RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

MAURO BILIOTTI
GIUSEPPE MICELLI

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Rendiconti del Seminario Matematico della Università di Padova, tome 73 (1985), p. 217-229

<http://www.numdam.org/item?id=RSMUP_1985__73__217_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 73 (1985)

On Translation Transversal Designs.

MAURO BILIOTTI - GIUSEPPE MICELLI (*)

SUMMARY - A translation transversal design is a transversal design with a point-regular group of automorphisms mapping each non-fixed block into a disjoint block. Such designs are investigated by making use of group partitions and a new construction method for those related to Frobenius groups is given.

Transversal designs and their dual structures, the so called nets, have been extensively investigated in the last years. Several authors devoted some attention to those, among these structures, possessing a group of automorphisms which acts regularly on the point-set (e.g. [6] for a survey and also [7]). In this connection recently Schulz [9], [10] considered transversal designs admitting a group of translations acting transitively—and hence regularly—on the point-set. In [9], after pointing out that the study of such designs, called translation transversal designs (TTD's), essentially turns into that of some special group partitions, he proves that Frobenius groups possess two different classes of partitions which are suitable for the construction of TTD's and carried out several investigations on one of these classes, the other being already considered by Jungnickel in [7]. [10] is devoted to TTD's arising from partitions of Hughes-Thompson groups. In this paper we give a further contribution to the study of TTD's. After giving a classification of those groups which possess

(*) Indirizzo degli AA.: Dipartimento di Matematica, Università, Via Arnesano, 73100 Lecce, Italia,
This research was supported by a grant from the M.P.I.

partitions connected to TTD's, we concentrate our attention to the partitions of Frobenius groups already investigated by Schulz [9] in order to settle some open questions. A construction method of Schulz's partitions is given which allows us to produce examples of such partitions also for Frobenius groups with non-abelian Frobenius kernel. Schulz's partitions of Frobenius groups with abelian Frobenius kernel may be obtained for any admissible value of parameters.

TTD's are very useful for producing examples of the so called « regular transversal designs » which are closely related to difference matrices and so to orthogonal arrays and Latin squares [4], [5].

1. Let $\Delta = (\mathfrak{P}, \mathfrak{B}, \in)$ be an incidence structure. We shall denote by $[p, q]$ the number of blocks joining p and q. Δ is called a transversal design of order s and degree k (or briefly an (s, k)-TD) if the following conditions are satisfied:

(i) the relation «$\sim$» on $\mathfrak{P}$ defined by

$$p \sim q \Leftrightarrow p = q \text{ or } [p, q] = 0$$

is an equivalence relation;

(ii) $p \nsim q \Rightarrow [p, q] = 1$ for all $p, q \in \mathfrak{P}$ [1];

(iii) each block meets each point class (equivalence class of $\sim$);

(iv) there are k point classes each containing s points.

It is easy to prove that for a (s, k)-TD we have

(v) $|B| = k$ for each $B \in \mathfrak{B}$ and there are exactly s blocks through each point;

(vi) $|\mathfrak{P}| = ks$, $|\mathfrak{B}| = s^2$.

A *translation* of a transversal design Δ is a fixed-point-free automorphism g of Δ such that $B = B^g$ or $B \cap B^g = \emptyset$ for each block $B \in \mathfrak{B}$. The identity map is also regarded as a translation. Clarly a translation maps point classes onto themselves. A (s, k)-TD with a translation group T acting transitively—and hence regularly—on the point-set

(1) The more general case of (s, k, λ)-TD's (where $[p, q] = \lambda$ for $p \nsim q$) will not be considered in this paper.

is called a *translation transversal design* (or briefly a (s, k)-TTD). As pointed out in [9], a (s, k)-TTD possesses a natural resolution, that is an equivalence relation « $\|$ » on the block-set such that each equivalence class (parallel class) is a partition of the point-set. Indeed it is enough to put $B_1 \| B_2$ if and only if $B_1^g = B_2$ for some translation $g \in T$. It is well known that for a (s, k)-TTD we have $k \leqslant s$.

We recall that a partition of a group G is a set $\mathcal{S} = \{G_i\}_{i\in I}$ of proper subgroups of G such that $G_i \cap G_j = \langle 1 \rangle$ for $G_i \neq G_j$ and $G = \bigcup_{i\in I} G_i$.

Translation transversal designs may be described by means of group partitions.

PROPOSITION 1.1. *Let* $\mathcal{S} = \{T_0, T_1, \dots, T_s\}$ *be a partition of a group* T *satisfying the condition*

$$(t) \qquad T_0 T_i = T \text{ for each } T_i \text{ with } 1 \leqslant i \leqslant s\,.$$

Then the incidence structure $\mathcal{F}(T, T_0, \mathcal{S})$ *defined as follows:*

1) *points are the elements of* T;

2) *blocks are the right cosets of the components of* $\mathcal{S}$-$\{T_0\}$;

3) *point classes are the right cosets of* T_0;

4) *the incidence is the* $\in$*-relation*;

is a translation transversal design of order s *and degree* k, *where* $k = |T_i|$, $1 \leqslant i \leqslant s$. *The translation group* T^* *consists of the maps* $a^*\colon T \to T$ *defined by* $x \mapsto xa$ *with* $a \in T$ *and we have* $T^* \simeq T$.

PROOF. For the proof of conditions (i) and (ii) see [9], Prop. 3.2. Note that condition (t) implies $|T_i| = |T_j|$ for all i, j with $1 \leqslant i, j \leqslant s$. Let $T_0 v$ be a point class and $T_i u$ a block. By (t) we may suppose $v \in T_i$, $u \in T_0$ and we get

$$|T_0 v \cap T_i u| = |T_0 v u^{-1} \cap T_i| = |T_0 \bar{u}\bar{v} \cap T_i| =$$
$$= |T_0 \bar{v} \cap T_i| = |T_0 \cap T_i \bar{v}^{-1}| = |T_0 \cap T_i| = 1\,,$$

where $\bar{u}\bar{v} = vu^{-1}$, $\bar{u} \in T_0$, $\bar{v} \in T_i$ (we have made still use of (t)). So (iii) is verified. We have $|T| = 1 + \sum_{i=0}^{s} (|T_i| - 1)$ and $|T| = k|T_0|$

since, by (t), $[T:T_0]=k$. These relations give $|T_0|=s$ and (iv) holds. It is easy to verify that maps a^* are the required translations (see [9], Prop. 3.2.).

A partition $\mathcal{S}=\{T_0, T_1, \dots, T_s\}$ will be called an (s, k)-partition if it satisfies (t) and $|T_0|=s$, $|T_i|=k$, $1 \leqslant i \leqslant s$.

Proposition 1.1 can be reversed as follows.

PROPOSITION 1.2. *Let $\Delta=(\mathcal{T}, \mathcal{B}, \in)$ be an (s, k)-TTD with transaltion group T and o a point of Δ, then $\mathcal{S}=\{T_B\colon o \in B \in \mathcal{B}\}$, together with the stabilizer T_0 in T of the point class containing o, is a (s, k)-partition of T and we have $\mathcal{F}(T, T_0, \mathcal{S}) \simeq \Delta$.*

For the proof see [9], Prop. 3.6. Note that condition (t) easily follows from conditions (iii), (v) and (vi) for a (s, k)-TD.

So the study of translation TD's is essentially reduced to that of (s, k)-partitions.

We assume the reader is familiar with the theory of group partitions due to Baer, Kegel and Suzuki. We only recall that given a prime p and a group G we denote by $H_p(G)$ the subgroup of G generated by the elements of order different from p. If $\langle 1\rangle \neq H_p(G) \neq G$ for some prime p and G is not a p-group, then $[G:H_p(G)]=p$ and G is called a group of Hughes-Thompson (regard to p). A group G with a proper subgroup H such that $H \cap H^x=\langle 1\rangle$ for all $x \in G\text{-}H$ is called a Frobenius group. The elements of G which do not lie in any H^x, $x \in G$, make a subgroup K and the subgroups H^x, $x \in G$, together with K, make a partition of G. K is called the Frobenius kernel, H^x a Frobenius complement and the above mentioned partition the (*minimal*) *Frobenius partition of G*.

Except perhaps for the case of Frobenius groups, the following theorem is a fairly immediate consequence of the classification of groups with partitions.

THEOREM 1.3. *Assume $\mathcal{S}=\{T_0, T_1, \dots, T_s\}$ is a partition of T satisfying condition (t), then one of the following holds:*

I) *T is a p-group and $H_p(T) \leqslant T_0$;*

II) *T is a group of Hughes-Thompson, $T_0=H_p(T)$ and T_i is cyclic of order p for all i with $1 \leqslant i \leqslant s$;*

(III) *T is a Frobenius group and*

1) *$\mathcal{S}$ is the Frobenius partition of T or*

2) *the Frobenius kernel of T is a p-group, $T_0 < K$, $T_i = K_i H$, where $K_i < K$ and H is a Frobenius complement of T, and $K = T_0 T_i$ for all i with $1 \leqslant i \leqslant s$.*

PROOF. If $T \simeq PGL(2, p^h)$, $p^h \geqslant 3$, then among the components of every partition of T there are cyclic groups of order $p^h - 1$, $p^h + 1$ and also p-groups, so no partition of T verifies (t). Similarly we cannot have $T \simeq PSL(2, p^h)$, $p^h \geqslant 4$. If $T \simeq Sz(q)$, $q = 2^{2r+1}$, every partition of T contains components of order q^2, $q - 1$, $q \pm r + 1$, where $r = \sqrt{2q}$, and hence condition (t) is unsatisfied. If T is a p-group and $H_p(T) \neq \langle 1 \rangle$, then $|H_p(T)| > [T : H_p(T)]$ (see [8]. Satz 2), and so we must have $H_p(T) \leqslant T_0$ since $|T_i| \leqslant |T_0|$ and $H_p(T)$ does not admit any partition. A group of Hughes-Thompson admits exactly one partition whose components are $H_p(T)$ and the cyclic groups of order p not contained in $H_p(T)$ and II easily follows. For the proof of III see [9], Th. 4.4. Since we have listed all the groups with partitions, the proof is then accomplished.

2. Theorem 1.3 gives some restriction on parameters s and k for the existence of a (s, k)-partition. Every Hughes-Thompson group as well as every p-group Q with $H_p(Q) = \langle 1 \rangle$ or $H_p(Q) \neq \langle 1 \rangle$ and $[Q : H_p(Q)] = p$ is suitable for the construction of (s, k)-partitions (see also [10]). Nevertheless every Frobenius group is suitable for the construction of a (s, k)- partition of type III.1. The case of Frobenius groups with partitions of type III.2. seems to present many more difficulties. Schulz deals with this case in [9] and his results will be mentioned during our subsequent investigation on this subject. We need some definitions.

Let K be a group, $\mathcal{A} = \{K_0, K_1, \ldots, K_s\}$ a set of subgroups of K and Φ a one-to-one map between the set $\{K_1, K_2, \ldots, K_s\}$ and K_0. We denote $K_i \Phi$ by u_i. Assume the following conditions are satisfied:

(a) $|K_0| = s$, $|K_i| = t$ for all i with $1 \leqslant i \leqslant s$;
(b) the elements of K which do not lie in any $u_i^{-1} K_i u_i$, $1 \leqslant i \leqslant s$, make a sugroup V_0 of K;
(c) $\{u_i^{-1} K_i u_i : 1 \leqslant i \leqslant s\} \cup \{V_0\}$ is a partition of K;
(d) $K = K_1 u_1 \dot\cup K_2 u_2 \dot\cup \ldots \dot\cup K_i u_i \dot\cup \ldots \dot\cup K_s u_s$ ($\dot\cup$ disjunct union),

then we call $(K, \mathcal{A}, \Phi)$ *an admissible triad* (*of parameters* s, t).

Let H be a group of automorphisms of K. We say that $(K, \mathcal{A}, \Phi)$ is an *H-admissible triad* if each of the subgroups of $\mathcal{A}$ is H-invariant.

Now let G be a group and α an automorphism of G. For each $g \in G$ define

$$\sigma_{g,\alpha}\colon G \to G$$

$$x \mapsto (x\alpha) g .$$

If H is a group of automorphisms of G, put

$$F(G, H) = \{\sigma_{g,\alpha}\colon g \in G,\ \alpha \in H\} .$$

$F(G, H)$ is nothing but the holomorph of G, with respect to H. The following Proposition holds.

PROPOSITION 2.1. *Let $\Sigma = (K, \mathcal{A}, \Phi)$ be an H-admissible triad of parameters s, t, where H is a group of f.p.f. automorphisms of K of order h; then*

(i) *$F(K, H)$ is a Frobenius group with Frobenius kernel*

$\bar{K} = \{\sigma_{k,I}\colon k \in K\}$ isomorphic to K and Frobenius complement

$\bar{H} = \{\sigma_{1,\alpha}\colon \alpha \in H\}$ isomorphic to H;

(ii) *the stabilizers $F(K, H)_{K_i u_i}$, $1 \leqslant i \leqslant s$, of $K_i u_i$ in $F(K, H)$, together with $\bar{V}_0 = \{\sigma_{v,I}\colon v \in V_0\}$, make a (s, k)-partition $\mathcal{S}(\Sigma, H)$ of $F(K, H)$ with $k = th$ and therefore of type* III.2.

PROOF. We point out that if $\alpha \in H$, $\alpha \neq I$, and $k \in K$, then the equation $k = (x\alpha)^{-1}x$ admits exactly one solution x since α is f.p.f.. From this (i) easily follows. Now let $\sigma_{k,\alpha} \in F(K, H)$ and assume $\alpha \neq I$. Take $x \in K$ such that $k = (x\alpha)^{-1}x$; by (d) we may find $K_i u_i$ such that $x = v u_i \in K_i\, u_i$ and thus we get

$$(K_i u_i)\sigma_{k,\alpha} = \big((K_i u_i)\alpha\big)\, k = (K_i\alpha)(u_i\alpha)(u_i\,\alpha)^{-1}(v\alpha)^{-1} v u_i =$$

$$= K_i (v\alpha)^{-1} v u_i = K_i u_i$$

since K_i is α-invariant. Therefore $\sigma_{k,\alpha} \in F(K, H)_{K_i u_i}$.

Assume $\alpha = I$. By (c) either $k \in u_i^{-1} K_i u_i$ for a suitable i with $1 \leqslant i \leqslant s$, or $k \in V_0$ and we have $\sigma_{k,\alpha} \in F(K, H)_{K_i u_i}$, resp. $\sigma_{k,\alpha} \in \bar{V}_0$.

Suppose $\sigma_{k,\alpha} \in F(K,H)_{K_iu_i} \cap F(K,H)_{K_ju_j}$ with $i \neq j$ and $\alpha \neq I$. From $\sigma_{k,\alpha} \in F(K,H)_{K_iu_i}$ we infer $(u_i\alpha)ku_i^{-1} \in K_i$, that is $k = (u_i\alpha)^{-1}vu_i$ with $v \in K_i$. Since K_i is α-invariant there exists exactly one element $v' \in K_i$ such that $v = (v'\alpha)^{-1}v'$ and hence $k = \big((v'u_i)\alpha\big)^{-1}(v'u_i)$. Similarly we get $k = \big((v''u_j)\alpha\big)^{-1}(v''u_j)$ with $v'' \in K_j$. It follows $v'u_i = v''u_j$, a contradiction by (d). If $\alpha = I$ we have $k \in u_i^{-1}K_iu_i \cap \cap u_j^{-1}K_ju_j$ and therefore $k = 1$ by (c). Lastly suppose $\sigma_{k,\alpha} \in \overline{V}_0 \cap \cap F(K,H)_{K_iu_i}$, then $\alpha = I$ and, as before, $k = 1$ by (c). This prove that $\{F(K,H)_{K_iu_i}: 1 \leqslant i \leqslant s\} \cup \{\overline{V}_0\}$ is a partition of $F(K,H)$. From that we have seen so far we get $F(K,H)_{K_iu_i} = \{\sigma_{k,\alpha}: \alpha \in H,\ k \in (u_i\alpha)^{-1}K_iu_i\}$ and hence $|F(K,H)_{K_iu_i}| = |H||K_i|$ and this completes the proof.

If K is abelian, and hence elementary abelian, for it possesses the partition $\mathcal{A} = \{K_0, K_1, \ldots, K_s\}$, it is straightforward to show that the construction of Prop. 2.1 yields the same result of Schulz's [9], Prop. 2.4, 2.6 and 2.7. Indeed it is enough to observe that in this case K may be regarded as a vector space over any subfield of the kernel of $\mathcal{A}$ (for the definition of « kernel » see [1]), so that K and $\mathcal{A}$ are nothing but the group T and the partition $\pi(T,\Sigma)$ of Def. 2.3 in the Schulz's work, while H is that Schulz calls « an admissible Fro_ benius group ». This situation is of some special interest since we have the following.

PROPOSITION 2.2 (Schulz [9]). *Let F be a Frobenius group with abelian Frobenius kernel and let $\mathcal{S}$ be a (s,k)-partition of F of type III.2, then there exists a suitable H-admissible triad $\Sigma = (K, \mathcal{A}, \Phi)$ and an isomorphism from F to $F(K,H)$ mapping $\mathcal{S}$ into $\mathcal{S}(\Sigma, H)$.*

Obviously the main problem is how finding H-admissible triads. To do this Schulz makes use of a suitable spread in a projective space, thus limiting himself to the case « k abelian » (for details see [9]). In this paper we give a procedure which tourns out to be more powerful than Schulz's method and includes also the non-abelian case.

If $\mathcal{S}$ is a partition of a group G and K is a subgroup of G such that $U \cap K \neq \langle 1 \rangle$ for each $U \in \mathcal{S}$, then we put $\mathcal{S}|_K = \{U \cap K: U \in \mathcal{S}\}$. An $\mathcal{S}$-automorphism of a group G is an automorphism of G leaving invariant each component of $\mathcal{S}$.

The following proposition is fundamental for our construction.

PROPOSITION 2.3. *Assume we have*

(I) *a group G and a partition $\mathcal{S} = \{U_0, U_1, \ldots, U_s\}$ of G satisfying condition (t);*

(II) *a subgroup K of G such that $U_0 < K \rhd G$;*

(III) *a non-trivial group H of $\mathcal{S}|_K$-automorphisms of K.*

Let g be a fixed element of G-K, then

1) $|U_i g \cap U_0| = 1$ *for all i with $1 \leqslant i \leqslant s$;*

2) *if we put $\mathcal{A} = \{K_i\colon K_i = U_i \cap K,\ 0 \leqslant i \leqslant s\}$ and $K_i \Phi = u_i$, $1 \leqslant i \leqslant s$, where u_i is the unique element of U_0 lying in $U_i g$, then $(K, \mathcal{A}, \Phi)$ is an H-admissible triad of parameters s, t with $t = |K_i|$, $1 \leqslant i \leqslant s$.*

PROOF. Note that we have $K_0 = U_0$. 1) Assume $x, x' \in U_i g \cap U_0$, then $x = kg = u$, $x' = k'g = u'$ for some $k, k' \in U_i$, $u, u' \in U_0$. We have $k^{-1}u = g = k'^{-1}u'$ and so $k'k^{-1} = u'u^{-1}$. But $k'k^{-1} \in U_i$, $u'u^{-1} \in U_0$, hence $k'k^{-1} = u'u^{-1} = 1$, that is $k' = k$ and $u' = u$. Therefore $|U_i g \cap U_0| \leqslant 1$. Nevertheless the number of distinct right cosets of U_i in G is $|U_0|$ and so the equality holds. 2) At first we point out that $\mathcal{S}|_K$ is a (s, t)-partition with $t = |K_i|$, $1 \leqslant i \leqslant s$. Indeed let $i \in \{1, \dots, s\}$ and $f \in K$, then $f = uk$ with $u \in U_0$ and $k \in U_i$, but $k \in K$ since $u \in U_0 < K$ and hence $k \in K_i$. So $K \leqslant U_0 K_i$. On the other hand $U_0 K_i \leqslant K$ and the equality holds. Now we shall prove that $(K, \mathcal{A}, \Phi)$ is an H-admissible triad of parameters s, t. Clearly condition (a) is verified. Let $u_i = \bar{g}g$ with $\bar{g} \in U_i$, $1 \leqslant i \leqslant s$, then $u_i^{-1} K_i u_i = g^{-1}\bar{g}^{-1} K \bar{g} g = g^{-1} K g$ since $\bar{g}^{-1} K_i \bar{g} = \bar{g}^{-1}(U_i \cap K)\bar{g} = U_i \cap K = K_i$. So condition ($b$) and ($c$) are satisfied if we put $V_0 = g^{-1} K_0 g$. Since $\mathcal{S}|_K$ is a (s, t)-partition, (d) is verified if we show that $K_i u_i \cap K_j u_j = \emptyset$ whenever $i \neq j$. Assume $x \in K_i u_i \cap K_j u_j$ and hence $x = k\bar{g}g = k'\bar{g}'g$ where $k \in K_i$, $k' \in K_j$ and $u_i = \bar{g}g$, $u_j = \bar{g}'g$ with $\bar{g} \in U_i$, $\bar{g}' \in U_j$. It follows $k\bar{g} = k'\bar{g}' = 1$, a contradiction since it is easily seen that $\bar{g}, \bar{g}' \notin K$.

3. This section is devoted to the construction of sets $(G, K, \mathcal{S}, H)$ satisfying conditions (I), (II) and (III) of Prop. 2.3.

The abelian case. Assume:

$F = GF(p^n)$, p a prime, $n > 1$;

$F_0 = GF(p^m)$ any proper subfield of F;

V a $(u + 1)$-dimensional vector space over F, $u \geqslant 1$;

U_0 a u-dimensional subspace of V.

Let consider the partition $\mathcal{S} = \{U_0, U_1, \dots, U_s\}$, $s = p^{nu}$, of $V(+)$, where $U_1, \dots, U_s$ are the 1-dimensional subspaces of V, not contained in U_0. Trivially $\mathcal{S}$ satisfies condition (t). Now take a set $\boldsymbol{v}_1, \dots, \boldsymbol{v}_l$ of elements of U_1 such that

— $\boldsymbol{v}_1, \dots, \boldsymbol{v}_l$ are independent over F_0, but they do not generate U_1, as a vector space over F_0,

and put $K = \left\{\boldsymbol{v} + \sum_{i=1}^{l} \boldsymbol{v}_i a_i \colon \boldsymbol{v} \in U_0,\ a_i \in F_0\right\}$. We have $U_0 \lhd K \lhd V$ (as additive groups). If $\overline{H}$ is any subgroup of the multiplicative group F_0^* of F_0, let denote by H the group of automorphisms of K of the form $\boldsymbol{v} \mapsto \boldsymbol{v}\alpha$ with $\alpha \in \overline{H}$. It is easy to see that each component of $\mathcal{S}|_K$ is H-invariant. Therefore $(V, K, \mathcal{S}, H)$ satisfies conditions (I), (II), (III) (with $G = V$). The associated triad $\Sigma = (K, \mathcal{A}, \Phi)$ gives rise to a (s, k)-partition of $F(K, H)$ of parameters p^{nu}, $p^{ml}h$ where $h = |H|$.

Proposition 3.1. *Let* $\mathcal{F}(T, T_0, \mathcal{S})$ *be a* (s, k)-TTD *where* T *is a Frobenius group with abelian Frobenius kernel and* $\mathcal{S}$ *a* (s, k)*-partition of type* III.2. *Then* $s = p^a$, $k = p^r h$ *for some prime* p, $1 \leqslant r < a$, $h|(p^a - 1, p^r - 1)$. *For any admissible value of parameters* s, k *such a* (s, k)-TTD *exists.*

Proof. It is enough to observe that by Prop. 2.2 we may suppose $T = F(K, H)$ and $\mathcal{S} = \mathcal{S}(\Sigma, H)$ for a suitable H-admissible triad $\Sigma = (K, \mathcal{A}, \Phi)$; furthermore, if $|K_0| = p^a$ and $|K_i| = p^r$, $1 \leqslant i \leqslant s$, we have $|H| | p^a - 1$ and $|H| | p^r - 1$ since every K_i, $1 \leqslant i \leqslant s$, is H-invariant and all elements of H are f.p.f.. Clearly $a > r$ since $s \geqslant k$. Now let be given a tern of numbers (p^a, p^r, h) with $1 \leqslant r < a$ and $h|(p^a - 1, p^r - 1)$. It is well known that $h | p^b - 1$ where $b = (a, r)$. In our previous construction assume $F = GF(p^a)$, $F_0 = GF(p^b)$, $u = 1$, $l = r/b$ and $|\overline{H}| = h$. The last assertion follows.

We point out that Schulz [9] produces examples of such (s, k)-TTD's only for parameters s, k satisfying some additional assumptions, this because the author is able to construct the projective spreads on which his method is founded only under some restrictive conditions. Actually, if we make use of a result of Bu ([2], Lemma 4) such conditions can be avoided, so that the last assertion of Prop. 3.1 may be also gained by a direct application of the Schulz's method.

The non-abelian case. Let $E = GF(q)$, q odd, V a finite dimensional vector space over E, W a subspace of V and γ an automorphism of E.

Assume $[,]\colon V\times V \to W$ is a map such that:

$$\left.\begin{aligned}&[\boldsymbol{u},\boldsymbol{v}] = -[\boldsymbol{v},\boldsymbol{u}]\\ &[\boldsymbol{u}_1+\boldsymbol{u}_2,\boldsymbol{v}] = [\boldsymbol{u}_1,\boldsymbol{v}]+[\boldsymbol{u}_2,\boldsymbol{v}]\\ &[\boldsymbol{u}x,\boldsymbol{v}] = [\boldsymbol{u},\boldsymbol{v}]x^{\gamma}\\ &[\boldsymbol{u},\boldsymbol{w}] = 0\end{aligned}\right|\begin{aligned}&\text{for all } \boldsymbol{u},\boldsymbol{u}_1,\boldsymbol{u}_2\in V\,,\\ &\boldsymbol{w}\in W,\ x\in E\end{aligned}$$

Following Herzer [3] we call [,] an « alternative bisemilinear map vanishing on $V\times W$ ». Clearly, with respect to a given basis $\boldsymbol{v}_1, \dots, \boldsymbol{v}_h$ of a complementary space of W, such a map takes the form

$$\Big[\boldsymbol{w}+\sum_i \boldsymbol{v}_i x_i,\ \boldsymbol{w}'+\sum_j \boldsymbol{v}_j y_j\Big] = \sum_{i<j} \boldsymbol{w}_{ij}(x_i y_j - x_j y_i)^{\gamma}$$

where $\boldsymbol{w}_{ij} = [\boldsymbol{v}_i, \boldsymbol{v}_j]$, $\boldsymbol{w}$, $\boldsymbol{w}'\in W$, x_i, $y_j\in E$, $\gamma\in \mathrm{Aut}(E)$.

Using [,] we may define a new operation « $\circ$ » on V as follows

$$\boldsymbol{u}\circ\boldsymbol{v} = \boldsymbol{u}+\boldsymbol{v}+[\boldsymbol{u},\boldsymbol{v}]\,,\qquad \boldsymbol{u},\boldsymbol{v}\in V\,.$$

$V(\circ)$ is a group of nilpotence class $\leqslant 2$ (see [3]).

We point out that whenever for a subspace U of V we have $[U, U]\leqslant U$, then U is also a subgroup of $V(\circ)$; so, because of $[\boldsymbol{v}E, \boldsymbol{v}E] = 0$ for all $\boldsymbol{v}\in V$, the 1-dimensional subspaces of V are subgroups of $V(\circ)$. Moreover if $W\leqslant U$, then U is a normal subgroup of $V(\circ)$.

Now suppose there exists $a\in E$, $0\neq a\neq 1$, such that $a^{2\gamma} = a$ and let $\mathcal{S}$ be a partition of $V(\circ)$ whose components are subspaces of V, then it is easily seen that the map $\delta_a\colon V\to V$, $\boldsymbol{v}\mapsto \boldsymbol{v}a$ is an $\mathcal{S}$-automorphism of $V(\circ)$. In a field $E = GF(q)$, $q = p^s$, p an odd prime, there exists such a pair (γ, a), where a is an element of E^* of (odd) order n and $\gamma\colon x\mapsto x^{p^r}$, $0<r<s$, if and only if for $t = (n+1)/2$ we have $p^r\equiv t \bmod n$. For the construction of pairs (γ, a) we may refer to [3]. Here we only recall that, for instance, such pairs exist for $n = 3$, r odd, s even and $p\equiv 2 \bmod 3$, thus $p = 5, 11, 17, 23, 29, \dots$

Let F, F_0, V and U_0 as in the abelian case. Suppose there exists a pair (γ, a) with $\gamma\in \mathrm{Aut}(F_0)$ and $a\in F_0$ such that $a^{2\gamma} = a$. Clearly γ extends to an automorphism of F which will be still denoted by γ. Now fix a subspace W of V with $\langle 0\rangle\neq W< U_0$ and a non-trivial alternative bisemilinear map [,] vanishing on $V\times W$, with associated automorphism γ, and let $V(\circ)$ be the related group. Furthermore assume $\mathcal{S}$ and K are defined as in the abelian case. Note that U_0 and K are normal subgroups of $V(\circ)$—they both contain W—and $\mathcal{S}$ is a partition of $V(\circ)$ consisting of subspaces of V. Moreover K is non-abelian whenever $[,]|_{K\times K}$ is non-trivial. Put $H = \langle\delta_a\rangle$, then H is a group of $\mathcal{S}|_K$-automorphisms of K and $(V, K, \mathcal{S}, H)$ satisfies conditions (I), (II) and (III). So we have constructed (s, k)-partitions of type III.2 in a Frobenius group with non-abelian Frobenius kernel.

4. Assume we have a TTD $\mathcal{F} = \mathcal{F}(T, T_0, \mathcal{S})$ with the property that $\mathcal{S}$-$\{T_0\}$ is invariant under the inner automorphisms of T and hence $T_0 \lhd T$. In this case the left translations of T also make a group $\overline{T} = \{\overline{a}: x \mapsto ax\}$ of automorphisms of $\mathcal{F}$. The group $\overline{T}$ is point-regular and respects the parallelism induced in $\mathcal{F}$ by T^*, moreover it is easily seen that $\mathcal{F}$ is a (s, k)-TTD with respect to $\overline{T}$, the new resolution « $\|'$ » being

$$T_i a \,\|'\, T_j b \text{ if and if } T_i^a = T_j^b,\ T_i, T_j \in \mathcal{S}\text{-}\{T_0\},\ a, b \in T\ .$$

We recall that two resolutions are called *orthogonal* if any two of their parallel classes have at most one block in common. We have the following

PROPOSITION 4.1. *Assume $\mathcal{F} = \mathcal{F}(T, T_0, \mathcal{S})$ has the property that $\mathcal{S}$-$\{T_0\}$ is invariant under* Inn(T), *then the resolutions $\|$ and $\|'$ induced in $\mathcal{F}$ by T^*, resp. $\overline{T}$, are orthogonal if and only if T is a Frobenius group with Frobenius kernel T_0 and $\mathcal{S}$ is the Frobenius partition of T.*

PROOF. Suppose $\|$ and $\|'$ are orthogonal. Let $T_i \in \mathcal{S}$-$\{T_0\}$, $k \in T$ and assume $T_i^k \cap T_i \neq \langle 1\rangle$. Since $T_i^k \in \mathcal{S}$-$\{T_0\}$, we have $T_i^k = T_i$ and so $T_i k = kT_i$. But the right cosets of T_i make a parallel class of $\|$, while the left cosets of T_i make a parallel class of $\|'$ and these classes have exactly the block T_i in common, so we must have $k \in T_i$. It follows without difficulty that T is a Frobenius group and the components of $\mathcal{S}$-$\{T_0\}$ are the Frobenius complements, while T_0 is the Frobenius kernel. For the converse see [7], Theorem 2.2 (III).

TTD's arising from Frobenius groups with partitions of type III.1 was widely investigated by Jungnickel in [7]. Another interesting case of TTD's $\mathcal{F}(T, T_0, \mathcal{S})$ such that $\mathcal{S}$-$\{T_0\}$ is invariant under Inn(T) occurs when T is a group of Hughes-Thompson, $T_0 = H_p(T)$ and $\mathcal{S}$ is the unique partition of T. It this connection we remark that a group T

of Hughes-Thompson is also a Frobenius group with Frobenius kernel $H_p(T)$ if and only if $p \nmid |H_p(T)|$ (see [11]).

We conclude with a remark concerning G-regular TD's [4]. Let Δ be a (s, k)-TD and G a group of order s. Δ is called *G-regular* if G acts as a collineation group of Δ wich is regular on each point class and semiregular on the block-set. Let $\mathcal{F}(T, T_0, \mathcal{S})$ be a TTD. Set any subset S_i, $|S_i| \geqslant 2$, of a component $T_i \in \mathcal{S}$-$\{T_0\}$ and define a new incidence structure $\Delta(\mathcal{F}, S_i)$ as follows:

1) the point-set $\mathcal{T}$ consists of the elements of T lying in some $T_0 s$ with $s \in S_i$;

2) the blocks are the intersections of the blocks of $\mathcal{F}$ with $\mathcal{T}$;

3) the incidence is that induced by the incidence of $\mathcal{F}$;

4) the point classes are the cosets $T_0 s$ with $s \in S_i$.

It is an easy exercise to show that $\Delta(\mathcal{F}, S_i)$ is a (s, k')-TD with $k' = |S_i|$.

PROPOSITION 4.2. *Assume $S_i \subseteq N_T(T_0)$, then the transversal design $\Delta(\mathcal{F}, S_i)$ is T_0^*-regular.*

PROOF. Let $s \in S_i$ and $t \in T_0$. We have $T_0 st = T_0 \bar{t} s$ with $\bar{t} \in T_0$ and so $T_0 st = T_0 s$, therefore T_0^* acts regularly on each point class. Now let B be a block of $\Delta(\mathcal{F}, S_i)$, then $B = B_j r$ for a suitable subset B_j of $T_j \in \mathcal{S}$-$\{T_0\}$ and a suitable $r \in T_0$. Suppose for some $t \in T_0$ we have $B_j rt = B_j r$, then since $|B_j| \geqslant 2$, we have also $T_j rt = T_j r$ and hence $t \in T_j^r$; but $t \in T_0 = T_0^r$ and so $t = 1$. It follows that T_0 acts semiregularly on the blocks of $\Delta(\mathcal{F}, S_i)$ and this completes the proof.

Note that the hypothesis of Prop. 4.2 is satisfied for any choice of the set S_i when T is abelian, T is a group of Hughes-Thompson or T is a Frobenius group and $\mathcal{S}$ is of type III.1, since $T_0 \lhd T$ in all these cases.

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Manoscritto pervenuto in redazione il 24 aprile 1984.