RENDICONTI
*del*
SEMINARIO MATEMATICO
*della*
UNIVERSITÀ DI PADOVA

G. DA PRATO
M. IANNELLI
L. TUBARO

# An existence theorem for a stochastic partial differential equation arising from filtering theory

*Rendiconti del Seminario Matematico della Università di Padova*, tome 71 (1984), p. 217-222
<http://www.numdam.org/item?id=RSMUP_1984__71__217_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 71 (1983)

# An Existence Theorem for a Stochastic Partial Differential Equation Arising from Filtering Theory.

G. DA PRATO - M. IANNELLI - L. TUBARO (*)

## 1. Introduction.

In this paper we consider the following stochastic partial differential problem:

$$(1.1)\qquad \begin{cases} du(t,x) = u_{xx}(t,x)\,dt + h(x)u(t,x)\,dW(t) \\ u(0,x) \;= u_0(x) \end{cases}$$

where $h$ is any polynomial of degree $n$ and $W(t)$ is a real Wiener process.

Our method consists in performing a transformation of the problem so to get a deterministic equation w.p.1. In fact, putting

$$(1.2)\qquad v(t,x) = \exp\left[-h(x)W(t)\right]u(t,x)$$

it is easy to see that $v$ formally satisfies the following problem w.p.1:

$$(1.3)\qquad \begin{cases} v_t = v_{xx} + \beta(t,x)v_x + \gamma(t,x)v \\ v(0,x) = u_0(x) \end{cases}$$

---

(*) Indirizzo degli AA.: G. DA PRATO: Scuola Normale Superiore di Pisa; M. IANNELLI e L. TUBARO: Dipartimento di Matematica, Libera Università di Trento, 38050 Povo (TN).

here

$$\beta(t, x) = 2W(t)h_x(x) \tag{1.4}$$

$$\gamma(t, x) = W(t)h_{xx}(x) + W(t)^2 h_x^2(x) - \tfrac{1}{2}h^2(x) \tag{1.5}$$

In the next section we will solve problem (2) by semigroups methods in order to get a solution to problem (1.1) by performing the « inverse » transformation

$$u = \exp[hW(t)]v .$$

We remark that the same procedure adopted for problem (1.1) allows to treat the more general problem

$$\begin{cases} du = (au_{xx} + bu_x + cu)\,dt + (gu_x + hu)\,dW(t) \\ u(0, x) = u_0(x) ; \end{cases} \tag{1.6}$$

a few details about it will be given at the end of section 3.

Problem (1.6) has been studied by Fleming-Mitter ([4]) using methods of dynamic programming. In a previous paper [1] we have studied a general method which applies to problem (1.6) assuming that $h$ is bounded.

Part of the results of the present paper have been reported in [2].

We are grateful to prof. Bove for useful discussions.

**2.** Here we solve problem (1.3). It can be written as an abstract Cauchy problem in the space $H = L^2(\mathbf{R})$

$$\frac{dv}{dt} = C(t)v , \qquad v(0) = u_0 . \tag{2.1}$$

where $C(t)\colon D_{C(t)} \subset H \to H$ is an operator family with constant domain

$$Y = H^2(\mathbf{R}) \cap L^2(\mathbf{R};\, x^{4n}dx) \;(^1) \tag{2.2}$$

---

(1) $H^2(\mathbf{R})$ is the usual Sobolev space and $L^2(\mathbf{R}, x^{4n}dx)$ denotes the space of square integrable functions with respect to the measure $x^{4n}dx$; here $n$ is the degree of the polynomial $h$.

defined by putting

$$C(t)v = v_{xx} + \beta(t,x)v_x + \gamma(t,x)v \qquad \forall v \in Y \tag{2.3}$$

In order to proceed for any $t \in [0, T]$ we consider $C(t)$ as the sum of the following two operators

$$C_1(t) \equiv \begin{cases} D_{C_1(t)} = Y \\ C_1(t)v = v_{xx} + \gamma(t,x)v \end{cases} \tag{2.4}$$

$$C_2(t) \equiv \begin{cases} D_{C_2(t)} = \{v \in H^1(\mathbf{R}), \beta(t,x)v \in L^2(\mathbf{R})\} \\ C_2(t)v = \beta(t,x)v_x \end{cases} \tag{2.5}$$

We have:

LEMMA 1. *For any $t \in [0, T]$ $C_1(t)$ is the infinitesimal generator of an analytic semigroup on $H$.*

PROOF. The proof can be found in [5] pag. 274. In fact here $\gamma(t,x)$ is bounded from above with respect to $x$ as it is polynomial of even order and the leading coefficient is negative (²).

REMARK 2. We remark that the graph norm induced in $Y$ by the operator $C_1(t)$ is equivalent to the norm:

$$|v|_Y^2 = \int_{-\infty}^{+\infty} v_{xx}^2\,dx + \int_{-\infty}^{+\infty} (1 + x^{4n})v^2\,dx\,, \qquad \forall v \in Y\,.$$

LEMMA 3. *For any fixed $t \in [0, T]$ and $\varepsilon > 0$ there exists $K_{\varepsilon,t} > 0$ such that*

$$|C_2(t)v|_H^2 \leqslant K_{\varepsilon,t}|v|_H^2 + \varepsilon|C_1(t)v|_H^2 \qquad [\text{w.p.1}] \tag{2.6}$$

PROOF. First we note

$$|C_2(t)v|_H^2 = 4W^2(t)\int_{-\infty}^{+\infty} h_x^2 u_x^2\,dx \tag{2.7}$$

---

(²) We actually remark that $C_1(t)$ is a self-adjoint operator.

Integrating by parts we have:

$$\int_{-\infty}^{+\infty} h_x^2 u_x^2\, dx = -\int_{-\infty}^{+\infty} 2h_x h_{xx} u_x u\, dx - \int_{-\infty}^{+\infty} h_x^2 u_{xx} u\, dx$$

Now it is

$$\int_{-\infty}^{+\infty} 2h_x h_{xx} u u_x\, dx \leqslant \frac{1}{2}\int_{-\infty}^{+\infty} h_x^2 u_x^2\, dx + 2\int_{-\infty}^{+\infty} h_{xx}^2 u^2\, dx$$

$$\int_{-\infty}^{+\infty} h_x^2 u_{xx} u\, dx \leq \frac{1}{4\varepsilon}\int_{-\infty}^{+\infty} h_x^4 u^2\, dx + \varepsilon\int_{-\infty}^{+\infty} u_{xx}^2\, dx$$

so that

$$\int_{-\infty}^{+\infty} h_x^2 u_x^2\, dx \leqslant 4\int_{-\infty}^{+\infty} h_{xx}^2 u^2\, dx + 2\varepsilon\int_{-\infty}^{+\infty} u_{xx}^2\, dx + \frac{1}{2\varepsilon}\int_{-\infty}^{+\infty} h_x^4 u^2\, dx$$

Denote by $a(\varepsilon)$ a suitable constant such that

$$h_{xx}^2 \leqslant a(\varepsilon) + \varepsilon x^{4n} \qquad h_x^4 \leqslant a(\varepsilon) + 4\varepsilon^2 x^{4n}$$

hence

$$\int_{-\infty}^{+\infty} h_x^2 u_x^2\, dx \leq 6\varepsilon\left[\int_{-\infty}^{+\infty} u_{xx}^2\, dx + \int_{-\infty}^{+\infty} x^{4n} u^2\, dx\right] + (1 + \tfrac{1}{2})a(\varepsilon)\int_{-\infty}^{+\infty} u^2\, dx$$

so that (2.6) follows from (2.7) and Remark 2.

We further remark that, though not necessary for the sequel, it is possible to prove that the constant $K_{\varepsilon,t}$ can be chosen independently of $t$.

LEMMA 4. *For any* $t \in [0, T]$, $C(t)$ *is the infinitesimal generator of an analytic semigroup. Moreover for any* $\alpha \in\, ]0, \frac{1}{2}[$ *there exists a constant* $K$ *such that*

$$|C(t)v - C(s)v|_H \leqslant K|t - s|^{\alpha}|v|_V \qquad [\text{w.p.1}] \tag{2.8}$$

**Proof.** The first statement follows by observing that $C_2(t)$ works as a perturbation of $C_1(t)$ (see for instance Kato [5], pag. 500). Finally (2.8) can be easily checked, taking in account that the Wiener process $W(t)$ is w.p.1 pathwise hölder-continuous with any exponent $\alpha \in ]0, \frac{1}{2}[$.

The previous results show that the assumptions of theorem 4.2 of [3] (³) for the existence of a solution to problem (2.1) are verified. Hence we can state the following result:

**Theorem 5.** *For any $u_0 \in H$ there exists a unique classical solution to problem* (2.1). *That is there exists a unique function*

$$v \in \mathbf{C}([0, T]; H) \cap \mathbf{C}^1(]0, T]; H) \cap \mathbf{C}(]0, T]; Y)$$

*such that* (2.1) *is verified. If moreover $u_0 \in Y$ then $v \in \mathbf{C}([0, T]; Y) \cap \cap \mathbf{C}^1([0, T]; H)$.*

**3.** Now we are ready to prove the following result on the equation (1.1).

**Theorem 6.** *For any $u_0 \in H = L^2(\mathbf{R})$ there exists a process $u$ which solves* (1.1) *in the following sense*:

i) $u \in \mathbf{C}([0, T]; L^2_{\mathrm{loc}}(\mathbf{R})) \cap \mathbf{C}(]0, T]; H^2_{\mathrm{loc}})$ [w.p.1]

ii) *for any $\varphi \in C_0^\infty(\mathbf{R})$ it is*

$$d(u, \varphi) = (u_{xx}, \varphi)\,dt + (hu, \varphi)\,dW(t) \qquad \textit{for } t > 0;$$

*if moreover $u_0 \in H$ then*

$$u_0 \in \mathbf{C}([0, T]; H^2_{\mathrm{loc}})$$

*and* ii) *is verified also for $t = 0$.*

**Proof.** To show the existence of a solution take $v$, the solution to problem (2.1), and put

$$u(t, x) = \exp[h(x)W(t)]v(t, x)$$

---

(³) The theorem is an improved version of the well-known result of Tanabe.

It is straightforward to check property i). For ii) consider $(u(t), \varphi)$, $\varphi$ being in $C_0^\infty(\mathbf{R})$; remark that

$$(u(t), \varphi) = (v(t), \exp[h(\cdot)\, W(t)]\varphi)_H; \tag{3.1}$$

by applying Itô formula at the right hand side of (3.1) it is easy to verify iii).

Concerning the more general problem (1.6) we consider the following assumptions:

$$\begin{cases} a \in C_b^1(\mathbf{R}); \qquad b, c \in C_b(\mathbf{R}), \quad g \in C_b^2(\mathbf{R}) \\ h \text{ any polynomial of order } n \\ 2a - g^2 \geqslant \varepsilon > 0 \end{cases}$$

Then (1.6) can be solved with the same procedure for problem (1.1) by using the following transform

$$v(t, x) = u\big(t, \varphi(W(t), x)\big) \exp\left[\int_0^{W(t)} h(\varphi(\xi), x)\, d\xi\right]$$

where $\varphi$ is the solution of the following problem

$$\frac{\partial \varphi}{\partial t} = g(\varphi) \qquad \varphi(0, x) = x\,.$$

## REFERENCES

[1] G. Da Prato - M. Iannelli - L. Tubaro, *Some results on linear stochastic differential equations in Hilbert spaces*, Stochastic, **6** (1982), pp. 105-116.

[2] G. Da Prato - M. Iannelli, L. Tubaro, *On a stochastic differential equation arising from filtering theory*, Libera Università degli Studi di Trento, U.T.M. 68.

[3] G. Da Prato - E. Sinestrari, *Hölder regularity for non autonomous abstract parabolic equations* (**42** (1982 pp. 1-19) « Israel Journal of Mathematics »).

[4] W. H. Fleming - S. K. Mitter, (to appear in « Stochastics »).

[5] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag (1976).

Manoscritto pervenuto in redazione il 6 Agosto 1982.