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On a synonymy relation for extensional 1st order theories. Part III. A necessary and sufficient condition for synonymy

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On a Synonymy Relation for Extensional 1st Order Theories.

PART III
A Necessary and Sufficient Condition for Synonymy.

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10. Intuitive description of the theory $\dot{\mathcal{T}}$. Semiotics for $\dot{\mathcal{T}}$ (1).

We want to associate theory $\mathcal{T}$ based on the language $\mathcal{L}$—see [1], § 6—with an interpreted theory $\dot{\mathcal{T}}$ in compliance with the following intuitive requirements, which refer to any model $\mathcal{I} = (\mathcal{D}, \mathcal{I}, \alpha)$ and any v-valuation V for $\mathcal{T}$, and hold for n, $i = 1, 2, \dots$.

(a) *The variable x_i of $\mathcal{T}$ acts, as far as its sense is concerned, as a proper name*, i.e. *as primitive constant.*

(b) *The relator R_i^n and connectives $\sim$ and $\supset$ are associated with three individual constants $\dot{R}_i^n$, $\dot{\sim}$, and $\dot{\supset}$ of $\dot{\mathcal{T}}$, that designate ($\mathcal{I}$, V)-senses of R_i^n, $\sim$, and $\supset$ respectively*—i.e. the senses of R_i^n to $\supset$ with respect to $\mathcal{I}$ and V.

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(1) The present paper is the third part of a work whose first and second part are [1] and [2] respectively. Therefore the numbering of its sections follows those for [1] and [2].

(*c*) *The functor* f_i^n *is associated with the individual constant* $\dot{f}_i^n$ *of* $\dot{\mathscr{T}}$, *that denotes the* $(\mathscr{I}, V)$*-sense of* f_i^n.

(*d*) [(*e*)] A_n [V_n] *is an* $(n+1)$*-ary functor of* $\dot{\mathscr{T}}$, *and wffs having the 1st [2nd] of the forms*

$$(10.1) \qquad \eta = A_n(\varrho, \xi_1, \ldots, \xi_n)\,, \qquad \zeta = V_n(\sigma, \xi_1, \ldots, \xi_n)$$

express that η [ζ] *is the* $(\mathscr{I}, V)$*-sense of the application* [*value*] *of the* n*-ary attribute* R [*functor* f], *of* $(\mathscr{I}, V)$*-sense* ϱ [σ], *to the terms* Δ_1 *to* Δ_n *of the respective* $(\mathscr{I}, V)$*-senses* ξ_1 *to* ξ_n.

(*f*) $\dot{\mathscr{T}}$ *has a predicate* $\mathscr{V}$ *such that any wff of* $\dot{\mathscr{T}}$, *of the form* $\mathscr{V}(\xi)$ *expresses that* ξ *is the* $(\mathscr{I}, V)$*-sense of a wff* $\mathscr{A}$ *of* $\mathscr{T}$, *and that* $\mathscr{A}$ *is true in* $\mathscr{I}$ *at* V, *i.e.* $\text{des}_{\mathscr{I},V}(\mathscr{A}) = 0$.

(*g*) *The variable* x_i *is associated with an operator* (Ωx_i) *of* $\dot{\mathscr{T}}$ *to be denoted with* $(\dot{x}_i)$, *such that* $(\dot{x}_i)\dot{\Delta}$ *denotes the* $(\mathscr{I}, V)$*-sense of the wff* $(x_i)\Delta$ *of* $\mathscr{T}$.

Now in order to define the theory $\dot{\mathscr{T}}$ rigorously, we stipulate, first, that, as well as $\mathscr{T}$, it is based on the language $\mathscr{L}$—see [1], §§ 2, 3—, so that $\dot{\mathscr{T}}$ has the same variables as $\mathscr{T}$.

It is not restrictive to assume $\mathscr{T}$ to have only constants c_i, R_i^n, or f_i^n with an odd value of i; for should this situation not occur, we can render it holding by performing the replacement $i \to 2i - 1$ $(i = 1, 2, \ldots)$ of all constants of $\mathscr{T}$. Thus denoting by $\mathscr{PC}$ (primitive constants and connectives) the set formed by the constants c_i, R_i^n, and f_i^n of $\mathscr{T}$ and by the connectives $\sim$ and $\supset$, we can choose

(i) an injection χ of $\mathscr{PC}$ into the new individual constants, i.e. c_{2e} $(e = 1, 2, \ldots)$, and

(ii) a predicate $\mathscr{V}$ and two $(n+1)$-ary functors A_n and V_n in $\mathscr{L}$ outside the counterdomain of χ $(n = 1, 2, \ldots)$.

We shall denote $\chi(c_i)$ to $\chi(\supset)$ by $\dot{c}_i$ to $\dot{\supset}$ respectively.

We also stipulate that the constants of $\dot{\mathscr{T}}$ are $\mathscr{V}$, A_n, V_n $(n = 1, 2, \ldots)$, the equality sign R_1^2, the χ-transformed of the elements in $\mathscr{PC}$, and a term-term operator sign Ω (2).

(2) $\mathscr{T}$ need not have any inexistent object.

We now associate every wfe Δ of $\mathscr{T}$ with the *corresponding* wfe $\dot{\Delta}$ of $\dot{\mathscr{T}}$ by means of the following recursive rules:

Rule	If Δ is	then $\dot{\Delta}$ is
s_1	x_i or c_i	$\dot{x}_i$ or $\dot{c}_i$ respectively
s_2	$f_i^n(\Delta_1, \ldots, \Delta_n)$	$V_n(\dot{f}_i^n, \dot{\Delta}_1, \ldots, \dot{\Delta}_n)$
s_3	$R_i^n(\Delta_1, \ldots, \Delta_n)$	$A_n(\dot{R}_i^n, \dot{\Delta}_1, \ldots, \dot{\Delta}_n)$
$s_{4\text{-}5}$	$\sim \Delta_1$ $[\Delta_1 \supset \Delta_2]$	$A_1(\dot{\sim}, \dot{\Delta}_1)$ $[A_2(\dot{\supset}, \dot{\Delta}_1, \dot{\Delta}_2)]$
s_6	$(x_i)\Delta$	$(\Omega x_i)\dot{\Delta}$.

Remark that *if $\mathscr{A}$ is a wff of $\mathscr{T}$, then $\dot{\mathscr{A}}$ is a term of $\dot{\mathscr{T}}$.*

11. Axioms for $\dot{\mathscr{T}}$. Some theorems relating $\mathscr{T}$ and $\dot{\mathscr{T}}$.

As proper axioms or axiom schemes of $\dot{\mathscr{T}}$, we take those on identity—see [4]—, i.e.

AA1-2 $\quad x = x,\ x = y \supset (\mathscr{A}(x) \supset \mathscr{A}(y))$,

the special axioms AA3-6 below on $\mathscr{V}$, where $\mathscr{B}$, $\mathscr{B}_1$, and $\mathscr{B}_2$ are arbitrary wffs of $\mathscr{T}$,

A3 $\quad \mathscr{V}[A_1(\dot{\sim}, \dot{\mathscr{B}})] \equiv \sim \mathscr{V}(\dot{\mathscr{B}})$,

A4 $\quad \mathscr{V}[A_2(\dot{\supset}, \dot{\mathscr{B}}_1, \dot{\mathscr{B}}_2)] \equiv \mathscr{V}(\dot{\mathscr{B}}_1) \supset \mathscr{V}(\dot{\mathscr{B}}_2)$,

A5 $\quad \mathscr{V}[(\Omega x_i)\dot{\mathscr{B}}] \equiv (x_i)\mathscr{V}(\dot{\mathscr{B}})$,

A6 $\quad \sim \mathscr{V}(\tau)$ *if* $\tau \neq \dot{\Delta}$ *for every wff* Δ *of* $\mathscr{T}$,

the following three axioms, connected with the synonymy conditions C_6), C_1), and C_7) in [1], § 6—see Def. 3.1.

AA7-8 $\quad (x_i)(\dot{p} = \dot{p}') \supset (\Omega x_i)\dot{p} = (\Omega x_i)\dot{p}', \quad \dot{D}'_\nu = \dot{D}''_\nu \ (0 < \nu < \omega)$,

A9 $\quad (\Omega x_i)\dot{\mathscr{B}} = (\Omega x_j)\dot{\mathscr{C}}$ *if* $\mathscr{B}$ *and* $\mathscr{C}$ *are* (x_i, x_j)*-similar wffs of* $\mathscr{T}$, and the following counterparts of $\mathscr{T}$'s axioms

A10 $\mathscr{V}(\dot{\mathscr{A}})$ *whenever* $\mathscr{A}$ *is an axiom of* $\mathscr{T}$.

Since $\dot{\mathscr{T}}$ is based on $\mathscr{L}$, its inference rules are MP and Gen. By AA1-2 theorems (11.1-4) below hold. They are connected with the synonymy conditions C_2 to C_5 in [1], and in them p, q, p', and q' are arbitrary wffs of $\mathscr{T}$, f and f' [R and R'] are arbitrary n-ary functor [relators] of $\mathscr{T}$ and $\Delta_1, \Delta_1', \ldots, \Delta_n, \Delta_n'$ are arbitrary terms of $\mathscr{T}$; n runs over Z^+ and $\Lambda_i^n p_i$ means $p_1 \wedge \ldots \wedge p_n$.

$$(11.1) \qquad \vdash_{\dot{\mathscr{T}}} \dot{f} = \dot{f}' \wedge \Lambda_i^n \dot{\Delta}_i = \dot{\Delta}_i' \supset V_n(\dot{f}, \dot{\Delta}_1, \ldots, \dot{\Delta}_n) = V_n(\dot{f}', \dot{\Delta}_1', \ldots, \dot{\Delta}_n'),$$

$$(11.2) \qquad \vdash_{\dot{\mathscr{T}}} \dot{R} = \dot{R}' \wedge \Lambda_i^n \dot{\Delta}_i = \dot{\Delta}_i' \supset A_n(\dot{R}, \dot{\Delta}_1, \ldots, \dot{\Delta}_n) = A_n(\dot{R}', \dot{\Delta}_1', \ldots, \dot{\Delta}_n'),$$

$$(11.3) \qquad \vdash_{\dot{\mathscr{T}}} \dot{p} = \dot{p}' \supset A_1(\dot{\sim}, \dot{p}) = A_1(\dot{\sim}, \dot{p}'),$$

$$(11.4) \qquad \vdash_{\dot{\mathscr{T}}} \dot{p} = \dot{p}' \wedge \dot{q} = \dot{q}' \supset A_2(\dot{\supset}, \dot{p}, \dot{q}) = A_2(\dot{\supset}, \dot{p}', \dot{q}').$$

THEOR. 11.5. *If* p *is a wff of* $\mathscr{T}$ *and* $\vdash_{\mathscr{T}} p$, *then* $\vdash_{\dot{\mathscr{T}}} \mathscr{V}(\dot{p})$.

Indeed let $\mathscr{B}_1$ to $\mathscr{B}_n$ be a proof of p in $\mathscr{T}$. We assume

$$(11.6) \qquad \vdash_{\dot{\mathscr{T}}} \mathscr{V}(\dot{\mathscr{B}}_j) \qquad \text{for } j = 1, 2, \ldots, i-1, \text{ where } i \leqslant n$$

(which holds vacously for $i = 1$) as the hypothesis of our (complete) induction. Then one of alternatives *a*) to *c*) below holds.

a) $\mathscr{B}_i$ *is an axiom of* $\mathscr{T}$. Then $\vdash_{\dot{\mathscr{T}}} \mathscr{V}(\dot{\mathscr{B}}_i)$ by A10.

b) *For some* r *and* s *smaller than* i, $\mathscr{B}_s$ *is* $\mathscr{B}_r \supset \mathscr{B}_i$ (*MP*). Then by rule s_5) in § 10 and A4 $\vdash_{\dot{\mathscr{T}}} \mathscr{V}(\dot{\mathscr{B}}_s) \equiv [\mathscr{V}(\dot{\mathscr{B}}_r) \supset \mathscr{V}(\dot{\mathscr{B}}_i)]$. Hence, by (11.6) and *MP* $\vdash_{\dot{\mathscr{T}}} \mathscr{V}(\dot{\mathscr{B}}_i)$.

c) *For some* $r < i$ *and some* k, $\mathscr{B}_i$ *is* $(x_k)\mathscr{B}_r$. By (11.6) $\vdash_{\dot{\mathscr{T}}} \mathscr{V}(\dot{\mathscr{B}}_r)$. Then, by Gen, $\vdash_{\dot{\mathscr{T}}} (x_k)\mathscr{V}(\dot{\mathscr{B}}_r)$. Furthermore, by rule s_6) in § 10 and A5 $\vdash_{\dot{\mathscr{T}}} (x_k)\mathscr{V}(\dot{\mathscr{B}}_r) \equiv \mathscr{V}(\dot{\mathscr{B}}_i)$. Hence $\vdash_{\dot{\mathscr{T}}} \mathscr{V}(\dot{\mathscr{B}}_i)$.

Then by the principle of complete induction, (11.6) holds for $j = n$ (and $\mathscr{B}_n = p$). q.e.d.

Theor. 11.6. *If a and b are wfes of $\mathcal{T}$ and $a \asymp b$ then* $\vdash_{\dot{\mathcal{T}}} \dot{a} = \dot{b}$.

Indeed let $\mathcal{R}$ be the relation such that *$a\mathcal{R}b$ iff a and b are wfes of $\mathcal{T}$ and* $\vdash_{\dot{\mathcal{T}}} \dot{a} = \dot{b}$. By AA1, 2, $\mathcal{R}$ is an equivalence relation. Furthermore by A8, theorems (11.1-4), A7, and A9, *relation $\mathcal{R}$ satisfies the synonymy conditions* C_1) *to* C_7) *in* [1], § 6.

To check the assertion above is obvious, except in connection with condition C_6). Therefore we now assume that $p\,\mathcal{R}\,p'$ where p and p' are wffs of $\mathcal{T}$. Then $\vdash_{\dot{\mathcal{T}}} \dot{p} = \dot{p}'$ by the definition of $\mathcal{R}$. Hence $\vdash_{\dot{\mathcal{T}}} (x_i)\dot{p} = \dot{p}'$ by Gen. Then by A7 we deduce $\vdash_{\dot{\mathcal{T}}} (\Omega x_i)\dot{p} = (\Omega x_i)\dot{p}'$, which by rule s_6 in § 10 is $\vdash_{\dot{\mathcal{T}}} \dot{a} = \dot{b}$, where a is $(x_i)p$ and b is $(x_i)p'$. Hence, by the definition of $\mathcal{R}$, $(x_i)p\,\mathcal{R}\,(x_i)p'$. We conclude that also condition C_7) in [1] is satisfied by $\mathcal{R}$, and the italicized assertion above is completely proved.

Since the synonymy relation $\asymp$ is the least equivalence relation that satisfies conditions $C_{1\text{-}7}$) in [1], $\asymp \subseteq \mathcal{R}$. Then our thesis holds.

q.e.d.

Theor. 11.7. $\vdash_{\dot{\mathcal{T}}} (x_i)\dot{p} = \dot{p}' \supset (\Omega x_i)\dot{p} = (\Omega x_j)\dot{p}''$ *where p' and p'' are $(x_i,\ x_j)$-similar.*

Proof. By the completeness of $\dot{\mathcal{T}}$ it is suffices to show that the wff $(x_i)\dot{p} = \dot{p}' \supset (\Omega x_i)\dot{p} = (\Omega x_j)\dot{p}''$ is true in every normal model of $\dot{\mathcal{T}}$. Let $\boldsymbol{M}$ be such a model. Note that

(i) V is an $\boldsymbol{M}$-valuation,

(ii) $\mathrm{des}_{\boldsymbol{M},V}\,[(x_i)\dot{p} = \dot{p}'] = 0$,

(iii) p' and p'' are $(x_i,\ x_j)$-similar.

By (ii), A7 yields $\mathrm{des}_{\boldsymbol{M},V}\,((\Omega x_i)\dot{p} = (\Omega x_i)\dot{p}') = 0$ and hence (since $\boldsymbol{M}$ is normal)

$$\mathrm{des}_{\boldsymbol{M},V}\,(\Omega x_i)\dot{p} = \mathrm{des}_{\boldsymbol{M},V}\,(\Omega x_i)\dot{p}' . \tag{11.7}$$

By (iii) and A9 we have

$$\mathrm{des}_{\boldsymbol{M},V}\,(\Omega x_i)\dot{p}' = \mathrm{des}_{\boldsymbol{M},V}\,(\Omega x_j)\dot{p}'' . \tag{11.8}$$

Hence by (11.7) and (11.8) we deduce

$$\mathrm{des}_{\boldsymbol{M},V}\,(\Omega x_i)\dot{p} = \mathrm{des}_{\boldsymbol{M},V}\,(\Omega x_j)\dot{p}'' , \tag{11.9}$$

i.e.

$$\text{(11.10)} \qquad \mathrm{des}_{M,V}[(\Omega x_i)\dot{p} = (\Omega x_j)\dot{p}''] = 0 \, .$$

Hence the wff $(x_i)\dot{p} = \dot{p}' \supset (\Omega x_i)\dot{p} = (\Omega x_j)\dot{p}''$ is true in $\boldsymbol{M}$. Since $\boldsymbol{M}$ is arbitrary, the thesis holds.

12. Statement of a necessary and sufficient condition for synonymy.

DEFINITION 12.1. *We say that a and b are $\dot{\mathscr{T}}$-equivalent, briefly $a \mathscr{E} b$, if a and b are wfes of $\mathscr{T}$ and*

$$\text{(12.1)} \qquad \mathrm{des}_{M,W}\, \dot{a} = \mathrm{des}_{M,W}\, \dot{b}$$

for every normal model $\boldsymbol{M}$ of $\dot{\mathscr{T}}$ and every $\boldsymbol{M}$-valuation W (3).

The main aim of Part 3 is the following equivalence theorem, in that it allows the inversion of Theor. 11.6.

THEOR. 12.1. *If a and b are wfes of $\mathscr{T}$, $a \asymp b \Leftrightarrow a \mathscr{E} b$.*

PROOF OF THE $\Rightarrow$-PART. Let $a \asymp b$. Then by Theor. 11.6,

$$\vdash_{\dot{\mathscr{T}}} \dot{a} = \dot{b} \, .$$

Since $\dot{\mathscr{T}}$ is a theory based on $\mathscr{L}$, by Theor. 3.4 in [1] the wff $\dot{a} = \dot{b}$ is true in every (normal) model $\boldsymbol{M}$ of $\dot{\mathscr{T}}$ and at any $\boldsymbol{M}$-valuation W. Then by the definition of normal model, (12.1) holds for arbitrary such $\boldsymbol{M}$ and W. Then, by Def. 12.1, $a \mathscr{E} b$.

PROOF OF THE $\Leftarrow$-PART. Let $a \mathscr{E} b$. In order to construct a suitable model $\dot{\mathscr{J}} = (\mathbf{D}, \dot{\mathscr{J}})$ of the theory $\dot{\mathscr{T}}$ (with a general operator Ω), associated to the one $\mathscr{J} = (\mathbf{D}, \mathscr{J}, \alpha)$ for the theory $\mathscr{T}$ (without general operators) we consider the set B formed by the wfes of $\mathscr{T}$, the functional and predicative letters of $\mathscr{T}$, and the connectives $\sim$ and $\supset$.

(3) Obviously $a \mathscr{E} b$ iff a and b are wfes of $\mathscr{T}$ and $\mathrm{des}_{M,W}(\dot{a} = \dot{b}) = 0$ for every model $\boldsymbol{M}$ of $\dot{\mathscr{T}}$ and every $\boldsymbol{M}$-valuation W—i.e. iff, for all such $\boldsymbol{M}$ and W, we have $\mathrm{des}_{M,W}\, a E_M \mathrm{des}_{M,W}\, b$, where E_M is the equivalence denoted in $\boldsymbol{M}$ by the identity sign of $\dot{\mathscr{T}}$.

Furthermore we set

$$(12.2)\qquad \dot{B} = \{\dot{\Delta}|\Delta \in B\}\,,\ [\Delta] = \{\dot{\Delta}_1|\Delta_1 \in B \,\&\, \Delta_1 \asymp \Delta\} \cup \{\dot{\Delta}\}\ (^4)\,.$$

Now we specify the domain $\mathbf{D}$ of $\mathscr{J}$:

$$(12.3)\qquad \mathbf{D} = \{[\Delta]\,|\Delta \in B\} \cup \{\emptyset\} \qquad (\emptyset = \text{the empty set})\,.$$

Note that $\mathbf{D}$ is *denumerable* (*hence infinite*). Indeed such is B ($x_i \in B$ for $i = 1, 2, \ldots$). Furthermore, by Theor. 8.1 in [2] $x_i \not\asymp x_j$ for $i \neq j$ and $i, j = 1, 2, \ldots$; and $(12.2)_2$ and (12.3) hold.

Let $\Delta = \hat{\Delta}(x_1, \ldots, x_n)$ be an expression of $\mathscr{T}$ in B, whose free variables, if any, occur among the n (distinct) variables x_1 to x_n; and let us consider the condition

COND. 12.1. For some wfes Δ_1 to Δ_n of $\mathscr{T}$, $\hat{\Delta}(\Delta_1, \ldots, \Delta_n)$ too is a wfe of $\mathscr{T}$ and $\Phi_i = [\Delta_i]$ $(\in \mathbf{D})$ $(i = 1, \ldots, n)$—cf. Convention 2.2 in [1].

STEP 1 (*in the proof of* $\Leftarrow$). *There is a mapping* $g_{\Delta;x_1,\ldots,x_n}$ *of* $\mathbf{D}^n$ *into* $\mathbf{D}$ *for which*

$$(12.4)\qquad g_{\Delta;x_1,\ldots,x_n}(\Phi_1, \ldots, \Phi_n) = \begin{cases} [\hat{\Delta}(\Delta_1, \ldots, \Delta_n)] & \textit{under } \text{Cond. 12.1}\,, \\ \emptyset & \textit{otherwise (for } \Phi_1, \ldots, \Phi_n \in \mathbf{D})\,. \end{cases}$$

To prove Step 1 it suffices to consider the case when Cond. 12.1 holds; (in the remaining case the proof is trivial). Therefore we assume that for $i = 1$ to n, $\Phi_i = [\Delta_i] = [\Delta_i']$ with Δ_i and Δ_i' wfes of $\mathscr{T}$; hence $\Delta_i \asymp \Delta_i'$. Hence, by Theor. 6.1 in [1], $\hat{\Delta}(\Delta_1, \ldots, \Delta_n) \asymp \hat{\Delta}(\Delta_1', \ldots, \Delta_n')$, which by (12.2) yields our thesis.

STEP 2. *There is an interpretation* $\xi^* = \mathscr{J}(\xi)$ *of* $\dot{\mathscr{T}}$ *that, under the definition*

$$(12.5)\qquad W_1(x_i) = [x_i] \qquad (i = 1, 2, \ldots)$$

(4) The simplification of $(12.2)_2$ into $[\Delta] = \{\Delta_1|\Delta_1 \in B$ and $\Delta_1 \asymp \Delta\}$ would imply $[\sim] = [\supset] = \emptyset$, since the synonymy relation is defined between wfes of $\mathscr{T}$, and $\sim$, $\supset$ are not wfes of $\mathscr{T}$.

of the v-valuation W_1 *for* $\dot{\mathscr{T}}$, *fulfils the (interpretation) conditions* (i_1) *to* (i_6) *below for all* $\Phi_0, \ldots, \Phi_n \in \mathbf{D}$, *for every element* Δ *of* B *that has at most* x_i *as a free variable, and for every v-valuation* W.

(i_1) $\dot{c}_i^* = \mathscr{I}(\dot{c}_i) = [c_i]$.

(i_2) [(i_3)] $A_n^*(\Phi_0, \ldots, \Phi_n)$ [$V_n^*(\Phi_0, \ldots, \Phi_n)$] *is* $[\Delta]$, *in case for some elements* Δ_0 *to* Δ_n *of* B, $\Phi_i = [\Delta_i]$ $(i = 0, \ldots, n)$ *and* $\Delta_0(\Delta_1, \ldots, \Delta_n)$ *is a wff* [*term*] Δ *of* $\mathscr{T}$ *respectively*; *it is* $\emptyset$ *otherwise.*

(i_4) $\Omega^*(g_{\Delta;x_i})$ *is* $[(x_i)\Delta]$ *if* Δ *is a wff of* $\mathscr{T}$; *it is* $\emptyset$ *otherwise.*

(i_5) $\mathscr{V}^* = \{\xi \in \mathbf{D} |$ *for some wff* Δ *of* $\mathscr{T}$ $\text{des}_{\mathscr{I},W}(\Delta) = 0$ *and* $[\Delta] = \xi\}$.

(i_6) $=^* =$ *the identity relation in* $\mathbf{D}$.

To prove Step 2 it is suffices to show that conditions (i_2) to (i_4) are good definitions of functions. To reach this aim in connection with (i_2) [(i_3)] we assume that (i) $\Delta_0, \Delta_0', \ldots, \Delta_n, \Delta_n'$ are elements of B, (ii) $\Delta_0(\Delta_1, \ldots, \Delta_n)$ and $\Delta_0'(\Delta_1', \ldots, \Delta_n')$ are two wffs [terms], say Δ and Δ' respectively, and (iii) $[\Delta_i] = [\Delta_i']$ $(i = 0, \ldots, n)$. Then by $(12.2)_2$ $\Delta_i \asymp \Delta_i'$ $(i = 0, \ldots, n)$, so that by Theor. 6.1 in [1] $\Delta \asymp \Delta'$. Then $[\Delta] = [\Delta']$ by $(12.2)_2$.

Thus our goal is reached in the first case considered in (i_2) [(i_3)]. In the remaining case this thesis is trivial.

To prove the acceptability of definition in (i_4) we assume that $\Delta_1 = \hat{\Delta}_1(x_i)$, $\Delta_2 = \hat{\Delta}_2(x_j)$ and

$$(12.6) \qquad g_{\Delta_1;x_i} = g_{\Delta_2;x_j}, \qquad \text{hence } g_{\Delta_1;x_i}([x_r]) = g_{\Delta_2;x_j}([x_r]) \qquad (r = 1, 2, \ldots).$$

Then, by (12.4) $[\hat{\Delta}_1(x_r)] = [\hat{\Delta}_2(x_r)]$ and hence, by $(12.2)_2$, $\hat{\Delta}_1(x_r) \asymp \hat{\Delta}_2(x_r)$ $(r = 1, 2, \ldots)$. By condition C_6) in [1], this yields

$$(12.7) \qquad (x_r)\hat{\Delta}_1(x_r) \asymp (x_r)\hat{\Delta}_2(x_r) \qquad (r = 1, 2, \ldots).$$

Now let x_h fail to occur in $\hat{\Delta}_1(x_i) \wedge \hat{\Delta}_2(x_j)$. Then, for $s = 1, 2$, $\hat{\Delta}_s(x_i)$ [$\hat{\Delta}_s(x_j)$] and $\hat{\Delta}_s(x_h)$ are (x_i, x_h)-similar [(x_j, x_h)-similar], so that by condition C_7) in [1],

$$(x_i)\,\Delta_1(x_i) \asymp (x_h)\,\Delta_1(x_h) \qquad \text{and} \qquad (x_j)\,\Delta_2(x_j) \asymp (x_h)\,\Delta_2(x_h)\,.$$

These results and (12.7) for $r = h$ yield

$$(x_i)\,\Delta_1(x_i) \asymp (x_j)\,\Delta_2(x_j)\,. \qquad \text{q.e.d.}$$

13. Completion of the proof of Theor. 12.1. A consequence of it.

STEP 3. *Assume that* (i) $\dot{\Delta} = \hat{\Delta}(x_1, \ldots, x_n)$ *is a wfe of* $\mathcal{T}$ *whose free variables, if any, are some among* x_1 *to* x_n, (ii) Δ_1 *to* Δ_n *are terms of* $\mathcal{T}$ *and* (iii) W *is a v-valuation for* $\dot{\mathcal{T}}$, *for which*

$$W(x_s) = [\Delta_s] \qquad (s = 1, \ldots, n)\,. \tag{13.1}$$

(*Note that* $\dot{\Delta}^* = \mathrm{des}_{\mathcal{T},W}(\dot{\Delta})$ *is independent of a such a choice of* W). Then

$$\dot{\Delta}^* = [\hat{\Delta}(\Delta_1, \ldots, \Delta_n)]\,. \tag{13.2}$$

For the proof we use induction on the length l of Δ. For $l = 1$, Δ is c_i or x_i, so that by (i_1) in § 12, or (13.1), $\dot{\Delta}^* = [c_i]$ or $\dot{\Delta}^* = W(x_i) = [\Delta_i]$ respectively. Hence (13.2) holds for $l = 1$.

As inductive hypothesis, assume that (13.2) holds for $l < \nu$; furthermore let l be ν. Only Cases 1 to 5 below can hold.

Case 1 [2]. Δ is $\varphi[\hat{\Delta}_1(x_1, \ldots, x_n), \ldots, \hat{\Delta}_m(x_1, \ldots, x_n)]$ where φ is R_i^m [f_i^m]. Then, for $\Psi = A_m$ [$\Psi = V_m$], $\dot{\Delta}$ is

$$\Psi[\dot{\varphi}, \hat{\Delta}_1(x_1, \ldots, x_n)^{\cdot}, \ldots, \hat{\Delta}_m(x_1, \ldots, x_n)^{\cdot}]$$

by s_3) [s_2)] in § 10, and $\dot{\Delta}^*$ is

$$\Psi^*[\dot{\varphi}^*, \hat{\Delta}_1(x_1, \ldots, x_n)^{\cdot *}, \ldots, \hat{\Delta}_m(x_1, \ldots, x_n)^{\cdot *}]\,.$$

By the inductive hypothesis—see (13.2)—,

$$\hat{\Delta}_s(x_1, \ldots, x_n)^{\cdot *} = [\hat{\Delta}_s(\Delta_1, \ldots, \Delta_n)] \qquad (s = 1, \ldots, m)\,.$$

Then by condition (i_2) [(i_3)] in § 12,

$$\dot{\Delta}^* = [\varphi[\hat{\Delta}_1(\Delta_1, \dots, \Delta_n), \dots, \hat{\Delta}_m(\Delta_1, \dots, \Delta_n)]] .$$

Thus (13.2) holds in this case.

Case 3 [4]: Δ is $\sim \hat{\Delta}_1(x_1, \dots, x_n)$ $[\hat{\Delta}_1(x_1, \dots, x_n) \supset \hat{\Delta}_2(x_1, \dots, x_n)]$. Then (13.2) follows as a particular case of Case 1 in that $\dot{\sim}$ and $\dot{\supset}$ are regarded as predicates of $\mathscr{T}$.

Case 5. Δ is $(y_0)\mathscr{A}(y_0, \dots, y_n)$ where y_0 to y_n are $n+1$ variables. We set

$$\text{(13.3)} \qquad \begin{cases} \mathscr{A}_1 \equiv_D \mathscr{A}_1(y_0) \equiv_D \mathscr{A}(y_0, \Delta_1, \dots, \Delta_n) , \\ \mathscr{A}' \equiv_D (y_0)\mathscr{A}_1 , \qquad W_1 = \begin{pmatrix} y_0 \\ [\Delta_0] \end{pmatrix} W , \end{cases}$$

where Δ_0 is any term of $\mathscr{T}$. By the inductive hypothesis

$$\operatorname{des}_{\mathscr{I}, W_1}(\mathscr{A}(y_0, \dots, y_n)^{\cdot}) = [\mathscr{A}(\Delta_0, \dots, \Delta_n)]$$

hence—see (13.1) and $(13.3)_4$—

$$\text{(13.4)} \qquad \varphi_{\mathscr{A}(y_0,\dots,y_n)^{\cdot}; y_0; \mathscr{I}, W}([\Delta_0]) = [\mathscr{A}(\Delta_0, \dots, \Delta_n)] .$$

By (12.4)

$$\text{(13.5)} \qquad g_{\mathscr{A}_1; y_0}([\Delta_0]) = [\mathscr{A}_1(\Delta_0)] = [\mathscr{A}(\Delta_0, \dots, \Delta_n)] .$$

Since Δ_0 is an arbitrary term of $\mathscr{T}$, by (13.4) and (13.5) the restriction of the functions $\varphi_{\mathscr{A}(y_0,\dots,y_n)^{\cdot}; y_0; \mathscr{I}, W}$ and $g_{\mathscr{A}_1; y_0}$ to $\mathbf{D} - \emptyset$ coincide. Furthermore, at $\emptyset$ both of them take the value $\emptyset$. Hence they coincide in $\mathbf{D}$, so that—see (i_4)—

$$\operatorname{des}_{\mathscr{I}, W}[(\Omega y_0)(\mathscr{A}(y_0, \dots, y_n))^{\cdot}] = \Omega^*(\varphi_{\mathscr{A}(y_0,\dots,y_n^{\cdot}); y_0; \mathscr{I}, W}) = \\ = \Omega^*(g_{\mathscr{A}_1; y_0}) = [(y_0)\mathscr{A}(y_0, \Delta_1, \dots, \Delta_n)] = [\hat{\Delta}(\Delta_1, \dots, \Delta_n)] .$$

Thus (13.2) holds in this case; and Step 3 is proved.

STEP 4. $\mathrm{des}_{\mathcal{J},W_1}(\dot{\Delta}) = [\Delta]$ *if* $W_1(x_i) = [x_i]$ $(i = 1, 2, \ldots)$. This is a simple corollary of Step 3.

One easily verifies that $\mathcal{J} = (\mathbf{D}, \mathcal{J})$ is a model for $\dot{\mathcal{T}}$.

Now, in order to complete the proof of the $\Leftarrow$-part of Theor. 12.1 we assume $a \mathscr{E} b$, so that (12.1) holds for every normal model $\boldsymbol{M}$ of $\dot{\mathcal{T}}$ and every $\boldsymbol{M}$-valuation.

Hence, for $\boldsymbol{M}$ equal to the above normal model $\mathcal{J}$ of $\dot{\mathcal{T}}$ and for $W = W_1$—see Step 4—, $\dot{a}^* = \dot{b}^*$. Furthermore, by Step 4, (13.2) holds for $\Delta = a$ and for $\Delta = b$: $\dot{a}^* = [a]$ and $\dot{b}^* = [b]$. Hence $[a] = [b]$ which by $(12.2)_2$ yield $a \asymp b$.

Thus the above completion has been performed, and also Theor. 12.1 is now completely proved. q.e.d.

Equality (12.1) is true in every normal model $\boldsymbol{M}$ of $\dot{\mathcal{T}}$ only if it is so every model $\boldsymbol{M}'$ (because $\boldsymbol{M}'$ can be contracted in an equivalent normal model); and by the completeness theorem for $\dot{\mathcal{T}}$—see [1], Theor. 3.4—this occurs iff $\vdash_{\dot{\mathcal{T}}} \dot{a} = \dot{b}$. Therefore Theor. 12.1 yields the following

THEOR. 13.1. *If a and b are wfes of $\mathcal{T}$, then*

α) $a \asymp b \Leftrightarrow \Vdash_{\dot{\mathcal{T}}} \dot{a} = \dot{b}$ (5) *and*

β) $a \asymp b \Leftrightarrow \vdash_{\dot{\mathcal{T}}} \dot{a} = \dot{b}$.

14 Proof of the assertion $\sim p \not\asymp \sim\sim\sim p$ made in [2], § 8.

We now prove assertion $(8.2)_3$ which, together with $(8.2)_2$—see [2]—shows that *the rather simple condition asserted by Theor.* 8.1 *in* [2] *to be sufficient for non-synonymy, is not necessary for this.*

We set $\mathcal{D}_1 = \mathbf{N}$ and consider a sequence $\{\mathcal{J}_\beta\}_{0<\beta<\omega}$ of functions that are defined (only) on the constants of $\dot{\mathcal{T}}_\beta$—see (ii) in [1], § 6—, have counterdomains in $\mathcal{D}_1$, and satisfy conditions i_1) to i_{16}) below. By these, if $0 < \beta < \gamma < \omega$ and a is a wfe of $\dot{\mathcal{T}}_\beta$ (so that $\mathrm{des}_{\mathcal{J}_\beta,V}(a)$ is meaningful for every mapping V of $\dot{\mathcal{T}}$'s variables into $\mathbf{N}$), then $\mathrm{des}_{\mathcal{J}_\gamma,V}(a) = \mathrm{des}_{\mathcal{J}_\alpha,V}(a)$, where $\mathcal{J}_\varrho = (\mathcal{D}_1, \mathcal{J}_\varrho)$ $(0 < \varrho < \omega)$. Therefore in conditions i_1) to i_{16}) $\mathcal{J}$ is written instead of $\mathcal{J}_\beta$. Furthermore it is

(5) « $\Vdash_{\dot{\mathcal{T}}} p$ » means $\mathrm{des}_{M',W'}\, p = 0$ at every model $\boldsymbol{M}'$ of $\dot{\mathcal{T}}$ and every $\boldsymbol{M}'$-valuation W'.

assumed that $\xi^* = \mathscr{J}(\xi)$, that p_i is the i-th prime number, and that m_1 to m_r run over N while β, n, m, r, and i run over $\mathrm{N} - \{0\}$.

i$_1$) $\dot{c}_i^* = 3^i$ if c_i is a primitive constant of $\mathscr{T}$.

i$_2$) $\dot{c}_i^* = 5^\beta$ if c_i is the constant of $\mathscr{T}$ defined by its β-th definition $\mathscr{D}_\beta$, i.e. $c_i \in S_{\mathscr{T}_\beta}$ and $c_i \notin S_{\mathscr{T}_\gamma}$ for $\gamma < \beta$.

i$_{3\text{-}4}$) $\dot{\sim}^* = 2 \cdot 7$, $\dot{\supset}^* = 2^2 \cdot 7^2$.

i$_5$) [i$_6$)] $\dot{R}_i^{n*} = 2^n \cdot 7^{i+2}$ [$\dot{f}_i^{n*} = 2^n \cdot 3^i$] if R_i^n [f_i^n] is a primitive predicative [functional] constant of $\mathscr{T}$.

i$_7$) [i$_8$)] $\dot{R}_i^{n*} = 2^n \cdot 11^\beta$ [$\dot{f}_i^{n*} = 2^n \cdot 5^\beta$] if R_i^n [f_i^n] is in $S_{\mathscr{T}_\beta}$ and is not in $S_{\mathscr{T}_\gamma}$ for $\gamma < \beta$—cf. i$_2$).

i$_9$) $V_n^*(2^n \cdot 3^i, m_1, \dots, m_n) = 2^n \cdot 3^i \cdot 5^{m_1} \dots p_{n+2}^{m_n}$.

i$_{10}$) $V_n^*(2^n \cdot 5^\beta, m_1, \dots, m_n) = 2^n \cdot 5^\beta \cdot 7^{m_1} \dots p_{n+3}^{m_n}$.

i$_{11}$) $A_n^*(2^n \cdot 7^i, m_1, \dots, m_n) = 2^n \cdot 7^i \cdot 11^{m_1} \dots p_{n+4}^{m_n}$.

i$_{12}$) $A_n^*(2^n \cdot 11^i, m_1, \dots, m_n) = \mathrm{des}_{\mathscr{J},V}\, D''_\beta$ where $V(x_i) = m_i$ for $1 \leqslant i \leqslant n$.

i$_{13}$) $$A_2^*(2^2 \cdot 7^3, n, n_0) = \begin{cases} \mathrm{des}_{\mathscr{J},V}\, D''_\beta \text{ where } V(x_i) = m_i \\ (i = 1, \dots, r) \text{ provided } n_0 = 2^m \cdot 5^\beta \cdot 7^{m_1} \dots \\ \dots p_{r+3}^{m_r} \text{ for some } m, \beta, r \text{ and } m_1 \text{ to } m_r; \\ 2^2 \cdot 7^3 \cdot 11^n \cdot 13^{n_0} \text{ otherwise}; \end{cases}$$

i$_{14}$) $=^*$ is the identity on N.

i$_{15}$) $(\Omega x_i)^*(\varphi) = k_\varphi$ where k is a function of φ (independent of i).

i$_{16}$) $\mathscr{V}^*$ is any subset $\mathscr{D}_0$ of $\mathscr{D}_1$.

Note that $2^2 \cdot 7^3$ is $\dot{=}^*$, because $=$ is the primitive predicative constant R_1^2. For the interpretation $\mathscr{J} = (\mathscr{J}, \mathscr{D}_1)$ of $\dot{\mathscr{T}}$ we have the following

THEOR. 14.1. *If* $a \asymp b$, *then* $\mathrm{des}_{\mathscr{J},V}\, \dot{a} = \mathrm{des}_{\mathscr{J},V}\, \dot{b}$ *for every* $\mathscr{J}$*-valuation* V.

Indeed let T be the equivalence relation among wffs of $\mathscr{T}$, for which $\Delta_1\, T\, \Delta_2$ iff, for all $\mathscr{J}$-valuation V, $\mathrm{des}_{\mathscr{J},V}\, \dot{\Delta}_1 = \mathrm{des}_{\mathscr{J},V}\, \dot{\Delta}_2$. Then T fulfils the conditions C$_1$) to C$_7$) in $\asymp$, written in [1] and used there to define $\asymp$. Hence, for the minimality property of $\asymp$, we have $\asymp \subseteq T$, and hence also the thesis. q.e.d.

COROLLARY 14.2. *$\sim p \not\asymp \sim\sim\sim p$, for every wff p of $\mathscr{T}$.*

Indeed assume $\sim p \asymp \sim\sim\sim p$, as an hypothesis for reduction ad absurdum. Then by Theor. 14.1, we have that

$$\mathrm{des}_{\mathscr{I},V} A_1(\dot{\sim}, \dot{p}) = \mathrm{des}_{\mathscr{I},V} A_1\big(\dot{\sim}, A_1(\dot{\sim}, A_1(\dot{\sim}, \dot{p}))\big)$$

hence, by i_3) and i_{11}), for $n = \mathrm{des}_{\mathscr{I},V}\,\dot{p}$ we arrive at the absurd result.

$$2\cdot 7\cdot 11^n = 2\cdot 7\cdot(11^{2\cdot 7\cdot(11^{2\cdot 7\cdot 11^n})})\,. \qquad \text{q.e.d}$$

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