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Finite Groups with a Standard-Component of Type $L_3(4)$, II.

CHENG KAI-NAH - DIETER HELD (*)

0. Introduction.

In this paper we finish the investigation of the $L_3(4)$-type standard-subgroup problem. Because of the result of [3] we have to treat here only the case in which the 2-rank of the center of the standard-subgroup is equal to 1, that is, we assume in what follows that the 2-part of the center is cyclic and different from $\langle 1 \rangle$.

The results obtained in [5] will be assumed; we shall retain the notations introduced there. As in [5], we consider a fixed standard-subgroup $\mathcal{A}$ of our group $G$ with $\mathcal{A}/\mathcal{Z}(\mathcal{A}) \cong L_3(4)$ and put $K = G(\mathcal{A})$. By $X$ we denote a fixed $S_2$-subgroup of $N(\mathcal{A})$ and put $X \cap \mathcal{A} = S$, $X \cap K = Q$. Thus, $X$ is contained in $\{QS, QS\langle \varphi \rangle, QS\langle \pi \rangle, QS\langle \varphi \varphi \rangle, QS\langle \varphi, \pi \rangle\}$; here $S = \langle Q \cap \mathcal{A}, \pi, \tau, \mu, \lambda, \zeta, \xi \rangle$, where the relations between the generators are those valid in $P \in \text{Syl}_2(L_3(4))$ but modulo $Q \cap \mathcal{A}$; of course $P \cong S/\mathcal{Q} \cap \mathcal{A}$.

The Schur-multiplier of $L_3(4)$ is isomorphic to $Z_4 \times Z_4 \times Z_3$. Thus, we have to handle the cases $Q \cap S \cong Z_2$ and $Q \cap S \cong Z_4$. The case $Q \cap S = \langle 1 \rangle$ has been treated in [3], and there it is proved that then $G$ is isomorphic to the sporadic simple group of Suzuki. Thus, making use of all earlier results we shall have proved the following theorem:

THEOREM. Let $G$ be a finite, nonabelian simple group which possesses a standard-subgroup $A$ such that $A/\mathcal{Z}(A)$ is isomorphic to $L_3(4)$. Then, $G$ is isomorphic to $Sz$, $He$, or $O'N$.

Here, $Sz$, $He$, and $O'N$ denote the sporadic simple groups discovered by Suzuki, Held, and O'Nan, respectively. We remark that by a result of Aschbacher, $Q$ is elementary abelian if the 2-rank of $K$ is greater than 1. In that case we put $Q \cong E_2$.

1. The case $Q \cap S \cong \mathbb{Z}_2$.

(1.1) Some properties of subgroups of $N(A)$.

We have $Q \cap A \cong \mathbb{Z}_2$; clearly $|O_3(A)| \in \{1, 3\}$. Now, $A$ is quasisimple, and so, $A$ is an epimorphic image of the full covering group of $L_3(4)$. Thus, $A$ is an epimorphic image of the perfect central extension of $Z_2 \times Z_2 \times Z_2$ by $L_3(4)$.

Since such an extension possesses an automorphism of order 3 acting fixed—point—free on the 2-part of its center, we see that $A/O_3(A)$ is uniquely determined up to isomorphism. Using the results of [5] we get the following relations:

$$[\mu, \xi] = \pi \tau, \quad [\lambda, \xi] = \tau, \quad [\mu, \zeta] = q \pi, \quad [\lambda, \zeta] = q \pi \tau,$$

$$R_1 = \langle q, \pi, \tau, \mu, \lambda \rangle \cong R_2 = \langle q, \pi, \tau, \zeta, \xi \rangle \cong E_2, \quad \langle q \rangle = Q \cap S.$$

From the results of [5], we get that $A$ possesses the «field»-automorphism $\varphi$ and the «transpose-inverse»-automorphism $\chi$. Thus, $\text{aut}(A)/A$ is a four-group. As in [5], we get

$$\varphi: \ q \rightarrow q, \quad \pi \rightarrow \pi, \quad \tau \rightarrow \pi \tau;$$

$$\chi: \ q \rightarrow q, \quad \pi \rightarrow \pi, \quad \tau \rightarrow \tau;$$

$$\varphi \chi: \ q \rightarrow q, \quad \pi \rightarrow \pi, \quad \tau \rightarrow \pi \tau.$$

Every involution of $S$ lies in $R_1$ or $R_2$. Set $S_i = \Omega_i(QR_i) = \Omega_i(Q)R_i$.

Then, $S_i = R_i$ if $m(Q) = 1$; and $S_i = QR_i \cong E_2$ if $m(Q) > 1$. As $q$ has no roots in $S$—see (1.3)—we get that $\Omega_i(QS) = S_1S_2$ with

$$(\Omega_2(QS))' = \langle q, \pi, \tau \rangle.$$
It is clear that $S/\langle q \rangle$ is isomorphic to a $S_2$-subgroup of $L_3(4)$ and that $N_A(S)/Z(A)$ is isomorphic to a $S_2$-normalizer of $L_3(4)$. There is an element $g \in N_A(S) \setminus Z(A)S$—defined as in [5]—such that $g$ operates on $S$ mod $\langle q \rangle$ in the following way.

$$g: \pi \to \pi \tau \to \tau, \quad \mu \to \mu \lambda \to \lambda, \quad \zeta \to \zeta \xi \to \xi.$$ Further, acting with $g$ on suitable commutators, one obtains

$$g: \pi \to q \pi \tau \to q \tau.$$ In particular, $Z(S)^g = \langle q, \pi, \tau \rangle$ splits into three conjugate classes under $N_A(S)$ with representatives $q, \pi, \text{ and } q\tau$.

Obviously, 3 does not divide the order of $S$, since an automorphism of order 3 of the full cover of $L_3(4)$ which is not inner acts fixed-point-free on the 2-part of the Schur-multiplier. Thus, $N(A) = A K X$ and $q^2 = Z(A)^2$.

(1.2) Lemma. The subgroups $S_1$ and $S_2$ are the only elementary abelian subgroups of $X$ of their orders.

Proof. This is a direct consequence of the structures of $S, Q,$ and $SQ$.

(1.3) Lemma. The involution $q$ has no root in $S$ and $X \in Syl_2(G)$. If $i$ is an involution in $QS$, then $i$ is contained in $S_1$ or $S_2$. Further, $i$ is conjugate to an involution in $\Omega_4(Q) \langle \pi \rangle$ under $A$.

Proof. Let $x$ be a root of $q$ in $S$; set

$$\bar{x} = \langle q \rangle x, \quad \bar{R}_j = R_j / \langle q \rangle,$$

$j = 1$ and 2, and $\bar{S} = S / \langle q \rangle$. Then, $\bar{x}$ is an involution of $\bar{S}$. The structure of $\bar{S}$ gives $x \in \bar{R}_1 \cup \bar{R}_2$. Hence, $x \in R_1$ or $x \in R_2$. Since $R_j$ is elementary abelian for $j \in \{1, 2\}$, we get $x^2 = 1$. Thus, $q$ has no root in $S$. In particular, $q$ has no root in $\Omega_4(Q)S = S_1 S_2$.

Let $X_1$ be a subgroup of $G$ which contains $X$ as a subgroup of index 2. Then, $X_1$ normalizes $\langle q, \pi, \tau \rangle = S' = (\Omega_4(Q)S)'$. Under the action of $N_A(S)$ the set $\langle q, \pi, \tau \rangle^g$ splits into three classes with representatives $q, \pi, q\pi$. Clearly, $X_1$ cannot centralize $q$, and $X_1$ normalizes $\Omega_4(Q)S$. Now, $\pi$ has the root $\mu \lambda \xi$ and $q\pi$ has the root $\mu \xi$, and both
\(\mu \lambda \xi\) and \(\mu \xi\) lie in \(S \subseteq \Omega_3(Q)S\). But \(q\) has no root in \(S\), and so, \(q\) has no root in \(\Omega_3(Q)S\). It follows \(X \in \text{Syl}_2(G)\).

An involution \(i\) of \(QS\) has the form \(i = us, u \in Q\) and \(s \in S\). Therefore, \(1 = i^2 = u^2s^2\), so that \(u^2 = s^{-2} \in Q \cap S = \langle q \rangle\). Since \(q\) has no root in \(S\), we get \(u^2 = s^{-2} = 1\). Thus, \(u \in \Omega_3(Q)\) and \(s \in R_1 \cup R_2\). Thus, \(i\) lies in \(S_1\) or \(S_2\), where \(S_i = \Omega_3(Q)R_j\) for \(j \in \{1, 2\}\). As \(A/\mathbb{Z}(A)\) possesses exactly one class of involutions and \([Q, A] = \langle 1 \rangle\), one gets that \(i\) is conjugate to an element of \(\Omega_3(Q)\).

(1.4) LEMMA. Depending on \(X\), one has:

\[
\begin{align*}
C_q(q) &= \langle q, \pi, \mu \lambda, \xi \rangle \cong Z_2 \times D_8; & \mathcal{U}^1(C_q(q)) &= \langle \pi \rangle; \\
C_q(\tau) &= \langle q, \pi, \tau \rangle \cong E_2; \\
C_q(q \tau) &= \langle q, \mu \lambda \xi \tau \rangle \cong Z_2 \times Z_4; & \mathcal{U}^1(C_q(q \tau)) &= \langle \pi \rangle.
\end{align*}
\]

PROOF. The first two assertions follow immediately from the structure of the automorphism group of \(I_3(4)\). Now, \(C_q(q \tau) \subseteq \langle q, \pi, \mu \lambda \xi \tau, \mu \xi \tau \rangle\).

We compute \(\mu \lambda \xi \tau \Rightarrow \mu \lambda \xi \tau \Rightarrow \xi \mu \lambda \tau = \mu \lambda \xi \tau, \mu \lambda \mu \lambda \tau = \mu \lambda \xi \tau\), and 
\[
\mu \xi \tau \Rightarrow \xi \mu \lambda \tau = \mu \xi \tau, \mu \lambda \tau = \mu \xi \tau.
\]

Note that \((\mu \lambda \xi \tau)^2 = \pi\). The lemma is proved.

(1.5) LEMMA. Let \(y \in X \setminus Q\) with \(y \in \{\varphi, \xi, q\xi\}\). Let \(z\) be an involution from \(QSy\). Then, \(Sz\) contains at most two classes of involutions under \(G\) with representatives \(z\) and \(qz\). If \(y = \varphi\), then \(\Omega^1(C_q(z)) = \langle \pi \rangle\) and \(z \cong \pi z\). If \(y = q\xi, \xi q\xi\), then \(\varphi, q\xi, \xi q\xi\). If \(y = \xi\), then \(C_q(z) = \langle \pi \rangle\) and \(z \cong \pi z \sim \pi z \sim q\pi z \sim q\pi z\) under \(S\).

PROOF. As in [5], one shows that \(C_q(z) \cong C_q(y)\). Let \(y \in \{\varphi, q\xi\}\). Then, we have \(\Omega^1(C_q(y)) = \langle \pi \rangle\). Since \(\pi \in Z(S)\), it follows \(\Omega^1(C_q(z)) = \langle \pi \rangle\). We have \(\tau^* = \pi \tau\). Because of \(\tau \in Z(S)\), we get \(\tau^* = \pi \tau\). Thus, \(z^* = \pi z\) and \(z \sim \pi z\) in \(S\).

Let \(y = \xi\). We know that \(C_q(\xi) = \langle q, \pi, \tau \rangle = \mathbb{Z}(S)\). Hence, \(C_q(z) = \langle q, \pi, \tau \rangle\). Put \(z = us\xi, \) where \(u \in Q\) and \(s \in S\). As in [5], one shows that \(s^2 = s^2 = [s, \xi] \in (Q \cap S)(s^2) \subset S' = \langle q, \pi, \tau, \mu \lambda \xi, \lambda \xi \rangle\). Denote the latter group by \(E\). We have \(E' = \langle 1 \rangle\). Compute: \([\xi, \lambda \xi] = \lambda^{-1} \xi \lambda \xi \lambda \xi = \lambda \xi \lambda \xi = q \pi \tau, [\xi, \mu \lambda \xi] = \pi, \) and \([\xi, \mu \lambda \xi] = \pi\).

It follows that

\[
\xi \sim \pi \xi \sim q \pi \xi \sim q \pi \xi
\]
holds under $E$. Since $E' = \langle 1 \rangle$ and $[u, E] = \langle 1 \rangle$, we get
\[ z \sim \pi z \sim q \pi z \sim q \pi \tau \]
under $E$. The Lemma is proved.

(1.6) **Lemma.** Two involutions of $Z(\Omega_4(Q)S)$ are conjugate in $G$ if, and only if, they are conjugate in $N(\Omega_4(Q)S) \subseteq N(A)$.

**Proof.** Note that $\Omega_4(Q)S = \Omega_4(Q)S$ is the subgroup of $X$ which is generated by all subgroups of $X$ which are isomorphic to $S_1$. Let $x$ and $y$ be two involutions of $Z(\Omega_4(Q)S)$. Then, $\Omega_4(Q)S$ lies in $C(x) \cap C(y)$. Assume that there is $g \in G$ such that $x^g = y$. Denote by $X_x$ a $S_2$-subgroup of $C(x)$ containing $\Omega_4(Q)S$ and by $X_y$ a $S_2$-subgroup of $C(y)$ containing $\Omega_4(Q)S$. Then, $X_x^h = X_y$ for some $h \in C(y)$. Clearly, $gh \in N(\Omega_4(Q)S)$ and $x^{gh} = y^g = y$. Since $\Omega_4(Q)S)' = S' = \langle q, \pi, \tau \rangle$, and since $q$ is the only element of $S'$ which has no root in $\Omega_4(Q)S$, the assertion follows.

(1.7) **Lemma.** (i) Let $m(Q) = 1$, and let $\langle q, s \rangle$ be a four-group contained in $QS$. Then, $q \sim qs \sim s \sim q$ in $G$. (ii) Let $m(Q) > 1$. Then, $\langle q \rangle$ is strongly closed in $QS$ with respect to $G$. If $i$ is an involution of $S$ and $i^g \in QS$ for some $g \in G$, then $i^{g} \in S$. Further, $\pi \sim q \pi$. In particular, $QS \subseteq X$.

**Proof.** Assume first that $m(Q) = 1$. Then $\langle q, s \rangle$ and $\langle q, \pi \rangle$ are conjugate via an element of $A$. We have $\langle q, \pi \rangle \subseteq Z(\Omega_4(Q)S)$, and by assumption $\Omega_4(Q)S = S$. Application of (1.6) gives that $G$-conjugates in $\langle q, \pi \rangle$ are conjugate under the action of $N(S)$ which lies in $N(A) = \langle A, KX \rangle$. Clearly, $KX \subseteq N(S)$ and $\langle \langle q, \pi \rangle, KX \rangle = \langle 1 \rangle$. So, a conjugation of two elements should be performed by an element of $A \cap N(S)$.

But $q, \pi$, and $q \pi$ are representatives of $N_a(S)$-classes. Assume now that $m(Q) > 1$. If $q$ is conjugate to an element $q'$ of $QS$, then—by the structure of $A$—we may assume that $q'$ lies in $Q \langle \pi \rangle$. We have $Q \langle \pi \rangle \subseteq Z(QS)$; note that $\Omega_4(Q)S = QS$. Application of (1.6) yields that $q \sim q'$ holds in $N(A)$. But $N(A) = AKX$, and so, we must have $q = q'$.

Let $i$ be an involution of $S$ and let $i^g \in QS$ for some $g \in G$. We may assume $m(Q) > 1$. There are elements $a, b \in A$ such that $i^{ga}, i^b$ lie in $Q \langle \pi \rangle \subseteq Z(QS)$. Application of (1.6) yields that $i^{ga}$ and $i^b$ are conjugate in $N(A)$; let $c$ be the conjugating element of $N(A)$ with
\(i^a = i^b\). Obviously, \(i^b\) lies in \(A\), and so, \(i^a \in A\). It follows \(i^a \in A \cap QS = S\). Assume that \(\pi \sim q\pi\). By (1.6) this conjugation is performed by an element of \(N(A)\). But \(N(A) = AKX\), and so, the conjugation \(\pi \sim q\pi\) is done by an element of \(A\). Since \(\pi, q\pi\) lie in \(Z(S)\), the conjugation is done by an element of \(N(S) \cap A\). But this is not the case. The element \(q\) is not conjugate to any element different from \(q\) in \(QS\). Application of a well-known result of Glauberman yields \(QS \subseteq X\).

(1.8) LEMMA. Let \(y \in X \setminus QS\) with \(y \in \{q, q\pi\}\). Then, \(q\) is not conjugate to an element of \(QS\).

PROOF. Assume that \(q \sim z\) for \(z \in QS\). Let \(y = q\). From (1.5) we get \(\mathfrak{U}^1(C(z)) = \langle \pi \rangle\) and \(z \sim \pi z\) under \(S\). Let \(\bar{X} \in \text{Syl}_2(C(z))\) with \(\bar{X} \supseteq C(z)\). Let \(\bar{A}\) be the unique standard-subgroup of \(C(z)\); note that \(\bar{A} \sim A\) in \(G\). Set \(\bar{Q} = \bar{X} \cap C(\bar{A})\) and \(\bar{S} = \bar{X} \cap \bar{A}\). Then, \(\bar{Q} \sim Q, \bar{S} \sim S, \) and \(\langle \pi \rangle = \bar{Q} \cap \bar{S}\). Further, \(\mathfrak{U}^1(X/QS) = \langle 1 \rangle\). Since \(\langle \pi \rangle = \mathfrak{U}^1(C(z))\), we get \(q \in QS\). Since \(z \sim \pi z\) and \(\pi z \in QS\), we get a contradiction to (1.7). In the case \(y = q\pi\) one arrives at a contradiction in the same way. The lemma is proved.

(1.9) LEMMA. The case \(X = QS\langle \pi \rangle\) does not occur.

PROOF. Assume by way of contradiction that \(X = QS\langle \pi \rangle\). Since \(X \in \text{Syl}_2(G)\), we get from (1.7) and a result of Glauberman that \(q\) is conjugate to an involution \(z\) of \(QS\). We know that \(C(z) = \langle q, \pi, \tau \rangle\) and that \(z \sim \pi z \sim q\tau z \sim q\pi\tau z\) holds under \(S\).

Let \(\bar{X} \in \text{Syl}_2(C(z))\) with \(C(x) \subseteq \bar{X}\). Define \(\bar{Q}, \bar{S},\) and \(\bar{A}\) as in (1.8). Then, \(\bar{X}: QS = 2, \) and so, \(\langle \pi, q\tau \rangle \cap QS = \langle 1 \rangle\). Assume that \(\pi\) lies in \(QS\). Then, we get \(\pi \in \bar{S}\) from (1.7), and we know that \(z \sim \pi z\). However, this contradicts (1.7) as \(\langle z \rangle = \bar{Q} \cap \bar{S}\). If \(q\tau\) or \(q\pi\tau\) is in \(QS\), then we get the same contradiction, since \(\langle q\tau, q\pi\tau \rangle \subseteq S\) and by (1.7).

(1.10) LEMMA. Under the assumptions of the theorem the case \(Q \cap S \cong Z_2\) does not occur.

PROOF. Application of (1.7), (1.8), (1.9) and a result of Glauberman yields that \(X = QS\langle \pi, \tau \rangle\), and that \(q\) is conjugate to an involution \(z\).
of $Q\times S$. We know that $q$ is not conjugate to an involution of $Q\times S$ or $\cup QS q$. We have $\tau^q = \tau^\pi = \pi \tau, [\varphi, \chi] \in Q$.

Let $Q$, $T$, $X$, and $\bar{A}$ be the subgroups of $C(\zeta)$ defined as in (1.8). Then, $\langle \zeta \rangle = \bar{Q} \cap \bar{T}$ and $\bar{X}/\bar{Q} \bar{S}$ is a four-group. We know that $C_S(\zeta) = \langle q, \pi, \tau \rangle$ and that $z \sim \pi z \sim q \pi z \sim q \pi \tau z$ holds under the action of $S$. As $z$ is isolated in $\bar{Q} \bar{S}$, we see as above that $\pi, q \tau, q \pi \tau, q \not\in \bar{Q} \bar{S}$.

If $\tau \not\in \bar{Q} \bar{S} \langle q \rangle$, then $\bar{Q} \bar{S} \langle q, \tau \rangle = \bar{X}$ and $\pi \in \bar{Q} \bar{S} q \cup \bar{Q} \bar{S} \tau$, since $q \pi \tau \not\in \bar{Q} \bar{S}$. If $\tau \in \bar{Q} \bar{S} \langle q \rangle$, then, as $q \tau \not\in \bar{Q} \bar{S}$, we must have $\tau \in \bar{Q} \bar{S}$. If in addition $\pi \in \bar{Q} \bar{S} q$, then we would obtain $q \pi \tau \in \bar{Q} \bar{S}$ which is not the case. Hence we have to handle the following two possibilities:

(a) $\bar{X} = \bar{Q} \bar{S} \langle q, \tau \rangle$ and $\pi \in \bar{Q} \bar{S} q \cup \bar{Q} \bar{S} \tau$; and

(b) $\bar{X} = \bar{Q} \bar{S} \langle q, \pi \rangle$ and $\tau \in \bar{Q} \bar{S}$.

Suppose that $\bar{X} = \bar{Q} \bar{S} \langle q, \tau \rangle$ and $\pi \in \bar{Q} \bar{S} q$. Then, $q \pi \tau \in \bar{Q} \bar{S} q$. Since $q \pi \tau \in \bar{S}$ and $\bar{S} \sim \bar{T}$, we get $q \pi \tau \in \bar{T}$. The $G$-fusion of the involutions of $\bar{Q} \bar{S} q$ yields $q \pi \tau \sim q \sim \tau$ by (1.7). Consider $q \pi \tau, z \tau$ in $\bar{S} \tau$. We know that $q \sim z \sim q \pi \tau z$ holds in $G$. It follows $q \pi \tau \sim q \pi \tau z \sim \tau$.

Since $S y$ with $y \in \{\varphi, \chi, \varphi \chi\}$ contains at most two $G$-classes of involutions, we get $\tau \sim q \pi \tau$ under $\bar{S}$. Using the structure of $N_A(\bar{S})$, we get $q \pi \sim \pi \tau \sim \tau$ and $\pi \sim q \pi \tau \sim \pi \tau$. It follows $\pi \sim q \pi \tau$ in $G$, against (1.7).

Suppose now that $\bar{X} = \bar{Q} \bar{S} \langle q, \tau \rangle$ and $\pi \in \bar{Q} \bar{S} \tau$. Then, $\pi \tau \in \bar{Q} \bar{S}$, and so, $\pi \tau \in \bar{S}$. Consider the set $\langle z, \pi \tau \rangle q \pi \tau$ in $\bar{S} q \pi \tau$. We know that $q \pi \tau \sim z \sim q \sim q \pi \tau$ and $q \sim q \pi \tau$. Hence, $q \tau \sim q \pi \tau \sim q \pi \tau$. Since in $\langle z, \pi \tau \rangle q \pi \tau$ there are at most two $G$-classes of involutions, we derive $q \tau \sim q \pi \tau$. However, $\pi$ is conjugate to $q \tau$ via a 3-element in $N_A(\bar{S})$, and this gives a contradiction.

Finally, we handle the case (b). Here, we have $\bar{X} = \bar{Q} \bar{S} \langle q, \pi \rangle$ and $\tau \in \bar{Q} \bar{S}$. Thus, $\tau \in \bar{S}$. Consider the set $\langle z, \tau \rangle q \pi \tau$ which lies in $\bar{S} q \pi \tau$. We know that $q \pi \tau z \sim q \pi \tau \sim q \sim z \sim q \pi \tau$ and $q \sim q \pi \tau$. Hence, in $\langle z, \tau \rangle q \pi \tau$ we have three $G$-classes of involutions against the fact that in $\bar{S} q \pi \tau$ there are at most two $G$-classes of involutions. This final contradiction proves the lemma.

2. The case $Q \cap S \cong Z_4$.

(2.1) Some properties of subgroups of $N(\bar{A})$.

We are interested in the possible structures for $S$. Set $Q \cap S = \langle t \rangle$ with $t^2 = q$ and $S = \langle t, \pi, \tau, \mu, \lambda, \zeta, \xi \rangle$. 
We put

\[ [\mu, \xi] = q^\alpha \pi \tau, \quad [\lambda, \xi] = q^\beta \tau, \quad [\mu, \zeta] = t q^\delta \pi \tau, \quad [\lambda, \zeta] = t q^\delta \pi \tau, \]

where \( \alpha, \beta, \gamma, \delta \in \{0, 1\} \) and \( t^{-1} = t q = t^0 \). If \( s \in S \), then \( o(s) = o(sq) \) if \( s \neq q \). We replace \( q^\alpha \pi \tau \) by \( \pi \tau \) and \( q^\delta \pi \) by \( \tau \) without changing the defining relations of \( S \). Interchanging \( t \) and \( t^{-1} \) if necessary, we may put \( [\mu, \xi] = t \pi \). Thus, we get:

\[ [\mu, \xi] = \pi \tau, \quad [\lambda, \xi] = \tau, \quad [\mu, \zeta] = t \pi, \quad [\lambda, \zeta] = t q^\delta \pi \tau, \]

where \( \alpha \in \{0, 1\} \). Furthermore, we have the freedom to choose \( \mu, \lambda, \zeta, \) and \( \xi \) to be involutions, since for each \( x \in \{\mu, \lambda, \zeta, \xi\} \) either \( o(x) = 2 \) or \( o(tx) = 2 \), and the commutator relations given above remain unchanged with \( tx \) in place of \( x \).

There is an element \( g \) in \( N_d(S) \setminus Z(A) S \) which acts fixed-point-free on \( S \) modulo \( \langle t \rangle \) in the following way:

\[ g: \pi \to \pi \tau \to \tau, \quad \mu \to \mu \lambda \to \lambda, \quad \zeta \to \zeta \xi \to \xi. \]

In fact, \( N_d(S) = Z(A) S \langle g \rangle \). We have \( t \pi = [\mu, \xi]^{-1} [\lambda, \xi] = \tau \), and so, \( q \pi^2 = \tau^2 \in \langle q \rangle \); this means that either \( o(\pi) = 4 \) and \( o(\tau) = 2 \), or \( o(\pi) = 2 \) and \( o(\tau) = 4 \). We compute:

\[ \pi \tau = [\mu, \xi]^{-1} [\mu \lambda, \zeta] = [\mu, \lambda, \zeta]^3 t \pi^3 \cdot t q^\delta \pi \tau = qq^a t^d \pi^a \pi \tau; \]

thus \( \pi \pi \pi \pi = \tau^2 \pi \pi \pi \pi = \tau^2 \), and so, \( \pi \pi \pi \pi = \tau \). One obtains two cases:

\begin{itemize}
  \item[a)] \( o(\pi) = 4 \) and \( o(\tau) = 2 \); then \( t \pi \pi \pi \pi = \pi^{-1}, \langle \pi, \tau \rangle \cong D_4 \), and \( \langle \pi, t \rangle \cong Q_8 \).
  \item[b)] \( o(\pi) = 2 \) and \( o(\tau) = 4 \); then \( [\pi, \tau] = 1 \), and \( \langle t, \pi, \tau \rangle \cong \langle t, \pi, t \rangle \cong Z_4 \times Z_2 \times Z_2. \)
\end{itemize}

Put \( R_1 = \langle t, \pi, \tau, \mu, \lambda \rangle \) and \( R_2 = \langle t, \pi, \tau, \zeta, \xi \rangle \). Then, \( S = R_1 R_2 \). Clearly, \( D(S) = S' = \langle t, \pi, \tau \rangle \). If \( S' = D(S) = Z(S) \), then \( S \) would be special, and hence, \( S' \) would be elementary abelian, namely: Let \( x, y \in S \); then \( [x, y] \in Z(S) \) and \( [x, y]^2 = x^{-2} y^{-1} x^2 y = 1 \), since \( \mathfrak{U}(S) \subseteq \subseteq D(S) \); hence every commutator of \( S \) has order 2 or 1 which implies that \( S' \) is elementary abelian. This is, however, not the case. Thus,
\[ Z(S) \subset \langle t, \tau, \pi \rangle, \text{ and since } g \text{ acts fixed-point-free on } \langle \pi, \tau, t \rangle \langle t \rangle, \text{ we get } Z(S) = \langle t \rangle. \]

It follows that not both \( R_1 \) and \( R_2 \) are abelian. We know that\[ \mathcal{U}^1_v(S) \subset \langle t, \pi, \tau \rangle = R_1 \cap R_2. \] Assume that \( R_1 \) was abelian. Put \( R_1^\circ = \Omega_3(R_1). \) We have \( R_1, D(R_1) \leq E_5, \) and so, \( \Omega_3(R_1) \) is elementary abelian of order \( 2^5. \) From the Jordan-canonical-form of \( \zeta \) and \( \xi \) on \( R_1^\circ, \) we get \( |C_{R_1^\circ}(\langle \zeta, \xi \rangle)| > 4. \) Since \( S = C_5(R_1) \langle \zeta, \xi \rangle, \) we see that \( C_{R_1^\circ}(\langle \zeta, \xi \rangle) \) lies in \( Z(S). \) But \( Z(S) = \langle t \rangle, \) and we have derived a contradiction. Thus, \( R'_1 \neq \langle 1 \rangle. \) Similarly, we get \( R'_2 \neq \langle 1 \rangle. \) It follows \( R'_1 = R'_2 = \langle q \rangle. \)

A subgroup of \( A \) involving \( A_5 \) acts transitively on \( R_i \langle t \rangle. \) This implies \( Z(R_i) = Z(R_2) = \langle t \rangle. \)

Clearly, \( Z(X) \subset Qs, \) and so, we have \( Z(X) \subset Q; \) note that \( Q \) is cyclic by a result of Aschbacher. It follows that \( X \in \text{Syl}_2(G) \) as \( C(q) \subset N(A) .\)

Since an element of order 5 of \( A \) acts fixed-point-free on \( R_i \langle t \rangle, \) and since \( R_i \langle t \rangle \) is elementary abelian, we deduce that \( R_1 = \langle t \rangle \leq E_1 \) and \( R_2 = \langle t \rangle \leq E_2, \) where \( E_i \) is extraspecial of order \( 2^5 \) and of type \( D_5 \leq Q; \) here \( \leq \) denotes the central product with amalgamated center of at least one factor. Clearly, \( E_i \) possesses 10 off-central involution and 20 elements of order 4. Thus, \( R_i \) possesses 30 off-central involutions and 30 off-central elements of order 4. The 2-rank of \( E_i \) is 2 as the maximal abelian subgrups of \( E_i \) are of type \((2, 4). \) Thus, the 2-rank of \( R_1 \) and of \( R_2 \) is equal to 3.

We know that \( t \langle t^2 \rangle, t^2 = q, \) does not possess a root in \( S \langle q \rangle. \) Hence, \( t \) has no root in \( S. \) Let \( i \) be an involution in \( QS. \) Then, \( i = us, \) \( u \in Q \) and \( s \in S. \) We have \( 1 = i^2 = u^2 s^2, \) and so, \( u^{-2} = s^2 \in Q_2 \cap \langle t \rangle = \langle t \rangle. \) Since \( t \) has no root in \( S, \) we get \( u^{-2} = s^2 \in \langle q \rangle. \) Since \( Q \) is cyclic and \( t \in Q, \) we have \( u \in \langle t \rangle. \) It follows \( i = us \in S. \) From the structure of \( S \) follows \( i \in R_1 \cup R_2. \)

Assume by way of contradiction that \( S \) had an elementary abelian subgroup \( E \) of order 16. From the structure of \( L_6(4) \) we get that if \( x \) is an element of \( R_i \setminus \langle t, \tau, \pi \rangle, \) then \( C_5(x) \subseteq R_i \) for \( i \in \{1, 2\}. \) Since the 2-rank of \( R_i \) is 3, we get \( |R_i, E| \geq 2^7 \) for \( i \in \{1, 2\}. \) Assume that \( |R_i, E| = 2^7. \) Then, \( R_1, E \in \{ R_1 \langle \zeta \rangle, R_1 \langle \xi \rangle, R_1 \langle \zeta \xi \rangle \}. \) There is an involution \( e \in E \) such that \( R_i, E = R_i, \langle e \rangle. \) We know that \( e \in R_2. \) Since \( C_5(e) \subseteq R_2, \) we get \( E \subseteq R_2, \) and this is a contradiction. Similarly, one sees that \( |R_2, E| = 2^7 \) does not happen. Assume now that \( R_i, E = S \) for \( i = 1 \) or \( i = 2. \) Then, there is an involution \( e \in E \setminus R_i, \) and so, \( E \subseteq C_5(e) \subseteq R_j, \) \( j \neq i; \) again we arrived at a contradiction. We have shown that the 2-rank of \( QS \) is precisely 3.
(2.2) **Lemma.** The $S_2$-subgroup $Q$ of $I$ is cyclic, $Z(S) = \langle t \rangle = Z(R_i)$ for $i \in \{1, 2\}$. If $i$ is an involution of $QS \setminus \langle q \rangle$, then $i$ is conjugate to an involution of $\langle t \rangle \pi$ under $A$. The involutions of $\langle t \rangle \pi$ are conjugate under $S$.

**Proof.** We have to prove only the last assertion. If $o(\pi) = 4$, then $o(t \pi) = 2$. Clearly, $\pi \sim \pi q$ under $R_i$, since $E_i$ is extraspecial.

(2.3) **Lemma.** The case (a) of (2.1) does not occur. Thus, we have $o(\pi) = 2$, $o(\tau) = 4$, and $\langle t, \pi, \tau \rangle$ is of type $(4, 2, 2)$.

**Proof.** Put $V = \langle t, \pi, \tau \rangle$, and assume that we are in case (a). Since $V = Z_2(S)$ and $Z(S) = \langle t \rangle$, we get $|S : C_s(V)| = 2^2$; note that $\langle t \rangle \pi$ and $\langle t \rangle \tau$ are both normal in $S$ and that $\langle t \rangle \pi$ contains precisely two involutions; the last assertion is also true for $\langle t \rangle \tau$. Since $V = \langle t \rangle \wedge \langle \pi, \tau \rangle$ with $\langle \pi, \tau \rangle \cong Q_8$, we get $V \cap C_s(V) = \langle t \rangle$, and so, $VC_s(V) = S$. But $S/V$ is elementary abelian of order 16, and $C_s(V)/\langle t \rangle \cong S/V$. Hence, $(V/\langle t \rangle)(C_s(V)/\langle t \rangle) = S/\langle t \rangle$ would be elementary abelian against the structure of a $S_2$-subgroup of $L_3(4)$.

(2.4) **Lemma.** The involutions of $A \setminus Z(A)$ form a single conjugate class. Further, $C_s(V) \setminus \langle q, \pi, \tau \rangle$ does not contain involutions; here and in what follows, we put $V = \langle q, \pi, \tau \rangle$. Clearly, $|C_s(V)| = 2^4$.

**Proof.** The first assertion follows from the fact that $A/\pi$ has exponent 4. The first assertion follows from the fact that $S_2(V) \setminus V$ does not contain involutions and that every involution of $QS$ lies in $R_1 \cup R_2$; note that $\mathfrak{S}^4(S) \subseteq (QS)' = \langle t, \pi, \tau \rangle$. Clearly, $S/C_s(V)$ is elementary of order 4, and $S$ and

Further,
$C_{s}(V)$ are $g$-invariant. The element $g$ acts fixed-point-free on $S/\langle t \rangle$, and so, $A$ induces an automorphism group isomorphic to $A_{4}$ of $V$. Clearly, $X$ normalizes $V$, $C_{s}(V)$, $S$, and $Z_{0}(S) = \langle t, \pi, \tau \rangle$. If $\kappa \in X$, then $[g, \kappa] = 1$; and also $[\kappa, V] = 1$, since the centralizer of $\kappa$ involves a section of $A$ isomorphic to $A_{4}$, and we know that $C_{s}(\kappa) \subseteq \langle t, \pi, \tau \rangle$. If $y \in \{q, \varphi \kappa \}$, then we get from the last section that $y \in N_{N_{A}(V)}(C(V))$. Clearly, $C(V) \subseteq C(q) = N(A) = KAX$ with $K \subseteq C(V)$. Since $N_{N_{A}(V)}/C(V)$ is a subgroup of $L_{3}(2)$ which has no element of order 7, the assertion of the lemma follows.

We want to get more information on the multiplication table of $S$. Clearly, $\mu \lambda$ or $\mu \lambda t$ is an involution. Compute $(\mu \lambda \xi)^{2} = \pi \bmod \langle q \rangle$; thus $o(\mu \lambda \xi) = 4$. It follows that $\langle \mu \lambda, \xi \rangle$ or $\langle \mu \lambda t, \xi \rangle$ is dihedral of order 8 with center in $\langle q, \pi \rangle \setminus \langle q \rangle$. We have shown that $\langle \varphi, \pi, \tau, \lambda \rangle = F \cong Z_{3} \times D_{6}$ and $Z(F) = \langle t, \pi \rangle$. Hence, $C_{n}(\pi) = \langle \varphi, \tau, \lambda, \mu \lambda \rangle$ and $C_{n}(\pi) = \langle \varphi, \pi, \tau, \xi \rangle$, and $|C_{n}(\pi)| = 2^{i}$ for $i \in \{1, 2\}$. It follows $\pi^{\mu} = \pi^{\lambda} = q \pi$ and $\pi^{\lambda} = \pi^{\varphi} = q^{2} \pi$. Further, since the 2-rank $m(S)$ is equal to 3, we get $(\tau^{4})^{\mu} = q \tau = (\tau^{4})^{\lambda}$, $(\tau^{4})^{t} = \tau^{\varphi} = t$. Compute $q = [\pi, \mu]^{\varphi} [\tau, \lambda] = q$, hence $[\tau, \lambda] = q$; also $1 = [\pi, \mu \lambda]^{\varphi} [\tau, \mu] = 1$, hence $[\tau, \mu] = 1$. Further, we have $1 = [\pi, \xi]^{\varphi} [\tau, \xi] = 1$, and so, $\tau^{\xi} = q \tau$. It follows $C_{s}(\pi) = \langle t, \varphi, \tau, \lambda, \mu \lambda, \lambda \xi, \xi \rangle$ has order 27. Thus, $C_{s}(\langle t, \tau, \varphi \rangle) = \langle t, \pi, \tau, \mu \lambda, \mu \xi, \mu \zeta, \xi \rangle = C_{s}(V)$, where $V = \Omega_{4}(Z_{6}(S)) = \langle q, \pi, t \tau \rangle$.

Put $W = C_{s}(V)$. We summarize:

(2.6) LEMMA. We have the following relations for the generators $t, \pi, \tau, \mu, \lambda, \zeta, \xi$ of $S$:

$t^{4} = \pi^{2} = \tau^{4} = \mu^{2} = \lambda^{2} = \xi^{2} = 1, \quad t^{2} = \pi^{2} = q, \quad [\pi, \tau] = 1$,

$\pi^{\mu} = \pi^{\lambda} = \pi^{\zeta} = \pi^{\xi} = \pi, \quad \tau^{\xi} = t^{\lambda} = \lambda = \tau^{\pi} = q \tau$,

$[\tau, \mu] = 1, \quad [\mu, \lambda] \in \langle q \rangle, \quad [\zeta, \xi] \in \langle q \rangle; \quad C_{s}(\pi) = \langle t, \tau, \mu \lambda, \mu \zeta, \xi \rangle$,

$C_{s}(\langle t, \pi, \tau \rangle) = \langle t, \pi, \tau, \mu \lambda \xi, \mu \zeta \xi \rangle; \quad [\mu, \xi] = \pi \tau, \quad [\mu, \zeta] = \tau \xi$,

$[\lambda, \xi] = t, \quad [\lambda, \zeta] = \pi^{\lambda} \tau \pi; \quad g: \pi \to q^{1+\lambda} \pi \tau \to q \tau \pi$.

From the action of the outer automorphism group of the full cover $A^{*}$ of $L_{3}(4)$ on $O_{3}(A^{*})$ one gets that our standard-subgroup $A$ possesses the « automorphism $\varphi \kappa$ ». Put $q^{i} = [\mu, \lambda]$ and compute $q^{i} = [\mu, \lambda]^{t} \cdot [\zeta, \xi] = q^{i}$. We want to determine under what conditions
the elements $\mu \lambda \xi$ and $\mu \zeta \xi$ commute. Compute: $\mu \lambda \xi \rightarrow \mu q \lambda \pi \tau \xi \rightarrow \mu \pi q \lambda q \pi \tau \pi \tau q \lambda \xi = q^a \mu \pi \lambda \xi \rightarrow q^a \mu \pi \tau \pi \lambda \xi = q^a \mu \lambda \xi$.

We get:

(2.7) Lemma. $[\mu \lambda \xi, \mu \zeta \xi] = 1$ if and only if, $\alpha = 0$. Here, $[\mu, \lambda] = [\zeta, \xi] = q^l$, $l \in \{1, 2\}$. Further,

$(\mu \lambda \xi)^2 = q^{4+1} \pi$, $(\mu \zeta \xi)^2 = q^a \pi \tau$, $(\mu \lambda \xi \mu \zeta \xi)^2 = q^{4+3} \pi \tau$.

Thus, $W$ is abelian of type $(4, 4, 4)$ if $\alpha = 0$, and $W' = \{q\}$ if $\alpha = 1$.

(2.8) Lemma. Let $y \in X \setminus QS$ with $y \in \{q, r, q r\}$. If $Q Sy$ contains an involution $y^*$, then there is an involution $z$ in $Q Sy$ conjugate to $y^*$ under $S$ which acts on $S$ in the same way as $y$ does.

Proof. The assertion follows from the proof of [5; Lemma 3.1].

(2.9) Here, we shall study the situation of a subgroup $\bar{W}$ of $X$ with $\bar{W} \cong W$.

If $W$ is abelian, then $\bar{W}$ is of type $(4, 4, 4)$; if $W' \neq \langle q \rangle$, then $\bar{W}' = \langle q \rangle$ and $Z(W) = \langle t, \pi, \tau \rangle$ and $\Omega_4(W) = \langle q, \pi, \tau \rangle$. Note that $\exp(W) = 4$. We denote by $\bar{W}$ a subgroup of $X$ isomorphic to $W$.

We assume first that $\bar{W}$ lies in $QS$. We know that $\Omega_4(\bar{W}) \subseteq S$, and since $QS/S$ is cyclic, we get $|\bar{W} \cap S| < 2^6$. Put $\bar{W} = \bar{W} \cap S$. We assume that $\bar{W} \not\subseteq S$. Then, there is an element $u s$ of order 4 of $\bar{W} \setminus S$, $u \in Q^s$ and $s \in S^t$. We compute: $u^4 = s^{-4} \in S \cap Q = \langle t \rangle$, and hence, $w^4 = s^{-4} = q$ as $t$ has no root in $S$, since otherwise $u^4 = 1$ and $u \in S$. Thus, $o(u) = o(s) = 8$. Since $|\bar{W}| = 2^8$ and $\exp(W) = 4$, we get $|\bar{W}\langle s\rangle| > 2^8$; clearly, $s$ centralizes $Z(\bar{W})$ and operates on $\bar{W}$ in the same way as $u s$ does. If $|\bar{W}\langle s\rangle| = 2^8$, then $\bar{W}\langle s\rangle = S$, and $Z(S)$ would contain $\Omega_4(\bar{W})$ which is not cyclic. If $|\bar{W}\langle s\rangle| = 2^7$, then, as $\Omega_4(\bar{W})$ lies in $Z(\bar{W}\langle s\rangle)$, we get a contradiction to $Z(S) = \langle t \rangle$ by the Jordan-canonical-form. Thus, we have $|\bar{W}\langle s\rangle| = 2^6$. If $\bar{W}\langle s\rangle W = S$, then $\bar{W}\langle s\rangle \cap W$ has order $2^4$, and from the structure of $W$, we see that the intersection contains a four-group, which lies in the center of $\bar{W}\langle s\rangle$ and of $\bar{W}$; note that the 2-rank of $QS$ is 3 and that $\bar{W} \cong W$. We get a contradiction to $Z(S) = \langle t \rangle$. If $|\bar{W}\langle s\rangle W| = 2^6$, then $|\bar{W}\langle s\rangle \cap W| = 2^5$, and $\bar{W}\langle s\rangle \cap W$ contains an elementary abelian subgroup of order 8. Thus, $\Omega_4(\bar{W})$ lies in $Z(\bar{W}\langle s\rangle W)$, and again we get a contradiction by the Jordan-canonical-form. The case $|\bar{W}\langle s\rangle W| = 2^6$ is not possible as $\exp(W) = 4$ and $o(s) = 8$. We have shown that $\bar{W}$ must lie in $S$. But then $|\bar{W} \cap W| > 2^4$, and hence, the center of $S$ would not be cyclic. It follows that if $\bar{W} \not\subseteq QS$ then $\bar{W} = W$. 
Finally, we have to consider the case that \( \overline{W} \) lies in \( X \) but not in \( QS \). Remember that \( |X:QS| < 4 \). Thus, \( QS \cap \overline{W} \) contains a subgroup of type \((2, 2, 4)\). Note that \( \Omega_4(\overline{W}) \subseteq Z(\overline{W}) \) and that \( \Omega_2(\overline{W}) = \langle \xi \rangle \). We know that \( \Omega_0(\overline{W}) \) must lie in \( S \) as \( X/QS \) is elementary and \( S \) contains the involutions of \( QS \). Put \( X^* = X/Q, S^* = SQ/Q \), and \( \overline{W}^* = \overline{W}Q/Q \). Then, \( |\overline{W}^*| > 2^4 \), since \( \exp(\overline{W}) = 4 \), and also \( |S^* \cap \overline{W}^*| > 2^4 \). As \( \Omega_0(\overline{W}) \subseteq S \), we see that there is a four-group in \( S^* \cap \overline{W}^* \) which is centralized by \( \overline{W}^* \). Let us assume first that \( X^* = S^* \overline{W}^* \) and \( |X^*:S^*| = 4 \). Then, we get a contradiction, because—computing in \( P(\varphi, \pi) \in \text{Syl}_2(\text{aut}(L_3(4))) \)—we see that \( C_p(s_1\varphi) \cap C(s_2\pi) \) is cyclic for \( s_1, s_2 \in P \); note that in \( \text{aut}(L_3(4)) \) we have \( C_p(\varphi) \subseteq \langle \pi, \tau, \mu, \xi \rangle \setminus \{\tau\} \), \( C_p(\pi) \subseteq \langle \lambda \rangle \), which is abelian of type \((4, 4)\), and

\[
C_p(s_1\varphi) \subseteq \langle \pi, \tau, \mu, \xi \rangle \subseteq \mathbb{Z}_2 \times D_8 \quad \text{and that} \quad \langle \pi, \tau, \mu, \xi \rangle \subseteq \mathbb{Z}_2 \times D_8.
\]

Note also that \( \langle \pi, \tau, \mu, \xi \rangle \cong \mathbb{Z}_2 \times D_8 \) and that \( \langle \pi, \tau, \mu, \xi \rangle \cong \mathbb{Z}_2 \times D_8 \).

Now, we consider the case that \( QS \overline{W}:QS \) is contained in the cyclic group \( QS/S \), and since \( C_p(\varphi) \subseteq \langle t, \pi, \tau, \lambda, \xi \rangle \) is contained in \( QS/S \). Since \( \exp(X/QS) = 2 \), we see that \( \mathbb{Z}_2(\overline{W}) \subseteq QS \), and so \( \mathbb{Z}_2(\overline{W}) = \Omega_0(\overline{W}) \subseteq S \). There is \( u \in Q, s \in S \) such that \( u \varphi \in \overline{W} \), and so, \( u \varphi \) centralizes \( \Omega_0(\overline{W}) = \Omega_0(\overline{W}) \), an elementary abelian group of order 8. It follows \( \Omega_0(\overline{W}) \subseteq W \), and so, \( \Omega_0(\overline{W}) = \Omega_0(\overline{W}) = \langle \eta, \pi, \tau \rangle \).

Next, we consider the case that \( |\overline{W} : QS| = 2 \). From the structure of \( \text{aut}(L_3(4)) \), we get that \( QS \overline{W} = QS/\langle \varphi \rangle \) is impossible, since \( C_p(\varphi) \) does not contain a four-group, but \( \mathbb{Z}_2(\overline{W}) \) contains a four-group in \( \overline{W}^* \cap S^* \). Thus, either \( QS \overline{W} = QS/\langle \varphi \rangle \) or \( QS \overline{W} = QS/\langle \pi \rangle \).

Assume that \( QS \overline{W} = QS/\langle \pi \rangle \). We know that \( C(s_1\varphi) \subseteq \langle t, \pi, \tau, \lambda, \xi \rangle \) is contained in \( QS/S \), and since \( \mathbb{Z}_2(\overline{W}) = \Omega_0(\overline{W}) \subseteq S \), we get \( |\overline{W} \cap S| > 2^4 \); note that \( S(\overline{W} \cap QS)/S \cong (\overline{W} \cap QS)/(\overline{W} \cap S) \). Now, \( \Omega_0(\overline{W}) \) is centralizes by an element \( \varphi \), where \( v \in Q, s \in S \); hence \( \Omega_0(\overline{W}) \) is centralizes by \( \varphi \). We know that \( C_p(\varphi) \subseteq \langle t, \pi, \tau, \lambda, \xi \rangle \) is contained in \( \langle t, \pi, \tau, \lambda, \xi \rangle \) which group we denote by \( B \). Also we know that \( \Omega_0(\overline{W}) \) lies in \( R_1 \) or \( R_2 \). Note that \( \langle \tau \rangle \neq \tau \) for any \( s \in S \). Hence, \( \Omega_0(\overline{W}) \neq \Omega_0(\overline{W}) \). We have \( |B \cap R_1| = 2^4 \), and so, \( \langle t, \pi, \tau \rangle \cap \Omega_0(\overline{W}) \rangle > 2^4 \); it follows \( \Omega_0(\overline{W}) \cap \Omega_0(\overline{W}) \rangle = 4 \). From the structure of \( L_3(4) \), it follows that \( C_p(\overline{W}) \) lies in \( R_1 \) or \( R_2 \). From the symplectic structure of \( R_1 \), it follows \( |C_p(\overline{W})| = 2^4 \); note that \( p \in \mathbb{Z}_2 \) as the 2-rank of \( S \) is 3. We have derived \( |S \cap \overline{W}| = 2^4 \). But \( |S \cap \overline{W}| = 2^4 \). Thus, there exists \( u \in Q, s \in S \) such that \( u \varphi \in \overline{W} \). This implies \( u \notin \langle t \rangle \). Since \( \Omega_0(\overline{W}) \subseteq S \), we have \( \varphi(u) = 4 \), and we know that \( s \in R_i, i = 1 \) or \( 2 \), as \( [s, \Omega_0(\overline{W})] = \langle 1 \rangle \). This implies \( s^4 = 1 \).
But then $u^4s^4 = 1$ implies $u^4 = 1$ and $u \in \langle t \rangle$ which is not possible. We have shown that in $QS\langle \varphi \rangle$ there is only one subgroup isomorphic to $W$, namely $W$ itself. We summarize:

(2.10) **Lemma.** Let $\tilde{W}$ be a subgroup of $X$ isomorphic to $W$ and assume $\tilde{W} \neq W$. Then $\tilde{W}$ is not contained in $QS$, $QS\langle \varphi \rangle$, or $QS\langle \varphi \kappa \rangle$. The case $QS\tilde{W} = QS\langle \varphi , \kappa \rangle$ is not possible. If $QS\tilde{W} = QS\langle \kappa \rangle$, then $\Omega_i(W) = \Omega_i(\tilde{W}) = \langle \varphi , \pi , t\tau \rangle$.

(2.11) **Lemma.** If $\pi \sim \varphi$ holds in $G$, then $\pi \sim \varphi$ holds in $N(\Omega_i(W))$.

**Proof.** Denote by $J$ the intersection of all subgroups $\tilde{W}$ of $X$ which are isomorphic to $W$. Then, $\Omega_i(W) = \Omega_i(Z(J))$.

Assume that $q \sim \pi$ holds in $G$. Denote by $X_{\pi}$ a $S_2$-subgroup of $C_0(\pi)$ which contains $X \cap C(\pi)$. We have $W \subseteq X \cap X_{\pi}$. Thus, $\Omega_i(W)$ is normalized by $X$ and $X_{\pi}$, and so, as $Z(X)$ is cyclic, we get $q \sim \pi$ in $\langle X, X_{\pi} \rangle \subseteq N(\Omega_i(W))$.

(2.12) **Lemma.** The case $QS = X$ does not occur.

**Proof.** Note that in $QS$ there are only two $N(A)$-classes of involutions with representatives $q$ and $\pi$. By a result of Glauberman we have $q \sim \pi$ in $G$. From (2.11) we get that $q$ and $\pi$ are conjugate under the action of $N(\Omega_i(W))$. Since $\pi$ has 6 conjugates under $N(S)$, we see that an element of order 7 of $N(V)/C(V)$ acts fixed-point-free on $V$. Thus, $G$ induces $L_3(2)$ on $V$, against $|QS:C_{QS}(V)| = 4$. The lemma is proved.

(2.13) **Lemma.** If $\varphi , \pi , \varphi \kappa$ are present in $X$, then $C_0(\varphi) \subseteq \langle t , \pi , \mu \lambda , \xi \rangle$, $C_0(\pi) \subseteq \langle t , \pi , t\tau \rangle$, $C_0(\varphi \kappa) \subseteq \langle t , \mu \lambda \xi \tau \rangle$; further $(\mu \lambda \xi \tau)^3 = q^4 \pi$.

**Proof.** The assertion is a consequence of (1.4).

(2.14) **Lemma.** Let $q$ be conjugate to an involution $z$ in $X \setminus QS$. If $[t, z] = 1$, then $q \sim \pi$ holds in $G$.

**Proof.** Let $q \sim z \in X \setminus QS$ and $[t, z] = 1$. Denote by $\bar{X}$ a $S_2$-subgroup of $C_0(z)$ with $C_0(z) \subseteq \bar{X}$ and by $\bar{A}$ the unique standard-subgroup of $C(z)$. Put $\bar{X} \cap \bar{A} = \bar{S}$ and $\bar{X} \cap C(\bar{A}) = \bar{Q}$. We have $z \in \bar{Q} \setminus \bar{S}$. Since $\bar{X}/\bar{Q}\bar{S}$ is elementary abelian and $t \in \bar{X}$, we get $t^2 = q \in \bar{Q}\bar{S}$, and so $q \in \bar{S}$ as $o(q) = 2$. Clearly, $q \neq z$. It follows that $q$ is conjugate to $\pi$ in $G$, since all involutions of $\bar{A}\langle z \rangle$ are conjugate to $\pi$; note that $q \in \bar{S} \subseteq \bar{A} \sim A$. 


(2.15) **Lemma.** We have \([\pi, \varphi] = [\pi, \kappa] = [\pi, \kappa] = 1\). Also \(\alpha = 0\) if, and only if \([t, \varphi] = 1\); further \(\alpha = 1\) if, and only if \([t, \kappa] = 1\); \(t^\varphi = t^{-1}\) always.

**Proof.** Since the centralizers of \(\varphi\) and \(\kappa\) involve \(L_3(2)\) and \(A_5\), respectively, we see easily that \(\pi \in C_2(\varphi\kappa)\) and \(\langle t^2, \pi, t \rangle \subseteq C_2(\varphi\kappa)\). Compute \(t\pi = [\mu, \xi] \overset{\psi}{\Rightarrow} [\lambda, \xi^s] = [\lambda, \xi][\lambda, \xi]^s = \tau(tq^s\pi t)^s = tq^s\pi\); thus \(\alpha = 0\) if, and only if \([t, \varphi] = 1\). Compute further \(t\pi = [\mu, \xi] \overset{\tau}{\Rightarrow} [\xi, \lambda] = [\xi, \lambda] = t\pi q^{1+\alpha}\); thus \(\alpha = 1\) if, and only if \([t, \kappa] = 1\). Finally, we have \(t\pi = [\mu, \xi] \overset{q^\pi}{\Rightarrow} [\xi, \mu] = [\mu, \xi]^{-1} = (t\pi)^{-1} = t^{-1}\pi\), and so, \(t^\varphi = t^{-1}\), since obviously \([\pi, \varphi \kappa] = 1\) as \(|C_2(q\kappa)| = 2^3\).

(2.16) **Lemma.** Let \(z\) be an involution of \(Q\kappa S\) which operates on \(S\) in the same way as \(\kappa\) does. If \(t^z = t^{-1}\), then all elements of \(\langle q, \pi, t \rangle\) are conjugate.

**Proof.** We prove the assertion by a series of computations:

\[
(\mu \lambda \xi)^z = \xi^t \mu \lambda \xi = \xi \mu \lambda q^t = \xi \mu q^{1+\gamma};
\]

thus

\[
(z \mu \lambda \xi z) \xi \lambda \mu = q^{1+\beta} \pi,
\]

and so,

\[
z \sim q^{1+\beta} \pi z \sim q^\beta z.
\]

Hence,

\[
z \sim q z \sim q \pi z \sim \pi z.
\]

Also,

\[
(\mu \xi \xi z)^z = (\xi \xi) \mu \xi \xi = \xi \mu q^{1+\gamma};
\]

thus \((z \mu \xi \xi z) \xi \lambda \mu = q^t \pi \gamma\). Hence,

\[
z \sim q^t \pi z \sim q^{1+\gamma} \pi z.
\]

Finally,

\[
(\mu \lambda \xi \mu \xi \xi)^z = \xi \mu q^{1+\beta} \xi \mu q^{1+\gamma},
\]

and it follows

\[
(z \mu \lambda \xi \mu \xi \xi z) \xi \lambda \mu = q^{1+\beta+\gamma} \pi \xi \pi;
\]

thus

\[
z \sim q^{1+\beta+\gamma} \pi \xi \pi \xi \sim q^{\beta+\gamma} \pi \xi \pi \xi.
\]
Here, $\beta$ and $\gamma$ are suitable exponents; the proof can also be done by looking at the structure of $S\langle g, z \rangle$.

(2.17) **Lemma.** The case $X = QS\langle \kappa \rangle$ is not possible.

**Proof.** By way of contradiction we assume $X = QS\langle \kappa \rangle$. As always put $V = \langle q, \pi, t \tau \rangle$. We know that $\langle q, \kappa \rangle$ centralizes $V$. Thus, $X/C_X(V)$ is a four-group and this implies that $G$ does not induce $L_q(2)$ on $V$. Hence, $\pi \sim q$ in $G$.

We know that all involutions of $QS\setminus \langle q \rangle$ are conjugate to $\pi$. Hence, by a result of Glauberman, there is $z \in QS\kappa$ such that $z \sim q$ in $G$ and such that $z$ operates in the same way as $\kappa$ does on $S$. Application of (2.14) yields that $[z, t] \neq 1$ as $\kappa \sim q$. Application of (2.15) gives $\alpha = 0$ as $[t, \kappa] \neq 1$.

Let $\bar{X}, \bar{Q}, \bar{S},$ and $\bar{A}$ as in the proof of (2.14). We have $z \in \bar{Q} \cap \bar{S}$. Obviously, all involutions of $QS\setminus \langle z \rangle$ are conjugate to $\pi$ in $G$. We have $C_{\bar{S}}(z) = C_{\bar{S}}(\kappa) \supset \langle q, \pi, t \tau \rangle$. Thus, $\langle q, \pi \rangle \subseteq \bar{X}$, and hence $\langle q, \pi \rangle \cap C_{\bar{S}}(z) \neq \{1\}$. Clearly, $q \notin \bar{Q} \bar{S}$, since $q \neq z$ and $q \sim \pi$ in $G$. It follows that $\pi$ or $q\pi$ lies in $\bar{Q} \bar{S}$. Application of (2.16) yields that $z \sim z\pi \sim zq\pi$. But $z\pi$ or $zq\pi$ is in $\bar{Q} \bar{S} \setminus \langle z \rangle$. This would give $z \sim q \sim \pi$ which is not possible. The lemma is proved.

(2.18) **Lemma.** The case $X = QS\langle \varphi \rangle$ is not possible.

**Proof.** We have $C_X(\pi) = QC_{\varphi}(\pi)\langle \varphi \rangle$ and $|X:C_X(\pi)| = 2$; clearly, $C_{\varphi}(\pi) = \langle t, \pi, \tau, \mu \lambda, \mu \zeta, \xi \rangle$, $S' = Z_4(S) = \langle t, \pi, t \tau \rangle$, $W = C_{\varphi}(S') = \langle t, \mu \lambda \xi, \mu \zeta \xi \rangle \subseteq C_X(\pi)$. We know that $W$ is the only subgroup of $X$ isomorphic to $W$.

**Case 1.** The subgroup $W$ is nonabelian. In that case, we have $W' = \langle q \rangle$ and $\alpha = 1$. Lemma (2.15) implies $[t, \varphi] \neq 1$.

Assume that $q \sim \pi$. Consider $C_{\varphi}(\pi)$, and let $\bar{X}$ be in $\text{Syl}_2(C_{\varphi}(\pi))$ such that $C_X(\pi) \subseteq \bar{X}$. Since $W \subseteq \bar{X}$ and since $\bar{X} \sim X$, we see that $W$ is the unique subgroup of $\bar{X}$ isomorphic to $W$. It follows that $q$ and $\pi$ are conjugate inside $N(W)$. But—as $W' = \langle q \rangle$—this is not possible. Hence, $\pi \sim q$ in $G$.

By a result of Glauberman there is an involution $z$ in $X \setminus QS$ such that $z \sim q$ in $G$. We choose $z$ so that $z$ operates on $S$ in the same way as $\varphi$ does. Denote by $\bar{X}, \bar{Q}, \bar{S},$ and $\bar{A}$ subgroups of $C_\varphi(z)$ as in the proof of (2.14). Clearly, all involutions of $QS\setminus \langle z \rangle$ are conjugate to $\pi$ in $G$. We have $\langle q, \pi \rangle \subseteq C_X(z) \subseteq \bar{X}$. But $q \notin \bar{Q} \bar{S}$. Since $|X:QS| = 2$, we get that $\pi$ or $q\pi$ lies in $\bar{Q} \bar{S}$. Thus, $\pi z$ or $q\pi z$ lies in $\bar{Q} \bar{S} \setminus \langle z \rangle$, and
this implies that \( nx \) or \( qnx \) is conjugate to \( x \) in \( G \). Compute \( \tau^t = [\lambda, \xi]^t = [\mu, \xi] = q\tau \). It follows \( \tau^t = nx \); but \( \tau^t = zq \), and so,

\[ z \sim nx \sim nqx. \]

This is not possible as \( z \sim x \) holds in \( G \).

**Case 2.** The subgroup \( W \) is abelian. In that case we have \( \alpha = 0 \). Lemma (2.15) gives \([t, q] = 1\).

We show that \( C_X(\pi) \) is normal in \( N_0(X) \). Let \( x \in N(X) \). Then, \( x \in N(A) \), and hence, \( x \) normalizes \( X \cap C(A) = Q \). But \( Z(X/Q) = \langle \pi Q \rangle \), and so, \( \pi^x = \pi \) or \( \pi^x = qa \). Clearly, \( C_X(\pi) = C_X(q\pi) \), and this implies \( x \in N_0(C_X(\pi)) \). We show further that \( \langle q \rangle \) char \( C_X(\pi) \). Put \( C = C_X(\pi) \); note that \( X = QS\langle q \rangle \) and \( C_\pi(q) = \langle t, \pi, \mu, \lambda, \xi \rangle \) and that \([t, q] = 1\) as \( \alpha = 0 \). Obviously, \( \langle t, \pi \rangle \subseteq Z(G) \), and \( Z(G) \subseteq QS\langle \pi \rangle \). Hence, \( \langle q \rangle \) char \( C \).

We assume that \( q \sim \pi \) holds in \( G \). Let \( \overline{A} \) be the unique standard-subgroup of type \( L_3(4) \) in \( C_\pi(\pi) \) and let \( \overline{X} \) be in \( Syl_2(C(\pi)) \) such that \( C_X(\pi) \subseteq \overline{X} \). There is \( g' \) in \( G \) such that \( q^g = \pi \) and \( X^g = \overline{X} \). We have \( C_X(\pi)^g^{-1} \) as a subgroup of index 2 in \( X \). Since \( X \in Syl_2(G) \), we may apply a theorem of Burnside, and get \( C_X(\pi)^g = C_\pi(\pi)^g \) for some \( g \in N(X) \). This implies \( g' \in N(C_X(\pi)) \). It follows \([g', q] = 1\) against \( q^t = \pi \). We have shown that \( \pi \sim q \) holds in \( G \). A result of Glauberman yields the existence of an element \( z \in X \setminus QS \) with \( q \sim z \) in \( G \). Application of (2.14) yields \( \pi \sim q \) in \( G \) which is a contradiction. The lemma is proved.

(2.19) **Lemma.** Let \( z \) be an involution in \( QS\varphi x \) which acts in the same way as \( \varphi x \) on \( S \). Then, \( C_\pi(z) = \langle q, \mu, \lambda, \xi, \tau \rangle \) or \( \langle q, \mu, \lambda, \xi, \tau \rangle \) and all involutions of \( Sx \) are conjugate to \( z \) under \( S \). Further, \( \mathfrak{S}^1(C_\pi(z)) = \langle q^* \pi \rangle \) for some \( e \in \{0, 1\} \).

**Proof.** From (2.13) we get \( C_\pi(z) \subseteq \langle t, \mu, \lambda, \xi, \tau \rangle \). Note that \( t^t = t^{-1} \) by (2.15). The coset \( \langle t \rangle x \) consists of four involutions. Computing in \( aut(L_3(4)) \) we have \( C_\pi(\varphi x) \cong C_8 \), and so, \( \varphi x \) has precisely \( 2^2 \cdot 2^3 = 2^5 \) conjugates under the action of \( P \) in \( P\varphi x \). Thus, the number of conjugates of \( z \) under the action of \( S \) is at most \( 8 \cdot 4 = 32 \). This forces \( |S : C_\pi(z)| \leq 2^5 \) which implies \( |C_\pi(z)| = 2^5 \). The lemma is proved.

(2.20) **Lemma.** If \( X = QS\langle \varphi x \rangle \), then \( W \) is abelian and \( G \) induces an automorphism group isomorphic to \( L_3(2) \) on \( W \).

**Proof.** Assume that \( q \sim \pi \) in \( G \). Then, \( q \sim z \in X \setminus QS \); we assume that \( z \) acts on \( S \) in the same way as \( \varphi x \) does. Let \( \overline{X}, \overline{A}, \overline{S}, \) and \( \overline{Q} \) be
subgroups of \( C_6(z) \) as in the proof of (2.14). Then, \( \pi \) or \( q\pi \) lies in \( 3 \), and so, \( \pi z \) or \( q\pi z \) is conjugate to \( \pi \) in \( G \). Application of (2.19) gives that \( z \sim \pi z \sim q\pi z \). But \( q \sim z \), and we have got a contradiction. Hence, \( q \sim \pi \) holds in \( G \). Now, \( W \) is the only subgroup of \( X \) isomorphic to \( W \), and \( \Omega_1(Z(W)) = \langle q, \pi, t\pi \rangle \). Hence, \( q \sim \pi \) holds in \( N_6(W) \). It follows \( N(W)/C(W) \cong L_3(2) \) and the lemma is proved, since \( W' = \langle q \rangle \) cannot happen.

(2.21) **Lemma.** Let \( X = QS\langle \varphi, \pi \rangle \) and \( \alpha = 0 \). If \( \pi \sim q \) in \( G \), then \( q \) is not conjugate to an involution of \( QS\varphi \).

**Proof.** Assume that \( q \sim z \in QS\varphi \); we choose \( z \) so that it operates on \( S \) as \( \pi \) does. Application of (2.15) yields that \( [t, z] \neq 1 \). We have \( C_6(z) = \langle q, \pi, t\pi \rangle \). Lemma (2.16) says that all involutions of \( Vz \) are conjugate to \( z \). Denote by \( \tilde{X}, \tilde{S}, \tilde{Q}, \) and \( \tilde{A} \) subgroups of \( C_6(z) \) as in the proof of (2.14). Since \( \langle X : QS \rangle = 4 \), we have \( \langle q, \pi, t\pi \rangle \cap \tilde{Q}\tilde{S} = S \cap \tilde{S} = \langle 1 \rangle \). Let \( x \) be a nontrivial element of that intersection. Then, \( x \sim xz \sim q \). Since \( \tilde{Q} \cap \tilde{S} \) contains \( z \), we have \( xz \sim \pi \). But this is against the assumption of the lemma.

(2.22) **Lemma.** Let \( X = QS\langle \varphi, \pi \rangle \) and \( \alpha = 0 \). If \( q \) is conjugate to an involution \( z \) of \( QS\varphi \), then \( q \sim \pi \) holds in \( G \).

**Proof.** From (2.15) we get \( [t, \varphi] = 1 \). Application of (2.14) yields the assertion.

(2.23) **Lemma.** Let \( X = QS\langle \varphi, \pi \rangle \) and \( \alpha = 0 \). Then, \( \pi \sim q \) in \( N_6(V) \).

**Proof.** By way of contradiction assume that \( q \sim \pi \) in \( G \). By a result of Glauberman, \( q \sim z \) for \( z \) in \( QS\varphi, QS\pi, \) or \( QS\varphi\pi \). Application of (2.21) and (2.22) yields that \( z \in QS\varphi\pi \). We get from (2.19) that \( z \sim z\pi \sim z\pi q \). Let \( \bar{A}, \bar{X}, \bar{Q}, \) and \( \bar{S} \) be subgroups of \( C(z) \) as in the proof of (2.14). We get \( \bar{S}'(C_6(z)) = \langle q^\pi \rangle \subseteq \bar{S} \). Thus, \( zq^\pi \pi \sim \pi \) in \( G \). This implies \( z \sim \pi \sim q \) which is against the assumption. The assertion is now a consequence of (2.11).

(2.24) **Lemma.** Let \( X = QS\langle \varphi, \pi \rangle \) and \( \alpha = 0 \). Then, \( Q = \langle t \rangle \) and \( |X| = 2^{10} \).

**Proof.** We know that \( q \sim \pi \) holds in \( N(V) \). Thus, \( N(V)/C(V) \cong L_3(2) \). Since \( C(V) \subseteq C(q) \), we get that \( QW\langle \pi \rangle \) is a \( S_\pi \)-subgroup of \( C(V) \). Clearly, \( QW\langle \pi \rangle \) is nonabelian, and since \( QW \) is abelian, we
get \( Z(QW\langle x \rangle) \subset QW \). Now, \( V \) lies in \( Z(QW\langle x \rangle) \), and since the 2-rank of \( QS \) is 3, we get \( V = \Omega_3(Z(QW\langle x \rangle)) \). Denote by \( uw \) an element of \( C_{qw}(x) \) with \( u \in Q \), \( w \in W \), and \( o(uw) = 4 \). Then, \( u^t w^t = 1 \) which implies \( u^t = 1 \), and this means \( u \in \langle t \rangle \). Thus, \( C_{qw}(x) = V = \langle g, \pi, t \rangle = Z(WQ\langle x \rangle) \). We have \( |QW| = 2^a4^b \). Assume there were a subgroup \( Q^*W^* \) in \( QW(x) \) isomorphic to \( QW \) and different from \( QW \). Then, \( (QW)(Q^*W^*) = QW\langle x \rangle \) and \( QW \cap Q^*W^* \) has order \( 2^a4^b \) and would be contained in the center of \( QW\langle x \rangle \); it would follow \( n = 0 \) which is not the case. Thus, \( QW \) is unique in \( QW\langle x \rangle \). By the Frattini-argument, \( N(V) \) induces an automorphism \( \sigma \) of order 7 of \( QW\langle x \rangle \) which acts fixed-point-free on \( V \), thus \( \sigma \) has no fixed-points on \( QW \) as \( \Omega_3(QW) = \langle g, \pi, t \rangle \). This implies that \( QW \) is homocyclic, and so, \( Q \subset W \). The lemma is proved.

(2.25) Lemma. If \( \alpha = 0 \), then the case \( X = QS\langle q, x \rangle \) is not possible.

Proof. We have \( C_x(V) = W\langle x \rangle \). Since \( \pi \sim q \) holds in \( N(V) \), we have \( N(V)/C(V) \cong L_3(2) \). Clearly, \( W\langle x \rangle \in \text{Syl}_2(C(V)) \). Since \( C_w(x) = V \), we see that \( W \) is the only subgroup of \( W\langle x \rangle \) of its type. We have \( N(W) \subset N(V) \). Now, \( C(V) = (O \times W)\langle x \rangle \). Denote by \( \tilde{W} \) a subgroup of \( C(V) \) isomorphic to \( W \) and assume \( \tilde{W} \neq W \). Then, \( (O \times W)\tilde{W} = C(V) \). Since \( W \unlhd C(V) \), we get that \( W\tilde{W} \) is a group of order \( 2^a \), and so, \( |W \cap \tilde{W}| = 2^a \). But then, a \( S_2 \)-subgroup of \( C(V) \) would have a center of order greater than 8 which is not the case. Hence, in \( C(V) \) the subgroup \( W \) is unique. It follows that \( N(V) \subset N(W) \), and hence, \( N(V) = N(W) \). By Frattini's argument there is an automorphism \( \sigma \) of order 7 of \( W\langle x \rangle \) induced by an element of \( N(V) \) which acts fixed-point-free on \( V \). Hence, \( C(\sigma) \cap W\langle x \rangle = \langle z \rangle \) has order 2. Since \( C(\sigma) \cap W = \langle 1 \rangle \), we get \( W\langle x \rangle = W\langle z \rangle \). It follows \( z \in Sx \). Clearly, all involutions of \( W \) are conjugate in \( N(V) \). From the structure of \( X \) we get that \( X/W \) is a direct product of \( \langle Wx \rangle \) and a dihedral group of order 8. Thus, \( N(W)/C(W) \) is isomorphic to \( L_3(2) \times Z_2 \).

Denote by \( N^* \) the subgroup of index 2 of \( N(W) \) which contains \( C(W) \) such that \( N^*/C(W) \cong L_3(2) \). Put \( X^* = X \cap N^* \). Then, \( X^* \cap \cap S \subseteq W \). Note that the involutions of \( N^*/C(W) \) are all conjugate in that factor group. Let \( s \) be an involution of \( (S \cap X^*) \setminus W \). If \( x \) is any involution of \( X^* \setminus W \), then \( sC(W) \sim xC(W) \) in \( N^*/C(W) \). We have \( sC(W) = s(W \times O) \subseteq S \times O \); so all involutions of \( sC(W) \) are conjugate as \( \pi \sim q \) in \( G \). It follows that all involutions of \( X^* \) are conjugate to \( q \) in \( G; \) as a matter of fact, \( S \) lies in \( X^* \) as \( S \) is normalized by \( g \). Note that \( X^* \) is a maximal subgroup of \( X \). A transfer lemma
of J. G. Thompson gives \( z \sim q \) in \( G \). The last statement produces a contradiction in the following way. In the normalizer of \( V \) in \( G \) there is an element \( \sigma' \) which centralizes \( z \) and conjugates all the elements of \( V' \). It follows that \( \langle z \rangle \times \Omega_{i}(W) \) lies in the unique standard-subgroup \( A_{z} \) of \( C(z) \). Thus, the 2-rank of \( S \) would be 4 which is a contradiction. The lemma is proved.

(2.26) **Lemma.** The case \( X = QS\langle p, \kappa \rangle \) and \( \alpha = 1 \) is not possible.

**Proof.** Assume by way of contradiction that \( q \sim \pi \) in \( G \). Then, \( q \sim \pi \) holds in \( N(V) \). We have \( C(V) = (QW\langle \kappa \rangle)O \), where \( O = O(N(A)) \). From Frattini’s argument we get \( N(V) = O(N(QW\langle \kappa \rangle) \cap N(V)) \). Since \( [O, V] = \langle 1 \rangle \), we see that \( q \sim \pi \) happens in \( N(QW\langle \kappa \rangle) \). However, \( t \in Z(QW\langle \kappa \rangle) \subseteq \langle \kappa, \pi, t, \kappa \rangle \), and therefore \( \langle \kappa \rangle \) char \( QW\langle \kappa \rangle \). It follows that \( q \sim \pi \) holds in \( G \).

From Glauberman’s result we get that \( q \) is conjugate to an involution \( z \) in \( X \setminus QS \). From (2.14) we get that \( z \notin QS\kappa \), since \( [t, x] = 1 \). Assume that \( z \in QS\kappa \). We assume also that \( z \) acts in the same way on \( S \) as \( \kappa \) does. Application of (2.19) yields that all involutions of \( Sz \) are conjugate to \( z \). We have \( S^{1}(C_{g}(z)) = \langle q^{e}\pi \rangle \) for some \( e \in \{0, 1\} \). Clearly, \( \pi \sim q^{e}\pi \). In \( C(z) \) we choose \( \tilde{A}, Q, S, \) and \( \tilde{X} \) as usual. Then, \( z \in \tilde{Q} \cap \tilde{S} \). We have \( \langle q^{e}\pi \rangle \subseteq \tilde{S} \), and so, \( z\pi q^{e} \sim \pi \) in \( G \). However, \( z\pi q^{e} \) lies in \( Sz \) and is an involution. Thus, \( z \sim z\pi q^{e} \sim \pi \), against \( q \sim \pi \) and \( q \sim \pi \).

We have still to treat the case that \( q \sim z \in QS\kappa \). Denote again by \( \tilde{A}, S, \tilde{X}, Q \) the usual subgroups of \( C(z) \). We have \( S^{1}(C_{g}(\kappa)) = \langle q^{e}\pi \rangle \) as \( t^{e} = t^{-1} \). Thus, \( \langle q^{e}\pi \rangle \in \tilde{S} \). It follows \( zq^{e}\pi \sim q^{e}\pi \sim \pi \sim z \sim q \) in \( G \). Now, \( z \) and \( q^{e}z \) are involutions of \( Sz \). We may assume that \( z \) acts in the same way on \( S \) as \( \kappa \) does. Compute: \( \tau^{e} = [\mu, \xi]^{e} = [\mu, \xi] = \pi \tau \). It follows \( \tau^{e} = \pi \tau \); but \( \tau^{e} = \pi \tau \), and so,

\[
z \sim \pi z \sim \pi qz .
\]

This is not possible as \( z \sim \pi \) in \( G \).

We are left with the situation of (2.20). We have \( W = \langle 1 \rangle \), and \( W \) is the only subgroup of its type in \( X = QS\langle \kappa \rangle \). Now, \( N(W)/C(W) = L_{6}(2) \) and \( C(W) = QWO \), where \( O = O(N(A)) \). Since \( N(W) \) operates transitively on \( \Omega_{i}(W) = V \), we see that \( |Q| < 4 \) as \( \langle q \rangle \) is not characteristic in \( WQ \). It follows \( X = S\langle \kappa \rangle \). Clearly, \( N(W)/O \) is a non-splitting extension of an abelian group of type \( (4, 4, 4) \) by \( L_{6}(2) \). By a result of Alperin, we see that \( X \) is isomorphic to a \( S_{z} \)-subgroup of
O'Nan's simple group. This is enough to get $G \cong O'N$; but we may invoke a result of G. Stroth [6] to identify $G$ with $O'N$.

The theorem is proved.

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