L. FUCHS

G. VILJOEN

A generalization of separable torsion-free abelian groups

Rendiconti del Seminario Matematico della Università di Padova, tome 73 (1985), p. 15-21

<http://www.numdam.org/item?id=RSMUP_1985__73__15_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 73 (1985)

A Generalization of Separable Torsion-Free Abelian Groups.

L. FUCHS - G. VILJOEN (*)

Recall that a torsion-free abelian group A is called *completely decomposable* if it is a direct sum of groups of rank 1, and *separable* if every finite set of elements of A is contained in a completely decomposable summand of A (see e.g. [6, p. 117]). There are two results on separable groups which are not easy to prove. One states that summands of separable groups are again separable [6, p. 120]. The other, due to Cornelius [4], asserts that for the separability of A it is sufficient to assume that every element of A can be embedded in a completely decomposable summand of A.

In this note, we generalize the notion of separability by replacing the class of rank 1 groups by a class of groups possessing some of the properties of the class of rank 1 groups. Our main purpose is to extend the two results mentioned above to groups which are separable in a wider sense. The result on the summands is based on a deep theorem of Arnold, Hunter and Richman [1], while Cornelius' own ideas are used to obtain a suitable generalization of his theorem in [4].

Needless to say, a further generalization is possible, in the spirit of [1], to certain additive categories. Since so far separability has had no application to general additive categories, we deal here only with abelian groups for which separability is of a great deal of interest.

All groups in this note are torsion-free and abelian. The notation and terminology are those of [5] and [6]. $E(A)$ will denote the endomorphism ring of A.

(*) Indirizzo degli AA.: L. FUCHS: Dept. of Mathematics, Tulane University, New Orleans, Louisiana 70118, U.S.A.; G. VILJOEN: Dept. of Mathematics, U.O.F.S., 9300 Bloemfontein, Republic of South Africa.

§ 1. Let $\mathcal{C}$ be a class of groups (always assumed to be closed under isomorphism) satisfying the following conditions:

(A) Every $G \in \mathcal{C}$ is torsion-free of finite rank.

(B) For each $G \in \mathcal{C}$, $E(G)$ is a principal ideal domain.

(C) If $A = \bigoplus_{n \in \mathbb{N}} G_n$ with $G_n \in \mathcal{C}$ and B is a summand of A, then $B \cong \bigoplus_{n \in J} G'_n$ with $G'_n \in \mathcal{C}$ and $J \subseteq \mathbb{N}$.

Examples of such classes $\mathcal{C}$ are abundant. The following are probably the most interesting ones.

1) The class of all rank 1 torsion-free abelian groups [6, p. 114, p. 216].

2) The class of indecomposable Murley groups [7, p. 662], [1, p. 239]. Recall that a torsion-free abelian group G of finite rank is called a *Murley group* if G/pG has order $\leqq p$ for every prime p.

3) The class of all torsion-free groups of finite rank whose endomorphism rings are P.I.D.

In the following definition, $\mathcal{C}$ denotes a class with properties (A)-(C).

DEFINITION 1. A group is said to be *completely* $\mathcal{C}$-*decomposable* if it is a direct sum of groups in $\mathcal{C}$. A group A is $\mathcal{C}$-*separable* if every finite subset of A is contained in a completely $\mathcal{C}$-decomposable summand of A. A $\mathcal{C}$-separable group A is G-*homogeneous* ($G \in \mathcal{C}$) if every summand $H \in \mathcal{C}$ of A is isomorphic to G.

Observe that if $\mathcal{C}$ is the class of rank 1 torsion-free groups, then these definitions coincide with the usual ones (where reference to $\mathcal{C}$ is omitted).

It is not hard to construct $\mathcal{C}$-separable groups which are not completely $\mathcal{C}$-decomposable. Let X be any $\mathbb{Z}$-homogeneous separable group which is not completely decomposable. If $G \in \mathcal{C}$, then from [5, pp. 93, 260, 262] it follows that $G \otimes X$ is G-homogeneous $\mathcal{C}$-separable.

If $\mathcal{C}$ denotes the class of indecomposable Murley groups, then for every $G \in \mathcal{C}$ and every separable torsion-free group X, the group $G \otimes X$ will be $\mathcal{C}$-separable.

§ 2. Our first aim is to prove that $\mathcal{C}$-separability is inherited by summands. The following result is crucial in our proof.

LEMMA 2. Let A be a $\mathfrak{C}$-separable group and assume $A = B \oplus C$. Given a finite rank summand M of A, there exists a pure subgroup N of A such that

(i) $M \leqq N$;

(ii) N is completely $\mathfrak{C}$-decomposable of countable rank;

(iii) $N = (N \cap B) \oplus (N \cap C)$.

PROOF. Let π and ϱ denote the projections of A onto B and C, respectively. Evidently, $M \leqq \pi M + \varrho M$. From the $\mathfrak{C}$-separability of A it follows that A has a direct summand $M_1 = H_1 \oplus \ldots \oplus H_n$ with $H_i \in \mathfrak{C}$ which contains a maximal independent set of elements in $\pi M + \varrho M$. Clearly, $M \leqq M_1$. Repeating this argument for M_1 rather than for M, and continuing in the same fashion we get a sequence M_n of completely $\mathfrak{C}$-decomposable summands of A such that

$$M_0 = M \leqq \pi M + \varrho M \leqq M_1 \leqq \pi M_1 + \varrho M_1 \leqq M_2 \leqq \ldots .$$

Manifestly, $N = \bigcup_n M_n$ is a pure subgroup of A satisfying $\pi N \leqq N$ and $\varrho N \leqq N$. Therefore $\pi N = N \cap B$, $\varrho N = N \cap C$, and (iii) holds. By condition (C), $M_{n+1} = M_n \oplus L_{n+1}$ implies that each L_n $(n = 0, 1, \ldots)$, including $L_0 = M_0$, is completely $\mathfrak{C}$-decomposable. Hence $N = \bigoplus L_n$ satisfies (ii). □

We are now able to prove one of our main results.

THEOREM 4. Let $\mathfrak{C}$ be a class of groups satisfying (A)-(C). Direct summands of $\mathfrak{C}$-separable groups are again $\mathfrak{C}$-separable.

PROOF. Let $A = B \oplus C$ be $\mathfrak{C}$-separable. Given a finite subset Δ of B, there exists a summand $M = G_1 \oplus \ldots \oplus G_k$ (with $G_i \in \mathfrak{C}$) of A such that $\Delta \subseteq M$. Embed M in a pure subgroup N of A satisfying conditions (i)-(iii) of Lemma 2. By hypothesis (C), $N \cap B$ is completely $\mathfrak{C}$-decomposable, hence there exists a finite rank summand $B^* = K_1 \oplus \ldots \oplus K_m$ of $N \cap B$ with $K_i \in \mathfrak{C}$ that contains Δ. Evidently, $B^* \leqq M_n$ for some M_n (see proof above) which is a summand of N. We conclude that B^* is a summand of M_n, and hence of A. Therefore B^* is a desired summand of B. □

We are indebted to Prof. Rangaswamy for pointing out to us that a similar argument has been used in his paper [8] in the proof of Theorem 6.

§ 3. Our next purpose is to show that, under a mild condition on $\mathcal{C}$, $\mathcal{C}$-separability follows if we know that every element is contained in some completely $\mathcal{C}$-decomposable summand.

We require the following result due to Botha and Gräbe [2].

LEMMA 3. Let G be a torsion-free abelian group of finite rank whose endomorphism ring is a principal ideal domain. If $M = G_1 \oplus \ldots \oplus G_k$ with $G_i \cong G$ for all i, then the kernel of each endomorphism of M is a summand of M and is itself a direct sum of copies of G.

We now proceed to prove a couple of preparatory lemmas. The class $\mathcal{C}$ is assumed to satisfy (A) and (B).

LEMMA 4. Let $A = B \oplus C = M \oplus H$ where $M = G_1 \oplus \ldots \oplus G_m$, $G_j \cong G \in \mathcal{C}$ for all j. Suppose that $\Delta = \{b_1, \ldots, b_n\} \subseteq B \cap M$ and m is minimal in the sense that Δ is not contained in any direct summand of A which is the direct sum of fewer than m copies of G. Then the projection of M in B is a summand of B, contains Δ and is isomorphic to M.

PROOF. Let π and σ denote the projections of A onto B and M, respectively. Evidently, $\sigma\pi b_i = b_i$ for $i=1, \ldots, n$, thus $\Delta \subseteq \mathrm{Ker}(\sigma\pi|M - 1_M)$. In view of Lemma 3, the minimality of m implies $\mathrm{Ker}\,(\sigma\pi|M - 1_M) = M$, i.e. $\sigma\pi|M = 1_M$. Consequently, $\pi\sigma\pi\sigma = \pi\sigma$ and $\pi\sigma$ is a projection of A onto a summand πM of B. This πM obviously contains Δ and $\pi|M$ is an isomorphism. □

LEMMA 6. Let $A = B \oplus C$ and $b \in B$. Suppose that $A = M \oplus H$ where $b \in M = G_1 \oplus \ldots \oplus G_m$ with $G_j \in \mathcal{C}$ for all j, but b is not contained in any summand of A which is the direct sum of fewer than m members of $\mathcal{C}$. If $G_1 \cong \ldots \cong G_k$ and $\mathrm{Hom}\,(G_i, C) = 0$ for $i = k+1, \ldots, m$, then b is contained in a completely $\mathcal{C}$-decomposable summand of B (isomorphic to M).

PROOF. Let π and ϱ denote the projections of A onto B and C, respectively. Our assumption implies that $\varrho(G_{k+1} \oplus \ldots \oplus G_m) = 0$ whence $G_{k+1} \oplus \ldots \oplus G_m \leqq B$ follows. Factoring out $G_{k+1} \oplus \ldots \oplus G_m$, we obtain

$$\bar{A} = \bar{B} \oplus \bar{C} = \bar{G}_1 \oplus \ldots \oplus \bar{G}_k \oplus \bar{H}$$

(bars indicate images mod $G_{k+1} \oplus \ldots \oplus G_m$) where $\bar{b} \in \bar{B} \cap (\bar{G}_1 \oplus \ldots \oplus \bar{G}_k)$. If $\bar{\pi}$, $\bar{\varrho}$ denote the projections onto $\bar{B}$, $\bar{C}$, then noting that here k is minimal in the sense of Lemma 5 (otherwise a contradiction to the minimality of m would arise), we can apply Lemma 5 to conclude that $\bar{\pi}$ maps $\bar{G}_1 \oplus \ldots \oplus \bar{G}_k$ isomorphically onto a summand of $\bar{B}$, say,

$$\bar{B} = \bar{\pi}(\bar{G}_1 \oplus \ldots \oplus \bar{G}_k) \oplus \bar{B}' .$$

The complete inverse image B' of $\bar{B}'$ satisfies $B' = G_{k+1} \oplus \ldots \oplus G_m \oplus B''$ for some B'', as $G_{k+1} \oplus \ldots \oplus G_m$ was a summand of A. We claim that

$$B = \pi M \oplus B'' .$$

On the one hand, clearly, $B = \pi M + B''$. On the other hand, as πM is the inverse image of $\bar{\pi}\bar{M}$, $\pi M \cap B'' \leqq (G_{k+1} \oplus \ldots \oplus G_m) \cap B'' = 0$. We infer that πM is a summand of B containing b. As $\bar{\pi}$ was an isomorphism on $\bar{G}_1 \oplus \ldots \oplus \bar{G}_k$, it follows at once that $\pi|M$ is likewise an isomorphism. □

For the remainder of this paper, we assume that $\mathcal{C}$ satisfies, in addition to (A)-(C), also the following condition [3]:

(D) $\mathcal{C}$ is a *semirigid system*, i.e. if $\{G_i | i \in I\}$ is the family of the non-isomorphic members of $\mathcal{C}$, then a partial ordering of I is obtained by declaring $i \leqq j$ if and only if $\operatorname{Hom}(G_i, G_j) \neq 0$.

Notice that if $\mathcal{C}$ is a semirigid system, then $\operatorname{Hom}(G_i, G_j) \neq 0 \neq$ $\neq \operatorname{Hom}(G_j, G_i)$ for $G_i, G_j \in \mathcal{C}$ implies $G_i \cong G_j$. Furthermore, $\operatorname{Hom}(G_i,$ $G_j) \neq 0 \neq \operatorname{Hom}(G_j, G_k)$ for $G_i, G_j, G_k \in \mathcal{C}$ implies $\operatorname{Hom}(G_i, G_k) \neq 0$.

Under the hypotheses (A)-(D) on $\mathcal{C}$, we have:

LEMMA 7. Suppose the group A has the property that each element of A is contained in a completely $\mathcal{C}$-decomposable summand of A. If $A = B \oplus C$ where $C = C_1 \oplus \ldots \oplus C_n (C_i \in \mathcal{C})$, then each element of B can be embedded in a completely $\mathcal{C}$-decomposable summand of B.

PROOF. Let $b \in B$, and assume

$$A = G_1 \oplus \ldots \oplus G_k \oplus H$$

with $G_j \in \mathcal{C}$, $b \in G_1 \oplus \ldots \oplus G_k$ and k is minimal. We induct on n.

First, let $n = 1$, i.e. $C = C_1 \in \mathcal{C}$. Denote by K the direct sum of the G_j's with $\operatorname{Hom}(C, G_j) = 0$, and by L the direct sum of those with $\operatorname{Hom}(C, G_j) \neq 0$. Thus $A = B \oplus C = K \oplus L \oplus H$. As the projection of A onto K carries C into 0, necessarily $C \leqq L \oplus H$. We can thus set $L \oplus H = B' \oplus C$ with $B' = (L \oplus H) \cap B$. Hence

$$A = B \oplus C = K \oplus B' \oplus C\,.$$

Write $b = b_1 + b_2$ with $b_1 \in K$, $b_2 \in L$, and $b_2 = b' + c$ with $b' \in B$, $c \in C$. Therefore, $b_2 - c = b' \in B'$. By hypothesis, there is a decomposition

$$A = E_1 \oplus \ldots \oplus E_m \oplus M$$

with $E_i \in \mathcal{C}$ and $b' \in E_1 \oplus \ldots \oplus E_m$. If $\varepsilon_i\colon A \to E_i$ $(i = 1, \ldots, m)$ denote the obvious projections, then $\varepsilon_i b' \neq 0$ may be assumed for $i = 1, \ldots, m$. Hence for each of $i = 1, \ldots, m$ we have either $\varepsilon_i b_2 \neq 0$ or $\varepsilon_i c \neq 0$.

If $\varepsilon_i b_2 \neq 0$, then there is an index j with G_j a summand of L such that $\varepsilon_i G_j \neq 0$. Thus $\operatorname{Hom}(C, G_j) \neq 0$, and $\operatorname{Hom}(G_j, E_i) \neq 0$ simultaneously, so by condition (D), we have $\operatorname{Hom}(C, E_i) \neq 0$. In the second alternative (i.e. when $\varepsilon_i c \neq 0$), we have obviously again $\operatorname{Hom}(C, E_i) \neq 0$. In either case, we must have $\operatorname{Hom}(E_i, K) = 0$ (otherwise $\operatorname{Hom}(C, K) \neq 0$ would follow).

Consequently, $\operatorname{Hom}(E_i, K \oplus C) \neq 0$ implies $\operatorname{Hom}(E_i, C) \neq 0$. But then, again by (D), $\operatorname{Hom}(C, E_i) \neq 0$ implies $E_i \cong C$. We conclude that for each $i = 1, \ldots, m$ either $E_i \cong C$ or $\operatorname{Hom}(E_i, K \oplus C) = 0$.

We may now apply Lemma 6 to the decomposition $A = (K \oplus C) \oplus B'$ and to the element $b' \in B'$ in order to obtain $B'' = F \oplus D$ with F completely $\mathcal{C}$-decomposable of finite rank and $b' \in F$. Hence $A = K \oplus C \oplus F \oplus D$ where $b \in K \oplus F \oplus C$ which group is completely $\mathcal{C}$-decomposable. We can write $K \oplus F \oplus C = B'' \oplus C$ where $b \in B'' = B \cap (K \oplus F \oplus C)$. In view of (C), B'' is completely $\mathcal{C}$-decomposable, completing the proof of case $n = 1$.

We now assume $n \geq 2$ and the statement true for summands C which are direct sums of less than n members of $\mathcal{C}$. Suppose $C = C_1 \oplus \ldots \oplus C_n$ $(C_i \in \mathcal{C})$. Induction hypothesis guarantees that every element of $B \oplus C_n$ is contained in a completely $\mathcal{C}$-decomposable summand of $B \oplus C_n$. A simple appeal to the case $n = 1$ completes the proof of Lemma 7. □

It is now easy to verify our second main result.

Theorem 8. Let $\mathcal{C}$ satisfy conditions (A)-(D). A group A is $\mathcal{C}$-separable if each element of A is contained in a completely $\mathcal{C}$-decomposable summand of A.

PROOF. As a basis of induction, suppose that every subset of A, containing at most $n \geq 1$ elements is embeddable in a completely $\mathcal{C}$-decomposable summand of A. Let $\Delta = \{a_1, \ldots, a_{n+1}\}$ be a subset of A. By induction hypothesis, there is a completely $\mathcal{C}$-decomposable summand B of A containing $\{a_1, \ldots, a_n\}$, say, $A = B \oplus C$. By Lemma 7, the C-coordinate of a_{n+1} belongs to a completely $\mathcal{C}$-decomposable summand C^* of C. Hence $B \oplus C^*$ is a completely $\mathcal{C}$-decomposable summand of A containing Δ. □

REFERENCES

[1] D. ARNOLD - R. HUNTER - F. RICHMAN, *Global Azumaya theorems in additive categories*, J. Pure Appl. Algebra, **16** (1980), pp. 223-242.

[2] J. D. BOTHA - P. J. GRÄBE, *On torsion-free abelian groups whose endomorphism rings are principal ideal domains*, Comm. in Alg., **11** (1983), pp. 1343-1354.

[3] B. CHARLES, *Sous-groupes fonctoriels et topologies, Etudes sur les Groupes Abéliens*, ed. B. Charles (Dunod, Paris, 1968), pp. 75-92.

[4] E. F. CORNELIUS Jr., *A sufficient condition for separability*, J. Algebra, **67** (1980), pp. 476-478.

[5] L. FUCHS, *Infinite abelian groups*, Vol. I, Academic Press, New York and London, 1970.

[6] L. FUCHS, *Infinite abelian groups*, Vol. II, Academic Press, New York and London, 1973.

[7] C. E. MURLEY, *The classification of certain classes of torsion-free abelian groups*, Pacific J. Math., **40** (1972), pp. 647-665.

[8] K. M. RANGASWAMY, *On strongly balanced subgroups of separable torsion-free abelian groups, Abelian Group Theory*, Lecture Notes in Math. 1006 (1983), pp. 268-274.

Manoscritto pervenuto in redazione il 23 settembre 1983 ed in forma riveduta il 5 marzo 1984.