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On Normalization of Nash Varieties.

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0. Introduction.

Let U be an open semialgebraic subset of $\mathbb{R}^n$, we can consider Nash subvarieties of U as the set of zeroes of ideals in $\mathcal{N}(U)$. If $Z \subset U$ is a normal Nash variety, then one can define the ring $\mathcal{N}(Z)$ of Nash functions on Z and show that $\mathcal{N}(Z)$ has good algebraic and geometric properties (cf. [4], [5]). We do not know any analogue for a Z with more general singularities.

In this direction, in the present paper, we show that it is always possible to normalize Nash varieties: more precisely we show that given a Nash variety $Z \subset U$, then there exists $Z' \subset U'$ with U' semialgebraic and Z' normal Nash subvariety together with a proper map $Z' \to Z$ with the usual properties of a normalization, except that it is not surjective in general.

Moreover we compare algebraic and Nash normalizations, we consider the compact case and we add a remark on seminormalization.

1. Preliminaries.

Let us first fix some notations. For an $\mathbb{R}$-algebra A we will denote $\Omega(A)$ (resp. $\Omega_{\mathbb{R}}(A)$) the set of all maximal (resp. real maximal) ideals of A

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If $\mathfrak{p} \subset A$ is any ideal we denote $Z(\mathfrak{p}) = \{\mathfrak{m} \in \Omega_{\mathbf{R}}(A) | \mathfrak{m} \supset \mathfrak{p}\}$. When A is an algebra of real valued functions defined on some set X, provided that there is a canonical bijection between X and $\Omega_{\mathbf{R}}(A)$, we will identify these two sets. Let A be any ring, we denote $\bar{A}$ the integral closure of A in the total ring of fractions and; if A is local, we denote $\hat{A}$ the completion of A with respect to its maximal ideal.

Unless contrarily specified, we will consider real algebraic varieties (and subsets of them) endowed with the real topology.

Through all the paper V will denote a real affine irreducible normal algebraic variety. On V we consider the sheaves $\mathcal{O}_V$, $\mathcal{A}_V$ and $\mathcal{N}_V$ respectively of regular rational, real analytic and Nash functions.

We will denote $\Gamma_V = \Gamma(V, \mathcal{O}_V)$ and $\Gamma_{V,x} = (\Gamma_V)_{\mathfrak{m}_x}$ where $\mathfrak{m}_x$ is the maximal ideal corresponding to a point $x \in V$.

Let $U \subset V$ be an open subset, and let $S \subset U$ be a real analytic set; we say that S is C-irreducible if it is a C-analytic set which is not proper union of two proper C-analytic subsets (cf. [6], ch. V). So, U is C-irreducible if and only if $\Gamma(U, \mathcal{A}_V)$ is an integral domain.

If U is C-irreducible we will consider the ring of Nash functions $\mathcal{N}(U)$ defined as the algebraic closure of Γ_V in $\Gamma(U, \mathcal{A}_V)$ (cf. [4]), if U is a disjoint union of U_1 and U_2 we set $\mathcal{N}(U) = \mathcal{N}(U_1) \times \mathcal{N}(U_2)$.

The theory developped in [4] allows us to treat also the compact case: let $K \subset V$ ba a compact set, $\Gamma(K, \mathcal{A}_V)$ will denote the ring of germs of real analytic functions defined in a neighborhood of K. If $\Gamma(K, \mathcal{A}_V)$ is an integral domain (i.e. when K is C-irreducible in the sense considered above), then $\mathcal{N}(K)$ will denote the ring of germs of Nash functions (defined in some neighborhood in V).

We will define Nash subsets and Nash subvarieties of U in the case when $\mathcal{N}(U)$ is noetherian (this happens if U is semialgebraic (cf. [4])).

We say that a set $Z \subset U$ is a Nash subset (resp. a Nash subvariety) of U if it is the locus of zeroes of some ideal (resp. prime ideal) of $\mathcal{N}(U)$. A Nash set (resp. variety) will be a Nash subset (resp. subvariety) of some open set U such that $\mathcal{N}(U)$ is defined and it is noetherian.

In the compact case we will consider Nash sets and Nash varieties defined in the same way.

Let us recall the following normalization theorem for algebraic varieties which is simply an adaptation to the real case of the normalization theorem given in [3], th. 1.5. in the case of varieties defined over any field.

THEOREM 1. Let X be a real affine algebraic variety and let S be the set of non normal points of X. Then there exists a normal affine variety X' together with a morphism $p: X' \to X$ such that:

i) p is proper, $\mathrm{Im}(p) = \{x \in X | \Omega_{\mathbf{R}}(\overline{\mathcal{O}_{X,x}}) \neq \emptyset\}$ and $p|: X' - p^{-1}(S) \to$ $\to X - S$ is a homeomorphism;

ii) p is a birational morphism with finite fibers;

iii) any morphism $f: Y \to X$ with Y normal and $\mathrm{Im}(f)$ Zariski dense in X factors uniquely through X'.

In [3] the set $\{x \in X | \Omega_{\mathbf{R}}(\overline{\mathcal{O}_{X,x}}) \neq \emptyset\}$ is denoted X_0. By analogy, let $Z \subset U$ be a Nash subvariety and let $\not{p} = \{f \in \mathcal{N}(U) | f(x) = 0, \forall x \in Z\}$, we will consider the set

$$Z_0 = \{x \in Z | \Omega_{\mathbf{R}}(\overline{\mathcal{N}_x / \not{p}\mathcal{N}_x}) \neq \emptyset\}.$$

LEMMA 2. Let S be the set of non normal points of Z and let $X \subset V$ be the Zariski closure of Z, then:

a) $Z - S \subset Z_0 \subset X_0 \cap U$;

b) Z_0 is Zariski dense in Spec $(\mathcal{N}(U)/\not{p})$.

PROOF. *a*) Let $x \in Z_0$ and consider $\Gamma_{X,x} \simeq \Gamma_{V,x}/\not{p} \cap \Gamma_{V,x} \hookrightarrow \mathcal{N}_{x}/\not{p}\mathcal{N}_x$. We get an induced map $\Omega_{\mathbf{R}}(\overline{\mathcal{N}_x/\not{p}\mathcal{N}_x}) \to \Omega_{\mathbf{R}}(\overline{\Gamma_{X,x}})$, therefore $x \in X_0$.

b) Since Z_0 contains the set of regular points of Z, a Nash function f vanishing on Z_0 is null on Z; hence $f \in \not{p}$.

2. The normalization theorem.

Let $U \subset V$ be an open semialgebraic C-irreducible subset, then $\mathcal{N}(U)$ is a noetherian normal domain. Let $Z \subset U$ be a Nash variety and let S be the set of non normal points of Z.

THEOREM 3. There exists an open semialgebraic C-irreducible subset $U' \subset U \times \mathbf{R}^m$ and there exists a Nash variety $Z' \subset U'$ together with an algebraic map $p: Z' \to Z$ such that:

i) Z' is normal;

ii) p is proper, finite to one map with $\mathrm{Im}(p) = Z_0 \supset \overline{Z - S}$ and $p|: Z' - p^{-1}(S) \to Z - S$ homeomorphism;

iii) any semialgebraic map $f\colon Y \to Z$ with Y normal Nash variety and $f(Y) \supset Z_0$, factors uniquely through Z'.

PROOF. Let X be the Zariski closure of Z in V, then $X = Z(\mathfrak{p} \cap \Gamma_V) \subset V$; and let $q\colon X' \to X$ be the normalization map. By [3], Prop. 1.3. we have that $X' \simeq \Omega_{\mathbb{R}}(\overline{\Gamma_X})$; moreover $X' \subset V \times \mathbb{R}^m$ and q is induced by the projection $V \times \mathbb{R}^m \to V$.

Let us consider $\mathfrak{p}^{ce} = (\mathfrak{p} \cap \Gamma_V)\mathcal{N}(U)$ By the proof of theorem 4.4 of [4] we have $\mathfrak{p}^{ce} = \bigcup_{j=1}^{s} \mathfrak{p}_j$ where $\operatorname{ht} \mathfrak{p}_j = \operatorname{ht} \mathfrak{p}$ for every j and we may assume $\mathfrak{p}_1 = \mathfrak{p}$. We obtain the following decomposition of $X \cap U$ into Nash varieties:

$$X \cap U = Z \cup Z(\mathfrak{p}_2) \cup \ldots \cup Z(\mathfrak{p}_s)\,.$$

Consider now the decomposition of $X' \cap (U \times \mathbb{R}^m)$ into Nash varieties:

$$X' \cap (U \times \mathbb{R}^m) = X'_1 \cup \ldots \cup X'_{s'}\,.$$

Since X' is normal, each X'_j is so. Moreover the X'_j's are connected, since they are Nash varieties, and they are mutually disjoint, since X' is normal. Hence the X'_j's are the connected components of $X' \cap (U \times \mathbb{R}^m)$ and they are analytically C irreducible by [5], th. 7. We obtain that:

$$\Gamma(X' \cap (U \times \mathbb{R}^m), \mathcal{A}) \simeq \prod_{j=1}^{s'} \Gamma(X'_j, \mathcal{A})$$

where the $\Gamma(X'_j, \mathcal{A})$'s are integral domains and thus:

$$\mathcal{N}(X' \cap (U \times \mathbb{R}^m)) \simeq \prod_{j=1}^{s'} \mathcal{N}(X'_j)$$

where the $\mathcal{N}(X'_j)$'s are noetherian domains which are integral and integrally closed.

Now, the projection map $U \times \mathbb{R}^m \to U$ gives a ring homomorphism $\mathcal{N}(U)/\mathfrak{p}^{ce} \to \mathcal{N}(U \times \mathbb{R}^m)/\mathcal{I}$ where $\mathcal{I}$ is the ideal of Nash functions vanishing on $X' \cap (U \times \mathbb{R}^m)$. Using Lemma 2 *b*) it is easy to see that it is injective; moreover $\mathcal{N}(U \times \mathbb{R}^m)/\mathcal{I} \hookrightarrow \mathcal{N}(X' \cap (U \times \mathbb{R}^m))$, hence we obtain:

$$\bar{\Gamma}_X \hookrightarrow \overline{\mathcal{N}(U)/\mathfrak{p}^{ce}} \simeq \overline{\mathcal{N}(U)/\mathfrak{p}} \times \Big(\prod_{i=2}^{s} \overline{\mathcal{N}(U)/\mathfrak{p}_i} \Big) \hookrightarrow \prod_{j=1}^{s'} \mathcal{N}(X'_j) \,.$$

Therefore:

$$X'_1 \coprod \cdots \coprod X'_{s'} \hookrightarrow \Omega_{\mathbb{R}}\big(\overline{\mathcal{N}(U)/\mathfrak{p}}\big) \coprod \Big(\coprod_{i=2}^{s} \Omega_{\mathbb{R}}\big(\overline{\mathcal{N}(U)/\mathfrak{p}_i}\big) \Big) \to \Omega_{\mathbb{R}}(\bar{\Gamma}_X) \simeq X' \,.$$

By the above discussion, eventually permuting indices, we obtain that $s' \leqslant s$ and that $X'_1 \hookrightarrow \Omega_{\mathbb{R}}\big(\overline{\mathcal{N}(U)/\mathfrak{p}}\big)$ and $X'_j \hookrightarrow \Omega_{\mathbb{R}}\big(\overline{\mathcal{N}(U)/\mathfrak{p}_j}\big)$ for all j, $2 \leqslant j \leqslant s'$.

We set $Z' = X'_1$ (which is normal) and $p = q|_{Z'}$.

As p is the composite $Z' \hookrightarrow \Omega_{\mathbb{R}}\big(\overline{\mathcal{N}(U)/\mathfrak{p}}\big) \to \Omega_{\mathbb{R}}(\mathcal{N}(U)/\mathfrak{p}) \simeq Z$ we have that $\operatorname{Im}(p) = Z_0$ and $p|: Z' - p^{-1}(S) \simeq Z - S$; moreover p is proper and finite to one since q is so.

Therefore to see that p is the required normalization map, we only have to prove the universal property (iii).

For this, let W be an open semialgebraic subset of some normal affine variety with $Y \subset W$ and let $\mathfrak{q}$ be the ideal defining Y in $\mathcal{N}(W)$. Shrinking eventually W, we may assume that $f\colon Y \to Z$ is induced by a map $F\colon W \to U$.

We obtain a ring homomorphism $\varphi\colon \mathcal{N}(U)/\mathfrak{p} \to \mathcal{N}(W)/\mathfrak{q}$, which, since $f(Y) \supset Z_0$ turns out to be injective (by lemma 2 *b*)). As $\mathcal{N}(W)/\mathfrak{q}$ is integrally closed, φ factors (uniquely) through $\varphi'\colon \overline{\mathcal{N}(U)/\mathfrak{p}} \to \mathcal{N}(W)/\mathfrak{q}$; let $f'\colon Y \to \Omega_{\mathbb{R}}\big(\overline{\mathcal{N}(U)/\mathfrak{p}}\big)$ be the induced map. Let us consider $\overline{\mathcal{N}(U)/\mathfrak{p}}$ as a subring of $\mathcal{N}(Z')$ and let $S = \{s \in \overline{\mathcal{N}(U)/\mathfrak{p}} \,|\, s(x) \neq 0,\ \forall x \in Z'\}$, we want to show that for every $s \in S$, $\varphi'(s)$ is a unit in $\mathcal{N}(W)/\mathfrak{q}$. To see this we observe that $f'(Y) \subset Z'$ (since Y is connected and $f'(Y) \cap \cap Z' \neq \emptyset$), then, for every $y \in Y$ we have

$$\varphi'(s)(y) = (s \cdot f')(y) = s\big(f'(y)\big) \neq 0.$$

Hence φ' extends (uniquely) to a $\varphi''\colon S^{-1}\big(\overline{\mathcal{N}(U)/\mathfrak{p}}\big) \to \mathcal{N}(W)/\mathfrak{q}$. Since $\Omega_{\mathbb{R}}\big(S^{-1}\big(\overline{\mathcal{N}(U)/\mathfrak{p}}\big) \simeq Z'$, our contention follows.

3. Algebraic and Nash normalizations.

Let $X \subset \mathbb{R}^n$ be an irreducible algebraic variety. We will consider on X the structures of Nash set and of real analytic variety, which we will denote respectively by X^n and X^a. Analogously, for a Nash variety Z, we denote Z^a the corresponding real analytic variety.

PROPOSITION 4. Let X and Z as above, then:

a) X is normal $\Leftrightarrow X^n$ is so $\Leftrightarrow X^a$ is so.

b) Z is normal $\Leftrightarrow Z^a$ is so.

PROOF. *a*) Let $x \in X$ and consider the following inclusions of noetherian local rings: $\mathcal{O}_x \hookrightarrow \mathcal{N}_x \hookrightarrow \mathcal{A}_x \hookrightarrow \hat{\mathcal{O}}_x$.

It turns out that all the homomorphisms are flat. Hence if $\mathcal{A}_x$ is normal, $\mathcal{N}_x$ is so and if $\mathcal{N}_x$ is normal $\mathcal{O}_x$ is so. On the other hand, since $\mathcal{O}_x$ is excellent, if $\mathcal{O}_x$ is normal then $\hat{\mathcal{O}}_x$ is so and therefore also $\mathcal{A}_x$ is normal.

The proof of *b*) is similar, using a suitable Zariski closure of Z.

PROPOSITION 5. Let us suppose that X^n is a Nash variety and let X' (resp. $(X^n)'$) be the normalization of X (resp. X^n), then $(X^n)' \simeq (X')^n$.

PROOF. Consider $(X^n)' \subset X \times \mathbb{R}^m$ and let Y be the normalization of the Zariski closure of $(X^n)'$. By the universal properties we have the existence (and uniqueness) of the following dotted maps:

$$\begin{array}{ccc} Y & \longrightarrow & X \\ & \searrow \; \nearrow & \\ & X' & \end{array} \qquad \begin{array}{ccc} (X')^n & \longrightarrow & X^n \\ & \searrow \; \nearrow & \\ & (X^n)' & \end{array}$$

We obtain: $(X')^n \to (X^n)' \subset Y \to (X')^n$, where the composed map is, by construction, the identity map on a dense subset. We get that it is the identity map. Similarly, the composed map of $(X^n)' \to$ $\to (X')^n \to (X^n)'$ is the identity. Hence $(X^n)' = Y$ and $(X^n)' \simeq (X')^n$.

4. The compact case.

We give an analogue of theorem 3 in the compact case.

Let $K \subset V$ ba a compact semialgebraic C-irreducible subset and let $\mathcal{N}(K)$ denote the ring of germs of Nash functions defined in some neighborhood of K; i.e. there exists a fundamental system U_i of open C-irreducible semialgebraic neighborhoods of K and $\mathcal{N}(K) = \varinjlim \mathcal{N}(U_i)$.

Let further $Z \subset K$ ba a Nash subvariety (i.e. $Z = Z(\mathfrak{p})$ with $\mathfrak{p} \subset \mathcal{N}(K)$ a prime ideal) and let $S \subset Z$ be the set of non normal points of Z.

PROPOSITION 6. There exist a compact semialgebraic C-irreducible set $K' \subset K \times \mathbb{R}^m$, a Nash subvariety $Z' \subset K'$ and an algebraic map $p: Z' \to Z$ such that:

i) Z' is normal;

ii) p is a proper finite to one map with $\operatorname{Im}(p) = Z_0 \supset \overline{(Z - S)}$ and $p|: Z' - p^{-1}(S) \simeq Z - S$ homeomorphism;

iii) let Y be a normal Nash subvariety in some compact semialgebraic set and let $f: Y \to Z$ be a semialgebraic map such that $f(Y) \supset Z_0$, then f factors uniquely through Z'.

PROOF. The proof runs in the same way of that of theorem 3 Let X be the Zariski closure of Z in V and let $q: X' \to X$ be the normalization map.

Since q is proper we have that $q^{-1}(X \cap K) = X' \cap (K \times \mathbb{R}^m)$ is compact, so that there exists a compact $F \subset \mathbb{R}^m$ such that

$$X' \cap (K \times \mathbb{R}^m) = X' \cap (K \times F) .$$

Then we can consider the decomposition $X' \cap (K \times F) = X'_1 \cup \ldots \cup X'_s$ into irreducible Nash varieties. The rest of the proof is identical with that of theorem 3, choosing $Z' = X'_j$ in the same way and choosing a suitable compact semialgebraic neighborhood K' of Z'.

5. A remark on seminarmolization.

Let us recall that, given a local ring A with finite integral closure, the seminormalization $^+A = \{f \in \bar{A} | f(\mathfrak{p}) \in A_{\mathfrak{p}} + \mathcal{R}(\bar{A}_{\mathfrak{p}}), \forall \mathfrak{p} \in \operatorname{Spec} A\}$ of A is the largest subring of $\bar{A}$ with trivial residue field extensions and such that Spec A is homeomorphic to Spec ^+A (cf. [2]). A is said seminormal if $A = {}^+A$. In [3] the seminormalization of real algebaaic varieties in studied, showing that for any affine variety X there exists a seminormalization $n: {}^+X \to X$, with ^+X seminormal and n homeomorphism (also in the real topology) but no «natural» universal property holds for n (cf. [3], th. 2.1. and ex. 2.6.).

The same result can be stated in our case. Let V, U and Z as in section 3.

PROPOSITION 7. There exist a semialgebraic set ${}^+U$ and a Nash variety ${}^+Z \subset {}^+U$ together with an algebraic map $n\colon {}^+Z \to Z$ zuch that ${}^+Z$ is seminormal (at each point) and n is a homeomorphism (in the usual topology).

PROOF. Let X be the Zariski closure of Z in V. By [3], th. 2.1. we have that $\Gamma_{+X} \simeq {}^+\Gamma_X \simeq \Gamma_X[f_1, \dots, f_s]$, hence the seminormalization map $m\colon {}^+X \to X$ is induced by the projection $p\colon V \times \mathbb{R}^s \to V$.

Let ${}^+U = p^{-1}(U)$, ${}^+Z = m^{-1}(Z)$ and $n = m|$ (note that ${}^+U \cap {}^+X$ is homeomorphic with $U \cap X$).

We have that ${}^+Z$ is a closed Nash set in ${}^+U$; in fact if $Z = \{h_1 = = \dots = h_r = 0\}$ then $m^{-1}(Z) = \{h_1 \cdot m = \dots = h_r \cdot m = 0\}$ and the $(h_i \cdot \cdot m)$'s are Nash functions on ${}^+U$.

Moreover, ${}^+Z$ is irreducible since it is locally so at every point and it is connected by Mostowski's theorem (cf. [5], Prop. 1). Finally by [2], Prop. 5.1 for every $y \in {}^+Z$, the local ring $(\mathcal{N}({}^+U)/J)_{m_y}$ is seminormal (J is the ideal of ${}^+Z$ and m_y is the maximal ideal corresponding to y).

As a final remark we raise the question of dealing with weak normalization (cf. [1] for the analytic case) and comparing weak normalization and seminormalization. At least in the case where $U \subset \mathbb{R}^n$ we think that a Nash variety $Z \subset U$ is seminormal if and only if Z^a is weakly normal.

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