RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

CONSTANTIN NĂSTĂSESCU
ŞERBAN RAIANU

Gabriel dimension of graded rings

Rendiconti del Seminario Matematico della Università di Padova, tome 71 (1984), p. 195-208

<http://www.numdam.org/item?id=RSMUP_1984__71__195_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 71 (1984)

Gabriel Dimension of Graded Rings.

CONSTANTIN NĂSTĂSESCU - ŞERBAN RAIANU (*)

One of the main problems in studying graded rings is to see whether a graded ring having a certain property has a similar property when regarded without grading. This problem has been attacked in [5], where the relation between the Krull dimension and the graded Krull dimension of a graded module is studied, among other properties.

The main goal of this paper is to give a relation between the Gabriel dimension and the graded Gabriel dimension of a graded module. We solve this problem completely in the commutative case and then apply the results to polynomial rings. We also add some remarks about the non-commutative case, in which the problem remains open.

1. Notation and preliminaries.

All rings considered in this paper will be commutative and unitary, unless explicitely mentioned otherwise. R will always denote such a ring and Mod-R will denote the category of all R-modules. When R will be supposed to be graded, this will mean that R is a graded ring of type $\mathbb{Z}$, and R-gr will denote the category of all graded R-modules.

We begin by recalling the notion of Gabriel dimension, introduced in [1] and then developped in [2]. One can define the following filtration on Mod-R, using transfinite recursion: denote by $(\text{Mod-}R)_0$ the

(*) Indirizzo degli AA.: Facultatea de Matematică, Str. Academiei 14, R 70109 Bucharest 1, Romania.

smallest localizing subcategory of Mod-R containing all simple R-modules.

If α is not a limit ordinal, $(\text{Mod-}R)_\alpha$ will denote the localizing subcategory of Mod-R such that the quotient category $(\text{Mod-}R)_\alpha / (\text{Mod-}R)_{\alpha-1}$ is the smallest localizing subcategory of $\text{Mod-}R/(\text{Mod-}R)_{\alpha-1}$ containing all simple objects.

If α is a limit ordinal, $(\text{Mod-}R)_\alpha$ will denote the smallest localizing subcategory of Mod-R containing all subcategories $(\text{Mod-}R)_\beta$, with $\beta < \alpha$. In this way, we obtain a transfinite sequence of localizing subcategories:

$$(\text{Mod-}R)_0 \subseteq (\text{Mod-}R)_1 \subseteq \ldots \subseteq (\text{Mod-}R)_\alpha \subseteq (\text{Mod-}R)_{\alpha+1} \subseteq \ldots .$$

If $M \in \text{Mod-}R$ is a module such that there exists an ordinal α, with $M \in (\text{Mod-}R)_\alpha$, we will say that the Gabriel dimension of M is defined. If this is the case, the least ordinal α for which $M \in (\text{Mod-}R)_\alpha$ will be called *the Gabriel dimension of* M and will be denoted by $\text{G.dim}(M)$.

From the definition of G.dim, it follows at once that if $(M_i)_{i\in I}$ is a family of modules having Gabriel dimension, then $M = \bigoplus_{i\in I} M_i$ has Gabriel dimension too, and we have:

$$\text{G.dim}(M) = \sup_{i\in I} \text{G.dim}(M_i) .$$

In particular, the above relation holds for $M = \bigcup_{i\in I} M_i$.

It is obvious that one can repeat the above construction in the graded case, obtaining thus the notion of *graded Gabriel dimension*, denoted in the sequel by gr-G.dim.

The set of all prime ideals of R (resp. graded prime ideals if R is graded) is denoted by $\text{Spec}\, R$ (resp. $\text{Spec}_g R$).

In [3] the following filtration on $\text{Spec}\, R$ is considered: $(\text{Spec}\, R)_0$ consists of all maximal ideals; if α is not a limit ordinal

$$(\text{Spec}\, R)_\alpha = \{P \in \text{Spec}\, R \mid \forall Q \in \text{Spec}\, R,\ P \subsetneq Q \Rightarrow Q \in (\text{Spec}\, R)_{\alpha-1}\} .$$

If α is a limit ordinal

$$(\text{Spec}\, R)_\alpha = \bigcup_{\beta<\alpha} (\text{Spec}\, R)_\beta .$$

It is clear that there exists an ordinal η such that

$$(\operatorname{Spec} R)_\eta = (\operatorname{Spec} R)_{\eta+1} = \dots .$$

If there exists an ordinal α such that $(\operatorname{Spec} R)_\alpha = \operatorname{Spec} R$ we say that the classical Krull dimension of R is defined. If this is the case, the least such ordinal is called the *classical Krull dimension of* R and is denoted by $\text{cl.K.dim}(R)$.

If R is a graded ring, we say that an ideal of R is *gr-maximal* if it is a maximal element in the set of all proper graded ideals of R. The following well-known lemma tells us more about gr-maximal ideals.

LEMMA 1.1. If P is a gr-maximal ideal, then P is prime and R/P is a graded field, i.e. $R/P \simeq K[X, X^{-1}]$ where K is a field and X is an indeterminate. In particular, R/P is a Noetherian principal ring, and hence of Krull dimension 1 (see [5] for a proof).

Now we can define as above a filtration on $\operatorname{Spec}_g R$ starting with $(\operatorname{Spec}_g R)_0 = \{P | P \text{ is gr-maximal}\}$, obtaining thus the notion of *graded classical Krull dimension*, denoted in the sequel by gr-cl.K.dim.

If $M \in \text{Mod-}R$, we will denote by $\operatorname{Ass}(M)$ the set

$$\operatorname{Ass}(M) = \{P \in \operatorname{Spec} R |\ \exists x \in M,\ x \neq 0,\ P = \operatorname{Ann}(x)\} .$$

It is well-known that if R is a graded ring and $M \in R\text{-gr}$ $\operatorname{Ass}(M)$ consists only of graded prime ideals [5].

In [3] it is proved the following

THEOREM 1.2. Let α be an ordinal and $M \in \text{Mod-}R$. The following assertions are equivalent:

1) $M \in (\text{Mod-}R)_\alpha$,

2) $\emptyset \neq \operatorname{Ass}(M/N) \subseteq (\operatorname{Spec} R)_\alpha$ for any $N \subsetneq M$.

A mere transcription for the graded case of the proof given in [3] enables us to state.

THEOREM 1.2′. If R is graded and α is an ordinal, the following assertions on $M \in R\text{-gr}$ are equivalent:

1) $M \in (R\text{-gr})_\alpha$,

2) $\emptyset \neq \operatorname{Ass}(M/N) \subseteq (\operatorname{Spec}_g R)_\alpha$ for any proper graded submodule N of M.

COROLLARY 1.3 [4]. Let R be graded and $M \in (R\text{-gr})_{\alpha_0}$. There exists an ordinal γ, and a filtration $(M_\alpha)_{\alpha<\gamma}$ with the following properties:

1) $M_0 = \sum_{\substack{x \in M \\ \operatorname{Ann}(x) \text{ prime}}} Rx = \sum_{\substack{x \in M \\ \operatorname{Ann}(x) \in (\operatorname{Spec}_g R)_{\alpha_0}}} Rx$.

2) If α is not a limit ordinal, then

$$M_\alpha/M_{\alpha-1} = \sum_{\substack{x \in M/M_{\alpha-1} \\ \operatorname{Ann}(x) \text{ prime}}} Rx = \sum_{\substack{x \in M/M_{\alpha-1} \\ \operatorname{Ann}(x) \in (\operatorname{Spec}_g R)_{\alpha_0}}} Rx .$$

If α is a limit ordinal, then

$$M_\alpha = \bigcup_{\beta<\alpha} M_\beta .$$

3) $M = \bigcup_{\alpha<\gamma} M_\alpha$.

COROLLARY 1.4. 1) If G.dim(R) exists, then cl.K.dim(R) also exists and G.dim$(R) =$ cl.K.dim(R).

2) If R is graded and gr-G.dim(R) exists then gr-cl.K.dim(R) also exists and gr-G.dim$(R) =$ gr-cl.K.dim(R).

PROOF. 1) Let $\alpha =$ G.dim(R). Then, for any $P \in \operatorname{Spec} R$, we have by Theorem 1.2 that $\{P\} = \operatorname{Ass}(R/P) \subseteq (\operatorname{Spec} R)_\alpha$. Hence $(\operatorname{Spec} R)_\alpha = \operatorname{Spec} R$. If there existed an ordinal β, $\beta < \alpha$, with $(\operatorname{Spec} R)_\beta = \operatorname{Spec} R$, then it would follow from Theorem 1.2 that $R \in (\text{Mod-}R)_\beta$, a contradiction. Thus G.dim$(R) =$ cl.K.dim(R).

2) Like 1), using Theorem 1.2′ instead of Theorem 1.2.

If R is graded and I is an ideal of R we denote by I_g the greatest graded ideal (w.r.t. inclusion) contained in I.

LEMMA 1.5. If P is prime, then P_g is graded prime and there is no prime between P_g and P (see [5] for a proof).

2. Gabriel dimension of graded modules.

Throughout this section R will denote a graded ring.

PROPOSITION 2.1. Let α be an ordinal. Then

$$(\mathrm{Spec}_g\, R)_\alpha \subseteq (\mathrm{Spec}\, R)_\alpha$$

if α is a limit ordinal, and

$$(\mathrm{Spec}_g\, R)_\alpha \subseteq (\mathrm{Spec}\, R)_{\beta+2n+1}$$

if $\alpha = \beta + n$, where $n \in \mathbf{N}$ and $\beta = 0$ or β is a limit ordinal and $n \neq 0$.

PROOF. Using transfinite recursion, we show first that $(\mathrm{Spec}_g\, R)_0 \subseteq \subseteq (\mathrm{Spec}\, R)_1$. To see this, let P be a maximal graded ideal and let Q, $Q' \in \mathrm{Spec}\, R$ such that $P \subsetneq Q \subseteq Q'$. Then $Q'_g = P$ and so $Q = Q'$ by Lemma 1.5, proving that Q is maximal.

Suppose now that the statement is true for all ordinals β, $\beta < \alpha$. If α is a limit ordinal, let $P \in (\mathrm{Spec}_g\, R)_\alpha = \bigcup_{\beta<\alpha} (\mathrm{Spec}_g\, R)_\beta$ and let β_0 be the least ordinal such that $P \in (\mathrm{Spec}_g\, R)_{\beta_0}$. Then $\beta_0 < \alpha$ is not a limit ordinal and so $\beta_0 = \beta'_0 + k$, where $\beta'_0 = 0$ or β'_0 is a limit ordinal and $k \neq 0$. Hence

$$P \in (\mathrm{Spec}_g\, R)_{\beta_0} \subseteq (\mathrm{Spec}\, R)_{\beta'_0+2k+1} \subseteq (\mathrm{Spec}\, R)_\alpha .$$

Now, if $\alpha = \beta + n$, β and n being as in the statement, we need to prove that $(\mathrm{Spec}_g\, R)_{\beta+n} \subseteq (\mathrm{Spec}\, R)_{\beta+2n+1}$ For this, let $P \in (\mathrm{Spec}_g\, R)_{\beta+n}$ and let Q, $Q' \in \mathrm{Spec}\, R$ such that $P \subsetneq Q \subsetneq Q'$. (We supposed that neither P nor Q are maximal.) We will show that $Q \in (\mathrm{Spec}\, R)_{\beta+2n}$. by proving that $Q' \in (\mathrm{Spec}\, R)_{\beta+2n-1}$. Indeed, by Lemma 1.5 $P \subsetneq Q'_g \subseteq Q'$. Hence $Q'_g \in (\mathrm{Spec}_g\, R)_{\beta+n-1} \subseteq (\mathrm{Spec}\, R)_{\beta+2n-1}$, and so $Q' \in (\mathrm{Spec}\, R)_{\beta+2n-1}$. Thus $Q \in (\mathrm{Spec}\, R)_{\beta+2n}$, i.e. we proved that $P \in (\mathrm{Spec}\, R)_{\beta+2n+1}$, and this completes the proof.

PROPOSITION 2.2. If R has limited grading and α is an ordinal, then

$$(\mathrm{Spec}_g\, R)_\alpha \subseteq (\mathrm{Spec}\, R)_\alpha$$

if α is a limit ordinal or $\alpha = 0$, and

$$(\mathrm{Spec}_g R)_\alpha \subseteq (\mathrm{Spec}\, R)_{\beta+2n-1}$$

if $\alpha = \beta + n$, where $n \in \mathbf{N}$, $n \neq 0$ and $\beta = 0$ or β is a limit ordinal.

PROOF. The same as the proof of Proposition 2.1. One uses for the proof of the case $\alpha = 0$ that if R has limited grading, then a gr-maximal ideal is maximal [5].

Now we are going to give a relation between the classical Krull dimension of R and the graded classical Krull dimension of R. We will use the following

LEMMA 2.3. If α is an ordinal, then

$$(\mathrm{Spec}\, R)_\alpha \cap \mathrm{Spec}_g R \subseteq (\mathrm{Spec}_g R)_\alpha .$$

PROOF. By transfinite recursion on α.

PROPOSITION 2.4. The following assertions hold:

1) If one of cl.K.dim(R) and gr-cl.K.dim(R) exists, then the other one also exists. Putting gr-cl.K.dim$(R) = \beta + n$, where $n \in \mathbf{N}$ and $\beta = 0$ or β is a limit ordinal, one has

$$\beta + n \leqslant \text{cl.K.dim}\,(R) \leqslant \beta + 2n + 1 .$$

2) If one of cl.K.dim(R) and gr-cl.K.dim(R) exists and is a limit ordinal, then

$$\text{gr-cl.K.dim}\,(R) = \text{cl.K.dim}\,(R) .$$

PROOF. 1) We suppose first that cl.K.dim(R) exists and show that gr-cl.K.dim(R) also exists and gr-cl.K.dim$(R) \leqslant$ cl.K.dim(R). To see this, it is enough to prove that if $(\mathrm{Spec}\, R)_\alpha = \mathrm{Spec}\, R$ for an ordinal α, then $(\mathrm{Spec}_g R)_\alpha = \mathrm{Spec}_g R$. Indeed, we have $\mathrm{Spec}_g R = (\mathrm{Spec}\, R)_\alpha \cap \mathrm{Spec}_g R \subseteq (\mathrm{Spec}_g R)_\alpha$ by Lemma 2.3.

Suppose now that $\alpha =$ gr-cl.K.dim(R) exists and $\alpha = \beta + n$, β and n as in the statement. We will show that $(\mathrm{Spec}\, R)_{\beta+2n+1} = \mathrm{Spec}\, R$. To see this, let $P \in \mathrm{Spec}\, R$. Then $P_g \in (\mathrm{Spec}_g R)_\alpha$ and so $P_g \in (\mathrm{Spec}\, R)_{\beta+2n+1}$ by Proposition 2.1. Hence $P \in (\mathrm{Spec}\, R)_{\beta+2n+1}$ and we are done.

2) Suppose first that $\alpha = \text{cl.K.dim}(R)$ is a limit ordinal. Suppose further that $\text{gr-cl.K.dim}(R) < \text{cl.K.dim}(R)$, i.e. $(\text{Spec}_g R)_\beta = \text{Spec}_g R$ for an ordinal β, $\beta < \alpha$. We put $\beta = \delta + n$, $n \in \mathbf{N}$, and $\delta = 0$ or δ is a limit ordinal, and let $P \in \text{Spec}\, R$. Now $P_g \in (\text{Spec}_g R)_\beta$ and so $P_g \in (\text{Spec}\, R)_{\delta+2n+1}$ by Proposition 2.1. Hence $P \in (\text{Spec}\, R)_{\delta+2n+1}$ and so we proved that $(\text{Spec}\, R)_{\delta+2n+1} = \text{Spec}\, R$. But since $\delta + 2n + 1 = \beta + n + 1$, $\beta < \alpha$ and α is a limit ordinal, it follows that $\delta + 2n + 1 < \alpha$, a contradiction. Thus $\text{gr-cl.K.dim}(R) = \text{cl.K.dim}(R)$, as required.

Suppose now that $\alpha = \text{gr-cl.K.dim}(R)$ is a limit ordinal. By 1) it is enough to show that $\text{cl.K.dim}(R) \leqslant \text{gr-cl.K.dim}(R)$. To see this, we prove that $(\text{Spec}\, R)_\alpha = \text{Spec}\, R$. Let $P \in \text{Spec}\, R$, then $P_g \in (\text{Spec}_g R)_\alpha$ and so $P_g \in (\text{Spec}\, R)_\alpha$ by Proposition 2.1. Hence $P \in (\text{Spec}\, R)_\alpha$ and this completes the proof.

We are now in a position to state the main result of this paper.

THEOREM 2.5. Let α be an ordinal and M a graded R-module such that $M \in (R\text{-gr})_\alpha$. Then

$$M \in (\text{Mod-}R)_\alpha$$

if α is a limit ordinal, and

$$M \in (\text{Mod-}R)_{\beta+2n+1}$$

if $\alpha = \beta + n$, where $n \in \mathbf{N}$ and $\beta = 0$ or β is a limit ordinal and $n \neq 0$.

PROOF. By transfinite recursion on α. The case $\alpha = 0$ may be achieved in an obvious manner by transfinite recursion on the gr-Loewy length of M. (Lemma 1.1 is used for the case M gr-semisimple.)

We suppose now that the statement is true for all ordinals β, $\beta < \alpha$, and prove it for α. By Corollary 1.3 we may suppose $M = R/P$, $P \in (\text{Spec}_g R)_\alpha$. Using Theorem 1.2, we must prove that for each ideal I, $P \subseteq I \subsetneqq R$. $\text{Ass}(R/I) \neq \emptyset$ and $\text{Ass}(R/I) \subseteq (\text{Spec}\, R)_\alpha$ if α is a limit ordinal, or $\text{Ass}(R/I) \subseteq (\text{Spec}\, R)_{\beta+2n+1}$ if $\alpha = \beta + n$, β and n as in the statement. By Proposition 2.1, it is sufficient to prove only that $\text{Ass}(R/I) \neq \emptyset$. Let I be a proper ideal of R. (We supposed that R is a graded domain having graded Gabriel dimension α.) If I contains a homogeneous element h, $h \neq 0$, then

$$\mathrm{Ass}_R(R/I) \neq \emptyset$$

($\mathrm{Ass}_{R/hR}(R/I) \neq \emptyset$ by the induction hypothesis, since gr-G.dim $(R/hR) <$ $<$ gr-G.dim (R)). Hence we may suppose that $I \cap S = \emptyset$, where

$$S = \{a \in R \mid a \text{ homogeneous and } a \neq 0\}\,.$$

Let

$$t_S(R/I) = \{x \in R/I \mid \mathrm{Ann}(x) \text{ contains a non-zero homogeneous element}\}$$

be the torsion submodule of R/I w.r.t. S. If $t_S(R/I) \leqslant 0$, let $x \in t_S(R/I)$, $x \neq 0$ and $h \in \mathrm{Ann}(x)$, $h \neq 0$ a homogeneous element. Then $\mathrm{Ass}_R(Rx) \neq \emptyset$ ($\mathrm{Ass}_{R/hr}(Rx) \neq \emptyset$ by the induction hypothesis, since gr-G.dim $(R/hR) <$ $<$ gr-G.dim (R)). If $t_S(R/I) = 0$, then R/I is a submodule of $S^{-1}(R/I)$. Since $S^{-1}R$ is a graded field, hence Noetherian (see Lemma 1.1) $\mathrm{Ass}_{S^{-1}R}(S^{-1}(R/I)) \neq \emptyset$, and it is straightforward to check that $\mathrm{Ass}_R(R/I) \neq \emptyset$.

Corollary 2.6. The following assertions on a graded module M hold:

1) If one of G.dim (M) and gr-G.dim (M) exists, then the other one also exists. Putting gr-G.dim $(M) = \beta + n$, where $n \in \mathbf{N}$ and $\beta = 0$ or β is a limit ordinal, one has

$$\beta + n \leqslant \mathrm{G.dim}(M) \leqslant \beta + 2n + 1\,.$$

2) If one of G.dim (M) and gr-G.dim (M) exists and is a limit ordinal, then

$$\text{gr-G.dim}(M) = \mathrm{G.dim}(M)\,.$$

Proof. 1) Suppose that $M \in (\text{Mod-}R)_\alpha$ where α is an ordinal. Let N be a graded proper submodule of M. Then

$$\emptyset \neq \mathrm{Ass}(M/M) \subseteq (\mathrm{Spec}\, R)_\alpha \cap \mathrm{Spec}_g R \subseteq (\mathrm{Spec}_g R)_\alpha$$

by Theorem 1.2 and Lemma 2.3 and hence $M \in (R\text{-gr})_\alpha$ by Theorem 1.2′. The rest follows at once from Theorem 2.5.

2) Suppose $\alpha = \text{gr-G.dim}(M)$ is a limit ordinal. Then $M \in (\text{Mod-}R)_\alpha$ by Theorem 2.5 and so $\text{gr-G.dim}(M) \geq \text{G.dim}(M)$. Hence $\text{gr-G.dim}(M) = \text{G.dim}(M)$ by 1). Now if $\delta = \text{G.dim}(M)$ is a limit ordinal, suppose that $\text{gr-G.dim}(M) < \delta$, i.e. $M \in (R\text{-gr})_{\beta+n}$ ($\beta = 0$ or β is a limit ordinal) with $\beta + n < \delta$. By Theorem 2.5 $M \in (\text{Mod-}R)_{\beta+2n+1}$. Since $\beta + 2n + 1 < \delta$ we have a contradiction, and so $\text{gr-G.dim}(M) = \text{G.dim}(M)$.

3. Application to polynomial rings.

Throughout this section R will denote a ring and $R[X]$ the ring of polynomials in one indeterminate over R. If $\mathfrak{p} \in \operatorname{Spec} R$, we will write $\mathfrak{p}^*$ for $\mathfrak{p}R[X]$. It is well known that if $P \in \operatorname{Spec}_g R[X]$, then either $P = \mathfrak{p}^*$ or $P = \mathfrak{p}^* + (X)$, where $\mathfrak{p} = R \cap P$ (see [5]).

PROPOSITION 3.1. Let α be an ordinal and $\mathfrak{p} \in \operatorname{Spec} R$. The following assertions are equivalent:

1) $\mathfrak{p}^* \in (\operatorname{Spec}_g R[X])_{\alpha+1}$,

2) $\mathfrak{p} \in (\operatorname{Spec} R)_\alpha$.

PROOF. 1) $\Rightarrow$ 2). By transfinite recursion on α. Suppose $\alpha = 0$ and let $\mathfrak{p}^* \in (\operatorname{Spec}_g R[X])_1$. If $\mathfrak{p}$ is not maximal, let $\mathfrak{q} \in \operatorname{Spec} R$, $\mathfrak{p} \subsetneq \mathfrak{q}$. Then we have the sequence $\mathfrak{p}^* \subsetneq \mathfrak{q}^* \subsetneq \mathfrak{q}^* + (X)$, contradicting the fact that $\mathfrak{q}^*$ must be a gr-maximal ideal of $R[X]$.

We suppose now that the assertion is true for all ordinals β, $\beta < \alpha$, and prove it for α. Let $\mathfrak{p}^* \in (\operatorname{Spec}_g R[X])_{\alpha+1}$ and $\mathfrak{q} \in \operatorname{Spec} R$ such that $\mathfrak{p} \subsetneq \mathfrak{q}$. If α is a limit ordinal, then $\mathfrak{q}^* \in (\operatorname{Spec}_g R[X])_\alpha = \bigcup_{\beta<\alpha} (\operatorname{Spec}_g R[X])_\beta$. Let β_0 be the least orinal β, $\beta < \alpha$, such that $\mathfrak{q}^* \in (\operatorname{Spec}_g R[X])_\beta$. Then β_0 is not a limit ordinal and $\beta_0 \neq 0$, hence $\mathfrak{q} \in (\operatorname{Spec} R)_{\beta_0-1}$ by the induction hypothesis. Since $\beta_0 - 1 < \alpha$, this proves that $\mathfrak{p} \in (\operatorname{Spec} R)_\alpha$.

If $\alpha = \beta + 1$, then $\mathfrak{q}^* \in (\operatorname{Spec}_g R[X])_{\beta+1}$ and so $\mathfrak{q} \in (\operatorname{Spec} R)_\beta$. Hence $\mathfrak{p} \in (\operatorname{Spec} R)_\alpha$ again and this finishes the proof.

2) $\Rightarrow$ 1). Again by transfinite recursion on α. Suppose first that $\alpha = 0$ and let $\mathfrak{p}$ be a maximal ideal of R. Then $R[X]/\mathfrak{p}^* \simeq (R/\mathfrak{p})[X]$ has Krull dimension 1, so that $\mathfrak{p}^* \in (\operatorname{Spec}_g R[X])_1$.

We suppose now that the assertion is true for all ordinals β, $\beta < \alpha$, and prove it for α. If α is a limit ordinal, then $\mathfrak{p} \in (\operatorname{Spec} R)_\alpha = \bigcup_{\beta<\alpha} (\operatorname{Spec} R)_\beta$. Ler $\beta < \alpha$ be such that $\mathfrak{p} \in (\operatorname{Spec} R)_\beta$. Then

$$\mathfrak{p}^* \in (\operatorname{Spec}_g R[X])_{\beta+1} \subseteq (\operatorname{Spec}_g R[X])_{\alpha+1} .$$

If $\alpha = \beta + 1$, let $Q \in \operatorname{Spec}_g R[X]$ such that $\mathfrak{p}^* \subsetneq Q$. Two cases arise:

a) If $Q = \mathfrak{q}^*$ then $\mathfrak{p} \subsetneq \mathfrak{q}$, and so $\mathfrak{q} \in (\operatorname{Spec} R)_\beta$. Hence $Q = \mathfrak{q}^* \in (\operatorname{Spec}_g R[X])_\alpha$.

b) If $Q = \mathfrak{q}^* + (X)$, and $\mathfrak{p} \subsetneq \mathfrak{q}$, then $\mathfrak{q}^* \in (\operatorname{Spec}_g R[X])_\alpha$ like in case *a*) and since $\mathfrak{q}^* \subsetneq Q$, it follows that $Q \in (\operatorname{Spec}_g R[X])_\alpha$ too. If $Q = \mathfrak{p}^* + (X)$, let $Q' \in \operatorname{Spec}_g R[X]$ such that $Q \subsetneq Q'$. We must prove that $Q' \in (\operatorname{Spec}_g R[X])_\beta$. Now $Q' = \mathfrak{r}^* + (X)$, where $\mathfrak{r} = R \cap Q'$ and $\mathfrak{p} \subsetneq \mathfrak{r}$. Hence $\mathfrak{r} \in (\operatorname{Spec} R)_\beta$ and so $\mathfrak{r}^* \in (\operatorname{Spec}_g R[X])_{\beta+1}$. Then, since $Q' \supsetneq \mathfrak{r}^*$, it follows that $Q' \in (\operatorname{Spec}_g R[X])_\beta$ and we are done.

COROLLARY 3.2. The following assertions on an ordinal α are equivalent:

1) $(\operatorname{Spec} R)_\alpha = \operatorname{Spec} R$.

2) $(\operatorname{Spec}_g R[X])_{\alpha+1} = \operatorname{Spec}_g R[X]$.

PROOF. 1) $\Rightarrow$ 2). Put $\mathfrak{q} = Q \cap R$. Then $\mathfrak{q} \in (\operatorname{Spec} R)_\alpha$ and so $\mathfrak{q}^* \in (\operatorname{Spec}_g R[X])_{\alpha+1}$ by Proposition 3.1. But $\mathfrak{q}^* \subseteq Q$ and so

$$Q \in (\operatorname{Spec}_g R[X])_{\alpha+1} \qquad \text{too} .$$

2) $\Rightarrow$ 1) follows directly from Proposition 3.1.

PROPOSITION 3.3. 1) gr-cl.K.dim $(R[X]) =$ cl.K.dim $(R) + 1$, if one of the two ordinals exists and neither of them is a limit ordinal.

2) gr-cl.K.dim $(R[X]) =$ cl.K.dim (R), if one of the two ordinals exists and is a limit ordinal.

PROOF. 1) Directly from Corollary 3.2.

2) Suppose first that $\alpha =$ gr-cl.K.dim $(R[X])$ is a limit ordinal. Let $\mathfrak{p} \in \operatorname{Spec} R$. Then $\mathfrak{p}^* \in (\operatorname{Spec}_g R[X])_\alpha = \bigcup_{\beta<\alpha} (\operatorname{Spec}_g R[X])_\beta$. Let β_0

be the least ordinal β, $\beta < \alpha$, such that $\mathfrak{p}^* \in (\mathrm{Spec}_g\, R[X])_\beta$. Then β_0 is not a limit ordinal and $\beta_0 \neq 0$. Thus, it follows from Proposition 3.1 that $\mathfrak{p} \in (\mathrm{Spec}\, R)_{\beta_0 - 1} \subseteq (\mathrm{Spec}\, R)_\alpha$, and so we proved that $(\mathrm{Spec}\, R)_\alpha = \mathrm{Spec}\, R$. Hence $\mathrm{cl.K.dim}\,(R) \leqslant \alpha$. If $\mathrm{cl.K.dim}\,(R) = \beta < \alpha$, then $\alpha \leqslant \beta + 1$ by Corollary 3.2, a contradiction.

Conversely, let $\alpha = \mathrm{cl.K.dim}\,(R)$ be a limit ordinal. If gr-cl.K.dim $(R[X])$ was not a limit ordinal, then it would be equal to $\alpha + 1$. We will show that $(\mathrm{Spec}_g\, R[X])_\alpha = \mathrm{Spec}_g\, R[X]$ and this will provide the desired contradiction. For this, let $Q \in \mathrm{Spec}_g R[X]$. Put $\mathfrak{q} = Q \cap R$. Then $\mathfrak{q} \in (\mathrm{Spec}\, R)_\alpha = \bigcup_{\beta < \alpha} (\mathrm{Spec}\, R)_\beta$. We choose $\beta < \alpha$ such that $\mathfrak{q} \in (\mathrm{Spec}\, R)_\beta$. Then $\mathfrak{q}^* \in (\mathrm{Spec}_g\, R[X])_{\beta+1} \subseteq (\mathrm{Spec}_g\, [RX])_\alpha$ by Proposition 3.1, and since $\mathfrak{q}^* \subseteq Q$ it follows that $Q \in (\mathrm{Spec}_g\, R[X])_\alpha$ too.

We are now in a position to give another proof for a result previously proved in [2] for the general case, and in [4] for the commutative case:

Proposition 3.4. 1) If one of $\mathrm{G.dim}\,(R)$ and $\mathrm{G.dim}\,(R[X])$ exists, then the other one also exists. If $\mathrm{G.dim}\,(R) = \beta + n$, $n \in \mathbb{N}$ and $\beta = 0$ or β is a limit ordinal and $n \neq 0$, then:

$$\beta + n + 1 \leqslant \mathrm{G.dim}\,(R[X]) \leqslant \beta + 2n + 1 .$$

2) If one $\mathrm{G.dim}\,(R)$ and $\mathrm{G.dim}\,(R[X])$ is a limit ordinal,

$$\mathrm{G.dim}\,(R) = \mathrm{G.dim}\,(R[X]) .$$

Proof. 1) Assume first that $\mathrm{G.dim}\,(R)$ exists. Then $\mathrm{G.dim}\,(R) = \mathrm{cl.K.dim}\,(R)$ by Corollary 1.4, and hence gr-cl.K.dim $(R[X])$ exists by Proposition 3.3. Thus, in order to prove the existence of $\mathrm{G.dim}\,(R[X])$, it is sufficient to show, by Corollary 2.6 and Theorem 1.2′, that $\mathrm{Ass}\,(R[X]/I) \neq \emptyset$ for any graded proper ideal I of $R[X]$. Now $I = I_0 \oplus I_1 X \oplus \ldots \oplus I_m X^m \oplus \ldots$, where the I_m are ideals of R and $I_0 \subseteq I_1 \subseteq \ldots \subseteq I_m \subseteq I_{m+1} \subseteq \ldots$: Two cases arise:

a) $I_0 = I_1 = \ldots = I_m = I_{m+1} \ldots$. Then let

$$\mathfrak{p} \in \mathrm{Ass}\,(R/I_0)\,, \qquad \mathfrak{p} = (I_0 : a)_R\,, \qquad a \in R\,.$$

It is easy to se that $\mathfrak{p}^* = (I : a)_{R[X]}$.

b) There exists $k \in \mathbf{N}$ such that $I_k \subsetneq I_{k+1}$. We pick then

$$\mathfrak{p} \in \operatorname{Ass}(I_{k+1}/I_k), \qquad \mathfrak{p} = (I_k : a)_R, \qquad a \in I_{k+1}.$$

It is straightforward to check that $\mathfrak{p}^* + (X) = (I : a)_{R[X]}$.

Now if G.dim $(R[X])$ exists, it is obvious that G.dim (R) also exists.

To prove the last part of the statement, let $\alpha =$ G.dim (R), $\alpha = \beta = n$, β and n as in the statement. Then $\alpha =$ cl.K.dim (R), and $\alpha + 1 = \beta + n + 1 =$ gr-cl.K.dim $(R[X]) =$ gr-G.dim $(R[X])$ by Proposition 3.3. Using now a result similar to Proposition 2.4, obtained for rings with limited grading by means of Proposition 2.2 we have

$$\beta + n + 1 \leqslant \text{G.dim}\,(R[X]) \leqslant \beta + 2(n+1) - 1 = \beta + 2n + 1\,.$$

This completes the proof.

2) If G.dim (R) is a limit ordinal, then

$$\text{G.dim}\,(R) = \text{cl.K.dim}\,(R) = \text{gr-cl.K.dim}\,(R[X]) =$$

$$= \text{gr-G.dim}\,(R[X]) = \text{G.dim}\,(R[X])$$

by Corollary 1.4, Proposition 3.3 and Corollary 2.6, and the same (reversed) argument may be used when G.dim $(R[X])$ is a limit ordinal.

REMARKS. 1) All evaluations obtained in Proposition 2.4, Corollary 2.6 and Proposition 3.4 are good, in the sense that each side can be effectively reached, as it is easy to see using in all cases the well-known examples of Seidenberg [6].

2) In the proof of Proposition 3.4 we showed the following general fact: if R has the property that Ass $(M) \neq \emptyset$ for all R-modules M, $M \neq 0$, then Ass $(N) \neq \emptyset$ for all graded $R[X]$-modules N, $N \neq 0$.

The following remarks might be useful to the study of the non-commutative case. From now on, R will be no longer supposed to be commutative.

3) Let $M \in R$-gr such that $\text{gr-G.dim}(M) = 0$. Then

$$\text{G.dim}(M) \leqslant 1 .$$

Indeed, using the Loewy series, we may suppose that M is gr-simple. Now from the structure of gr-simple modules it follows that M is simple or 1-critical (see Theorem 7.5 p. 61 of [5]). Hence $\text{K.dim}(M) \leqslant 1$ and so $\text{G.dim}(M) \leqslant 1$.

4) Let R be a graded ring and $M \in R$-gr,

$$R = \bigoplus_{i \in \mathbf{Z}} R_i \qquad \text{and} \qquad M = \bigoplus_{i \in \mathbf{Z}} M_i .$$

Assume that $\alpha = \text{gr-G.dim}_R(M)$ exists. Then $\text{G.dim}_{R_0}(M_i) \leqslant \alpha$ for all $i \in \mathbf{Z}$.

This can be proved by transfinite recursion on α. If $\alpha = 0$, then using the Loewy series we may suppose that M is gr-simple. But in this case, either $M_i = 0$ or M_i is a simple R_0-module for all i.

We suppose now that the assertion is true for all graded R-modules N with $\text{gr-G.dim}_R(N) < \alpha$. It is easy to see that we may suppose that M is α-simple (see [2] for the definition). Let $N_i \subseteq M_i$, $N_i \neq 0$. Then $N_i = M_i \cap RN_i$. Now M being α-simple implies that $\text{gr-G.dim}_R(M/RN_i) < \alpha$, and so $\text{G.dim}_{R_0}\big((M/RN_i)_i\big) < \alpha$ by the induction hypothesis. But $(M/RN_i)_i = M_i/N_i$ and so $\text{G.dim}_{R_0}(M_i/N_i) < \alpha$. Hence $\text{G.dim}_{R_0}(M_i) \leqslant \alpha$.

5) Let R and M be as in 4). Put $M^+ = \bigoplus_{i \geqslant 0} M_i$, $M^- = \bigoplus_{i \leqslant 0} M_i$. M^+ is a graded R^+-module and M^- is a graded R^--module. Assume that $\text{gr-G.dim}_R(M) = \alpha$. Then

$$\text{gr-G.dim}_{R^+}(M^+) \leqslant \alpha + 1 \qquad \text{and} \qquad \text{gr-G.dim}_{R^-}(M^-) \leqslant \alpha + 1 .$$

To see this, one uses transfinite recursion to reduce the problem to the case when M is gr-α-simple. The rest follows as in Lemma 4.11 p. 50 of [5].

REFERENCES

[1] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France, **90** (1962), pp. 323-448.
[2] R. Gordon - J. C. Robson, *The Gabriel dimension of a module*, J. Algebra, **29** (1974), pp. 459-473.
[3] C. Năstăsescu, *La filtration de Gabriel*, Ann. Scuola Norm. Sup. Pisa, **27** (1973), pp. 457-470.
[4] C. Năstăsescu, *La filtrazione di Gabriel II*, Rend. Sem. Mat. Univ. Padova, **50** (1973), pp. 189-195.
[5] C. Năstăsescu - F. Van Oystaeyen, *Graded and filtered rings and modules*, Lect. Notes in Math., 758, Springer-Verlag, 1979.
[6] A. Seidenberg, *On the dimension theory of rings II*, Pacific J. Math., **4** (1954), pp. 603-615.

Manoscritto pervenuto in redazione il 23 luglio 1982; in edizione riveduta il 15 ottobre 1982.