is the class of abelian groups and \mathcal{S}_π is the class of π -groups, π is a set of prime numbers. This class is a n -closed saturated formation but it is neither a Fitting class nor

E_z -closed. Of course if we take $\pi = \{2\}$ and $G = C_5 \times Q$ where Q is the quaternion group of order 8, then $G/Z(G) \simeq C_2 \times C_2 \in \mathcal{F}$ but $G \notin \mathcal{F}$. On the other hand if we consider:

$$\begin{aligned}\pi &= \{2, 5\}, \quad G = [C_5 \times C_5] \cdot Q_8, \\ C &= [C_5 \times C_5] \langle a \rangle \quad \text{and} \quad D = [C_5 \times C_5] \langle b \rangle\end{aligned}$$

where $Q = \langle a, b | a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle$ then $C, D \in \mathcal{F}$ but $G \notin \mathcal{F}$ because G' is not nilpotent.

c) The class of solvable groups with absolute arithmetic 3-rank a 3'-number is a not saturated Fitting formation [3]; but it is an E_z -closed class, because the 3-chief factors under $Z(G)$ have order 3.

d) Following Cossey, we consider inside solvable groups, the class $\mathfrak{X} = \mathbb{S}_p \mathbb{S}_{p'} \mathbb{S}_p$ where $\mathbb{S}_{p'}$, \mathbb{S}_p are the class of p' -groups and p -groups respectively. We denote by $c_p(G)$ the least common multiple of the absolute degrees of the complemented p -chief factors. Thus the class:

$$\mathbb{Y} = \{G \in \mathfrak{X} | c_p(G) \text{ is coprime to } p\}$$

is both a Fitting class and a Schunck class but it is not a formation [1]. This class verifies A and also satisfies B by a similar argument to that of c).

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