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An Existence Theorem for a Stochastic Partial Differential Equation Arising from Filtering Theory.

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1. Introduction.

In this paper we consider the following stochastic partial differential problem:

\[ \begin{align*}
    du(t, x) &= u_{xx}(t, x) \, dt + h(x) u(t, x) \, dW(t) \\
    u(0, x) &= u_0(x)
\end{align*} \tag{1.1} \]

where \( h \) is any polynomial of degree \( n \) and \( W(t) \) is a real Wiener process.

Our method consists in performing a transformation of the problem so to get a deterministic equation w.p.1. In fact, putting

\[ v(t, x) = \exp \left[ -h(x)W(t) \right] u(t, x) \tag{1.2} \]

it is easy to see that \( v \) formally satisfies the following problem w.p.1:

\[ \begin{align*}
    v_t &= v_{xx} + \beta(t, x)v_x + \gamma(t, x)v \\
    v(0, x) &= u_0(x)
\end{align*} \tag{1.3} \]

In the next section we will solve problem (2) by semigroups methods in order to get a solution to problem (1.1) by performing the «inverse» transformation

\[ u = \exp [\hat{h} W(t)] v. \]

We remark that the same procedure adopted for problem (1.1) allows to treat the more general problem

\[ \begin{cases} 
  d\mathbf{u} = (a \mathbf{uu} + b \mathbf{u} + c \mathbf{u}) \, dt + (g \mathbf{u} + h \mathbf{u}) \, dW(t) \\
  \mathbf{u}(0, x) = \mathbf{u}_0(x); 
\end{cases} \tag{1.6} \]

a few details about it will be given at the end of section 3.

Problem (1.6) has been studied by Fleming-Mitter ([4]) using methods of dynamic programming. In a previous paper [1] we have studied a general method which applies to problem (1.6) assuming that \( \hat{h} \) is bounded.

Part of the results of the present paper have been reported in [2]. We are grateful to prof. Bove for useful discussions.

2. Here we solve problem (1.3). It can be written as an abstract Cauchy problem in the space \( H = L^2(\mathbb{R}) \)

\[ \frac{dv}{dt} = C(t) v, \quad v(0) = u_0. \tag{2.1} \]

where \( C(t): D(c(t)) \subset H \to H \) is an operator family with constant domain

\[ Y = H^2(\mathbb{R}) \cap L^2(\mathbb{R}; x^{4n} \, dx) \tag{1} \]

\( (1) \) \( H^2(\mathbb{R}) \) is the usual Sobolev space and \( L^2(\mathbb{R}, x^{4n} \, dx) \) denotes the space of square integrable functions with respect to the measure \( x^{4n} \, dx \); here \( n \) is the degree of the polynomial \( \hat{h} \).
In order to proceed for any $t \in [0, T]$ we consider $C(t)$ as the sum of the following two operators

\begin{equation}
C_1(t) \equiv \begin{cases}
D_{C_1(t)} = Y \\
C_1(t)v = v_{xx} + \gamma(t, x)v
\end{cases}
\end{equation}

\begin{equation}
C_2(t) \equiv \begin{cases}
D_{C_2(t)} = \{v \in H^1(\mathbb{R}), \beta(t, x)v \in L^2(\mathbb{R})\} \\
C_2(t)v = \beta(t, x)v_x
\end{cases}
\end{equation}

We have:

**Lemma 1.** For any $t \in [0, T]$ $C_1(t)$ is the infinitesimal generator of an analytic semigroup on $H$.

**Proof.** The proof can be found in [5] pag. 274. In fact here $\gamma(t, x)$ is bounded from above with respect to $x$ as it is polynomial of even order and the leading coefficient is negative $(^2)$.

**Remark 2.** We remark that the graph norm induced in $Y$ by the operator $C_1(t)$ is equivalent to the norm:

\[
|v|_Y^2 = \int_{-\infty}^{+\infty} v_{xx}^2 dx + \int_{-\infty}^{+\infty} (1 + x^4)v^2 dx, \quad \forall v \in Y.
\]

**Lemma 3.** For any fixed $t \in [0, T]$ and $\varepsilon > 0$ there exists $K_{\varepsilon, t} > 0$ such that

\begin{equation}
|C_2(t)v|_H^2 \leq K_{\varepsilon, t}|v|_Y^2 + \varepsilon |C_1(t)v|_H^2 \quad [w.p.1]
\end{equation}

**Proof.** First we note

\begin{equation}
|C_2(t)v|_H^2 = 4 \int_{-\infty}^{+\infty} W^2(t) \int_{-\infty}^{+\infty} h_x^2 u_x^2 dx
\end{equation}

$(^2)$ We actually remark that $C_1(t)$ is a self-adjoint operator.
Integrating by parts we have:

\[ \int_{-\infty}^{+\infty} h_x^2 u_x^2 \, dx = -\int_{-\infty}^{+\infty} 2 h_x h_{xx} u_x \, dx - \int_{-\infty}^{+\infty} h_x^2 u_{xx} \, dx \]

Now it is

\[ \int_{-\infty}^{+\infty} 2 h_x h_{xx} u_x \, dx \leq \frac{1}{2} \int_{-\infty}^{+\infty} h_x^2 u_x^2 \, dx + 2 \int_{-\infty}^{+\infty} h_x^2 u_x^2 \, dx \]

\[ \int_{-\infty}^{+\infty} h_x^2 u_{xx} \, dx \leq \frac{1}{4\varepsilon} \int_{-\infty}^{+\infty} h_x^2 u_x^2 \, dx + \varepsilon \int_{-\infty}^{+\infty} u_x^2 \, dx \]

so that

\[ \int_{-\infty}^{+\infty} h_x^2 u_x^2 \, dx \leq 4 \int_{-\infty}^{+\infty} h_x^2 u_x^2 \, dx + 2\varepsilon \int_{-\infty}^{+\infty} u_x^2 \, dx + \frac{1}{2\varepsilon} \int_{-\infty}^{+\infty} h_x^4 u_x^2 \, dx \]

Denote by \( a(\varepsilon) \) a suitable constant such that

\[ h_x^2 \leq a(\varepsilon) + \varepsilon x^4 \quad h_x^4 \leq a(\varepsilon) + 4\varepsilon^2 x^4 \]

hence

\[ \int_{-\infty}^{+\infty} h_x^2 u_x^2 \, dx \leq 6\varepsilon \left[ \int_{-\infty}^{+\infty} u_x^2 \, dx + \int_{-\infty}^{+\infty} x^4 u_x^2 \, dx \right] + (1 + \frac{1}{2})a(\varepsilon) \int_{-\infty}^{+\infty} u_x^2 \, dx \]

so that (2.6) follows from (2.7) and Remark 2.

We further remark that, though not necessary for the sequel, it is possible to prove that the constant \( K_{s,t} \) can be chosen independently of \( t \).

**Lemma 4.** For any \( t \in [0, T] \), \( C(t) \) is the infinitesimal generator of an analytic semigroup. Moreover for any \( \alpha \in ]0, \frac{1}{2} [ \) there exists a constant \( K \) such that

\[ |C(t)v - C(s)v|_R \leq K|t - s|^{\alpha}|v|_R \quad \text{[w.p.1]} \]
PROOF. The first statement follows by observing that $C_{1}(t)$ works as a perturbation of $C_{0}(t)$ (see for instance Kato [5], pag. 500). Finally (2.8) can be easily checked, taking in account that the Wiener process $W(t)$ is w.p.1 pathwise Hölder-continuous with any exponent $\alpha \in [0, \frac{1}{2}]$.

The previous results show that the assumptions of theorem 4.2 of [3] (3) for the existence of a solution to problem (2.1) are verified. Hence we can state the following result:

THEOREM 5. For any $u_{0} \in H$ there exists a unique classical solution to problem (2.1). That is there exists a unique function

$$v \in C([0, T]; H) \cap C^{1}([0, T]; H) \cap C([0, T]; Y)$$

such that (2.1) is verified. If moreover $u_{0} \in Y$ then $v \in C([0, T]; Y) \cap C^{1}([0, T]; H)$.

3. Now we are ready to prove the following result on the equation (1.1).

THEOREM 6. For any $u_{0} \in H = L^{2}(\mathbb{R})$ there exists a process $u$ which solves (1.1) in the following sense:

i) $u \in C([0, T]; L^{2}_{loc}(\mathbb{R})) \cap C([0, T]; H_{loc}^{2})$ [w.p.1]

ii) for any $\varphi \in C_{0}^{\infty}(\mathbb{R})$ it is

$$d(u, \varphi) = (u_{x}, \varphi) dt + (hu, \varphi) dW(t) \quad \text{for } t > 0;$$

if moreover $u_{0} \in H$ then

$$u_{0} \in C([0, T]; H_{loc}^{2})$$

and ii) is verified also for $t = 0$.

PROOF. To show the existence of a solution take $v$, the solution to problem (2.1), and put

$$u(t, x) = \exp [h(x)W(t)] v(t, x)$$

(3) The theorem is an improved version of the well-known result of Tanabe.
It is straightforward to check property i). For ii) consider \((u(t), \varphi)\), \(\varphi\) being in \(C^\infty_0(\mathbb{R})\); remark that

\[(3.1) \quad (u(t), \varphi) = (v(t), \exp [h(\cdot) W(t)] \varphi)_\mathbb{H};\]

by applying Itô formula at the right hand side of (3.1) it is easy to verify iii).

Concerning the more general problem (1.6) we consider the following assumptions:

\[
\begin{aligned}
    & a \in C^1_0(\mathbb{R}), \\
    & b, c \in C_0(\mathbb{R}), \quad g \in C_0(\mathbb{R}) \\
    & h \text{ any polynomial of order } n \\
    & 2a - g^2 \geq \varepsilon > 0 
\end{aligned}
\]

Then (1.6) can be solved with the same procedure for problem (1.1) by using the following transform

\[
v(t, x) = u(t, \varphi(W(t), x)) \exp \left[ \int_0^{w(t)} h(\varphi(\xi), x) \, d\xi \right]
\]

where \(\varphi\) is the solution of the following problem

\[
\frac{\partial \varphi}{\partial t} = g(\varphi) \quad \varphi(0, x) = x.
\]

REFERENCES


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