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LADISLAV BICAN

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Factor-Splitting Length of Torsionfree Abelian Groups of Rank Two.

LADISLAV BICAN (*)

The factor-splitting length of a torsionfree abelian group is defined as the supremum of the splitting lengths of all its factor-groups. Describing the splitting length of a given factor-group we present a characterization of the splitting length of a rank two torsionfree group in terms of its bases. For every positive integer n there is a rank two torsionfree group G having the factor-splitting length n and for each $k \leq n$ there is a factor-group of G with the splitting length k . If the factor-splitting length of a given rank two group G is infinite then two cases can occur: Either G has a factor-group of infinite splitting length and factor-groups of all finite splitting lengths, or G has a factor-group of infinite splitting length and factor-groups of finite splitting lengths up to some n , only.

By the word «group» we shall always mean an additively written abelian group. As in [1], we use the notions «characteristic» and «type» in the broad meaning, i.e. we deal with these notions in mixed groups. The symbols $h_p^g(g)$, $\tau^g(g)$ and $\hat{\tau}^g(g)$ denote respectively the p -height, the characteristic and the type of the element g in the group G . π will denote the set of all primes. If $\pi' \subseteq \pi$ and if G is a torsionfree group then for each subset $M \subseteq G$ the symbol $\langle M \rangle_{\pi'}^g$ denotes the π' -pure closure of M in G . Any maximal linearly independent set of elements of a torsionfree group G is called a basis. The set

(*) Indirizzo dell'A.: MFF UK, Sokolovská 83, 18600 Praha 8-Karlín, Cecoslovacchia.

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of all positive integers is denoted by \mathbb{N} , $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Other notation will be essentially that as in [7].

For a mixed group G with the torsion part $T = T(G)$ we denote by \bar{G} the factor-group G/T and for $g \in G$ \bar{g} is the element $g + T$ of \bar{G} . The rank of a mixed group G is that of \bar{G} .

LEMMA 1 ([1; Theorem 2]): *A mixed group G of rank one splits if and only if each element $g \in G \setminus T(G)$ has a non-zero multiple mg such that $\hat{\tau}^g(mg) = \hat{\tau}^{\bar{g}}$ and mg has p -sequence in G whenever $h_p^g(\bar{g}) = \infty$ (i.e. there exist elements $h_0^{(v)} = mg, h_1^{(v)}, \dots$ such that $ph_{n+1}^{(v)} = h_n^{(v)}, n = 0, 1, \dots$).*

Recall [5], that the p -height sequence of an element g of a mixed group G is the double sequence $\{k_i, l_i\}_{i=0}^\infty$ of elements of $\mathbb{N}_0 \cup \{\infty\}$ defined inductively in the following way: Put $k_1 = k_0 = l_0 = 0$ and $l_1 = h_p^g(g)$. If k_i, l_i are defined and either $h_p^g(p^{k_i}g) = l_i = \infty$, or $l_i < \infty$ and $h_p^g(p^{k_i+k}g) = l_i + k$ for all $k \in \mathbb{N}$ then put $k_{i+1} = k_i$ and $l_{i+1} = l_i$. If $l_i < \infty$ and there are $k \in \mathbb{N}$ with $h_p(p^{k_i+k}g) > l_i + k$ then let k_{i+1} be the smallest positive integer for which $h_p^g(p^{k_{i+1}}g) = l_{i+1} > l_i + k_{i+1} - k_i$.

Let p be a prime and n an integer, $n > 1$. We say that an element g of a mixed group G has the (p, n) -property if for its p -height sequence $\{k_i, l_i\}_{i=0}^\infty$ the sequence $\{(n-1)(l_i - k_i) - k_{i+1}\}_{i=0}^\infty$ has non-negative elements and $\lim_{i \rightarrow \infty} \{(n-1)(l_i - k_i) - k_{i+1}\} = nh_p^g(\bar{g}) - \lim_{i \rightarrow \infty} l_i$, where we put $\infty - m = \infty$ for every $m \in \mathbb{N}_0 \cup \{\infty\}$.

Recall [10], that the splitting length of a mixed group G is the infimum of the set of all positive integers n such that the n -th tensor power $G^n = G \otimes G \otimes \dots \otimes G$ splits.

LEMMA 2 ([5; Theorem]): *A mixed group G of rank one has the splitting length $n > 1$ if and only if G does not split and $n > 1$ is the smallest integer such that every element $g \in G \setminus T(G)$ has a non-zero multiple mg which has the (p, n) -property for every prime p .*

DEFINITION 3: Let H be a subgroup of a torsionfree group G . For each prime p we define the p -height $h_p^g(H)$ of H in G as the minimum of the p -heights of all elements of H in G , i.e. $h_p^g(H) = \min \{h_p^g(h) | h \in H\}$.

LEMMA 4: *Let H be a rank one subgroup of a torsionfree group G , $S = \langle H \rangle_x^g$, $g \in G \setminus S$. If $h_p^g(g) \geq h_p^g(H)$ then $h_p^{g|S}(g + H) = h_p^{g|S}(g + S)$.*

PROOF: Without loss of generality we can assume that $h_p^g(g) = l < \infty$. Denoting $h_p^g(H) = r$ we can choose an element $h \in H$ with $h_p^g(h) = r$. If the equation $p^k x = g + s$, $k > l$, is solvable in G , then $ah = bs$ for some relatively prime integers a, b . From $p|b$ we get $h_p^g(s) < h_p^g(ah) = h_p^g(h) = r \leq h_p^g(g)$ — a contradiction. Hence there are integers u, v with $p^k u + bv = 1$ and $p^k(bvx + ug) = bvg + bvs + p^k u g = g + avh$, from which the assertion follows.

LEMMA 5: Let H be a rank one subgroup of a torsionfree group G , $S = \langle H \rangle_\pi^g$, $g \in G \setminus S$. If $h_p^{g/H}(p^k g + H) = h_p^g(g) + k$ for every $k \in \mathbb{N}_0$ then $h_p^{g/H}(g + H) = h_p^{g/S}(g + S)$. The converse holds provided $h_p^g(g) < h_p^g(H)$.

PROOF: Without loss of generality we can assume that $h_p^g(g) = l < \infty$. If the equation $p^r x = g + s$, $s \in S$, $r > l$, is solvable in G , then $mp^k s = h \in H$ for some $k \in \mathbb{N}_0$ and $(m, p) = 1$. Then $mp^{k+r} x = mp^k g + h$ and $k + r \leq h_p^g(mp^k g + h) \leq h_p^{g/H}(mp^k g + H) = l + k$ which contradicts the inequality $l < r$.

Conversely, by the hypothesis we have $h_p^{g/H}(g + H) = h_p^g(g) = l$. Let the equation $p^{r+l+1} x = p^r g + h$, $h \in H$, be solvable in G . Then $p^r(p^{l+1} x - g) = h$, hence $p^{l+1} x - g = s \in S$ and $h_p^{g/S}(g + S) \geq l + 1$, which contradicts the hypothesis.

LEMMA 6: Let H be a rank one subgroup of a torsionfree group G of rank two, $S = \langle H \rangle_\pi^g$, $g \in G \setminus S$. If $(\langle g \rangle_\pi^g \oplus S) \otimes Z_p = G \otimes Z_p$ then $h_p^{g/H}(g + H) = h_p^{g/S}(g + S)$. The converse holds provided $h_p^g(g) < h_p^g(H)$.

PROOF: We can suppose that $h_p^g(g) = l < \infty$. If the equation $p^k x = g + s$ is solvable in G then $p^k(x \otimes 1) = g \otimes 1 + s \otimes 1$. If $p^l g' = g$ then from the hypothesis it easily follows that $p^k(g' \otimes \alpha) = g \otimes 1$ for some $\alpha \in Z_p$. From this and from $p^l(g' \otimes 1) = g \otimes 1$ we get $g' \otimes (p^k \alpha - p^l) = 0$, hence $p^k \alpha = p^l$; $k \leq l$, from which the direct part follows immediately.

Suppose that $h_p^g(g) < h_p^g(H) < \infty$, the case $h_p^g(H) = \infty$ being trivial. Choose $s \in S$ with $h_p^g(s) = 0$ and take an arbitrary element $x \otimes m/n \in G \otimes Z_p$. Then $kp^r x = ug + vs$ for some $r \in \mathbb{N}_0$, $(k, p) = (kp^r, u, v) = 1$. For $r = 0$ we have $x \otimes m/n = g \otimes um/kn + s \otimes vm/kn \in (\langle g \rangle_\pi^g \oplus S) \otimes Z_p$. If $r > 0$ then $p|u$ yields $p|vs$, hence $p|v$ which is impossible. Thus $ua + p^r b = 1$ for some $a, b \in \mathbb{N}$ and consequently $p^r(kax + bg) = g + vas$. So, $r \leq h_p^g(g + vas) \leq h_p^{g/S}(g + S) = h_p^g(g)$ and from $p^r g' = g$, $kp^r x = up^r g' + vs$ we get $v = p^r v'$ and consequently $lx = ug' + v's$, from which the assertion follows.

LEMMA 7: *Let H be a rank one subgroup of a torsionfree group G , $S = \langle H \rangle_\pi^g$, $g \in G \setminus S$, and let $\{k_i, l_i\}_{i=0}^\infty$ be the p -height sequence of the element $g + H$ in G/H . Then:*

- (i) *If $h_p^{g/H}(g + H) < h_p^{g/S}(g + S)$ then $h_p^g(H) - h_p^g(g) = k_2 = k_3 = \dots$, $l_1 = h_p^g(g)$ and $h_p^g(H) + \alpha = l_2 = l_3 = \dots$, $\alpha \in \mathbb{N} \cup \{\infty\}$;*
- (ii) *If $h_p^{g/H}(g + H) = h_p^{g/S}(g + S)$ then $0 = k_0 = k_1 = \dots$ and $h_p^{g/H}(g + H) = l_1 = l_2 = \dots$.*

PROOF: (i) By Lemma 4 we have $h_p^g(g) < h_p^g(H) < \infty$. Putting $k_2 = h_p^g(H) - h_p^g(g)$ we obviously get $h_p^{g/H}(p^r + H) = h_p^g(g) + r$ for each $r \in \mathbb{N}_0$, $r < k_2$. Further, again by Lemma 4, $h_p^{g/H}(p^{k_2}g + H) = h_p^{g/S}(p^{k_2}g + S) = h_p^{g/S}(g + S) + k_2 > h_p^{g/H}(g + H) + k_2$ and finally

$$\begin{aligned} h_p^{g/H}(p^{k_2+r}g + H) &= h_p^{g/S}(p^{k_2+r}g + S) = h_p^{g/S}(p^{k_2}g + S) + r = \\ &= h_p^{g/H}(p^{k_2}g + H) + r. \end{aligned}$$

(ii) For each $r \in \mathbb{N}$ we have

$$\begin{aligned} h_p^{g/H}(p^r g + H) &\leq h_p^{g/S}(p^r g + S) = h_p^{g/S}(g + S) + r = \\ &= h_p^{g/H}(g + H) + r \leq h_p^{g/H}(p^r g + H). \end{aligned}$$

PROPOSITION 8: *Let H be a rank one subgroup of a torsionfree group G of rank two, $S = \langle H \rangle_\pi^g$. The following conditions are equivalent:*

- (i) *The factor-group G/H splits;*
- (ii) *For every $g \in G \setminus S$ and for almost all primes p with $h_p^g(g) < h_p^g(H) < \infty$ it is $h_p^{g/H}(p^k g + H) = h_p^g(g) + k$ for each $k \in \mathbb{N}_0$;*
- (iii) *There is $g \in G \setminus S$ such that for all primes p with $h_p^g(g) < h_p^g(H) < \infty$ it is $h_p^{g/H}(p^k g + H) = h_p^g(g) + k$ for each $k \in \mathbb{N}_0$;*
- (iv) *There is $g \in G \setminus S$ such that for all primes p with $h_p^g(g) < h_p^g(H) < \infty$ it is $h_p^{g/H}(p^k g + H) = h_p^g(g) + k$ for $k = h_p^g(H) - h_p^g(g)$.*

PROOF: (i) \Rightarrow (ii). Since S/H is the torsion part of G/H , it follows from Lemma 1 that for almost all primes p it is $h_p^{g/H}(g + H) = h_p^{g/S}(g + S)$ and it suffices to use Lemma 5.

(ii) \Rightarrow (iii). If p_1, p_2, \dots, p_n are all primes with $h_{p_i}^g(g) < h_{p_i}^g(H) < \infty$ for which the equality in question do not hold, consider the element $p_1^{l_1} p_2^{l_2} \dots p_n^{l_n} g$ where $l_i = h_{p_i}^g(H) - h_{p_i}^g(g)$, $i = 1, 2, \dots, n$.

(iii) \Rightarrow (iv). Obvious.

(iv) \Rightarrow (i). For each prime p with $h_p^g(g) < h_p^g(H) < \infty$ and for $k = h_p^g(H) - h_p^g(g)$ we have by Lemma 4 and the hypothesis

$$\begin{aligned} h_p^{g/H}(g + H) &= h_p^g(g) + k - k = h_p^{g/H}(p^k g + H) - k = \\ &= h_p^{g/S}(p^k g + S) - k = h_p^{g/S}(g + S). \end{aligned}$$

Since the same equality holds by Lemma 4 for all other primes, it remains with respect to Lemma 1 to show that for each prime p with $h_p^{g/S}(g + S) = \infty$ the element $g + H$ has a p -sequence in G/H .

The p -primary component of S/H is obviously either a cyclic group $C(p^r)$ or a quasicyclic group $C(p^\infty)$. In the either case put $g_0 = g$ and assume that g_0, g_1, \dots, g_n are such elements that

$$p(g_{i+1} + H) = g_i + H, \quad i = 0, 1, \dots, n-1,$$

and that each $g_i + H$ is of infinite p -height in G/H . Thus for each $s \in \mathbb{N}$ there is $g^{(s)} \in G$ and $h_s \in H$ with $p^{r+s} g^{(s)} = g_n + h_s$. Now $p^{r+1}(p^{s-1} g^{(s)} - g^{(1)}) = h_s - h_1$ and so

$$p^{s-1} g^{(s)} - g^{(1)} \in S \quad \text{and} \quad p^r(p^{s-1} g^{(s)} - g^{(1)}) \in H.$$

Setting $g_{n+1} = p^r g^{(1)}$ we get $p(g_{n+1} + H) = p^{r+1} g^{(1)} + H = g_n + H$ and $p^{r+s-1}(g^{(s)} + H) = p^r(p^{s-1} g^{(s)} - g^{(1)} + g^{(1)} + H) = p^r g^{(1)} + H = g_{n+1} + H$.

In the respective case the factor-group S/H is p -divisible. Again, put $g_0 = g$ and assume that g_0, g_1, \dots, g_n are such elements that

$$p(g_{i+1} + H) = g_i + H, \quad i = 0, 1, \dots, n-1.$$

Since G/S is p -divisible, there are elements $g'_{n+1} \in G$ and $s \in S$ such that $pg'_{n+1} = g_n - s$. Further, $ps' = s + h$ for suitable elements $s' \in S$, $h \in H$. Setting $g_{n+1} = g'_{n+1} + s'$ we get $p(g_{n+1} + H) = g_n + H$ and the proof is complete.

THEOREM 9: *Let H be a rank one subgroup of a torsionfree group G of rank two, $S = \langle H \rangle_\pi^a$. The following conditions are equivalent:*

- (i) G/H has the splitting length $n \geq 1$;
- (ii) n is the smallest integer such that for each element $g \in G \setminus S$ the inequality $nh_p^g(g) \geq h_p^g(H)$ holds for almost all primes p with $h_p^{g/H}(g + H) < h_p^{g/S}(g + S)$;

- (iii) $G \setminus S$ contains an element g such that n is the smallest integer satisfying $nh_p^g(g) \geq h_p^g(H)$ for all primes p with $h_p^{g/H}(g + H) < h_p^{g/S}(g + S)$.

PROOF: (i) \Rightarrow (ii). Let $g \in G \setminus S$ be arbitrary, $n \geq 2$. It follows from Lemma 2 that $g + H$ has the (p, n) -property for almost all primes p . If p is such a prime with $h_p^{g/H}(g + H) < h_p^{g/S}(g + S)$ then Lemma 7(i) yields $(n-1)(l_1 - k_1) - k_2 \geq 0$, i.e. $nh_p^g(g) \geq h_p^g(H)$. If $n = 1$ then $h_p^{g/H}(g + H) = h_p^{g/S}(g + S)$ for almost all primes p by Lemma 1 and condition (ii) is trivially satisfied.

(ii) \Rightarrow (iii). If p_1, p_2, \dots, p_n are all primes with $h_p^{g/H}(g + H) < h_p^{g/S}(g + S)$ and $nh_p^g(g) < h_p^g(H)$ consider the element $g' = p_1^{l_1} p_2^{l_2} \dots p_n^{l_n} g$ where $l_i = h_{p_i}^g(H) - h_{p_i}^g(g)$, $i = 1, 2, \dots, n$. Then for each p_i $h_{p_i}^{g'}(g') = h_{p_i}^g(H)$ so that $h_{p_i}^{g'/H}(g' + H) = h_{p_i}^{g'/S}(g' + S)$ by Lemma 4, while for the remaining primes in question the inequality considered remains valid.

(iii) \Rightarrow (i). For $n = 1$ the inequality $h_p^g(g) \geq h_p^g(H)$ yields the equality $h_p^{g/H}(g + H) = h_p^{g/S}(g + S)$ so that this equality holds for each prime p . If $h_p^g(g) < h_p^g(H) < \infty$ then Lemma 5 gives $h_p^{g/H}(p^k g + H) = h_p^g(g) + k$ for each $k \in \mathbb{N}_0$ and G/H splits by Proposition 8.

Let $n > 1$. If we put $\pi_1 = \{p \in \pi \mid h_p^{g/H}(g + H) < h_p^{g/S}(g + S)\}$ and $\pi_2 = \pi \setminus \pi_1$ then the set π_1 is infinite (for otherwise G/H splits and $n = 1$ by the first part of this proof). For each prime $p \in \pi_1$ the p -height sequence of the element $g + H$ is given by Lemma 7(i). Then

$$(n-1)(l_1 - k_1) - k_2 = nh_p^g(g) - h_p^g(H) \geq 0,$$

$$\begin{aligned} (n-1)(l_2 - k_2) - k_3 &= (n-1)(h_p^g(g) + \alpha) - h_p^g(H) + h_p^g(g) = \\ &= nh_p^g(g) - h_p^g(H) + (n-1)\alpha \geq 0 \end{aligned}$$

and by the hypothesis n is the smallest integer for which the element $g + H$ has the (p, n) -property for all primes $p \in \pi_1$. Concerning the primes $p \in \pi_2$ the p -height sequence of $g + H$ is given by Lemma 7(ii) and $g + H$ has the (p, n) -property trivially. Thus G/H is of the splitting length n by Lemma 2.

For the sake of brevity we shall denote the splitting length of a group G by $\text{sl}(G)$.

COROLLARY 10: *Let G be a torsionfree group of rank two and $H_1 \subseteq H_2$ be rank one subgroups. Then $\text{sl}(G/H_2) \leq \text{sl}(G/H_1)$.*

PROOF: Since $H_1 \subseteq H_2$, $h_p^g(H_2) \leq h_p^g(H_1)$ and for $\text{sl}(G/H_1) < \infty$ it suffices to use Theorem 9, while for $\text{sl}(G/H_1) = \infty$ the assertion is trivial.

THEOREM 11: *Let G be a torsionfree group of rank two and H be its rank one subgroup such that $\text{sl}(G/H) = n < \infty$. Then for each $k \in \mathbb{N}$, $k \leq n$, there is a rank one subgroup K of G with $\text{sl}(G/K) = k$. Moreover, K can be selected such that $H \subseteq K$.*

PROOF: Put $S = \langle H \rangle_\pi^g$ and $\pi' = \{p \in \pi \mid h_p^{g/H}(g + H) < h_p^{g/S}(g + S)\}$. By Theorem 9 we can choose an element $g \in G \setminus S$ in such a way that $nh_p^g(g) \geq h_p^g(H)$ for each $p \in \pi'$. For $k = 1$ it suffices to put $K = S$ (or take some subgroup of finite index in S). So, suppose $n > k \geq 2$ and let $p \in \pi'$ be arbitrary. Denoting $h_p^g(H) = s_p$ define $l_p \in \mathbb{N}_0$ to be the smallest integer with $l_p + kh_p^g(g) \geq s_p$.

First show, that $h_p^g(g) < s_p - l_p$. By the choice of l_p we have $l_p - 1 + kh_p^g(g) < s_p$. Now the assumption $h_p^g(g) \geq s_p - l_p$ yields $(k - 1) \cdot h_p^g(g) < 1$ - a contradiction.

By hypothesis, for each prime $p \in \pi'$ there is $y_p \in H$ with $h_p^g(y_p) = s_p$ and with respect to the preceding item there is $x_p \in G$ with $p^{l_p}x_p = y_p$. Now consider the group $K = \langle H \cup \{x_p \mid p \in \pi'\} \rangle$.

Let $p \in \pi'$ be arbitrary and let $y \in K$ be such that $h_p^g(y) = r < s_p - l_p$. Then

$$y = h + \alpha_p x_p + \sum_{i=1}^n \alpha_{p_i} x_{p_i}, \quad h \in H, p_i \in \pi'$$

pairwise different $p_i \neq p$, and $\rho h = \sigma y_p$ for some integers $(\rho, \sigma) = 1$. For $m = p^{l_p} p_1^{l_1} p_2^{l_2} \dots p_n^{l_n} = p^{l_p} \bar{m} = p_i^{l_i} m_i$, where $l_i = l_{p_i}$, we have

$$\rho m y = m \sigma y_p + \bar{m} \rho \alpha_p y_p + \sum_{i=1}^n m_i \rho \alpha_{p_i} y_{p_i} \in H.$$

Hence $h_p(\rho) = h_p^g(\rho m y) - l_p - r \geq s_p - l_p - r > 0$ and so $h_p(\sigma) = 0$. On the other hand,

$$h_p^g(h) = h_p^g(\rho h) - h_p(\rho) = h_p^g(\sigma y_p) - h_p(\rho) = s_p - h_p(\rho) < s_p.$$

This contradiction shows that $h_p^g(K) \geq s_p - l_p$ and, in fact, $h_p(K) = s_p - l_p$ since $h_p^g(x_k) = s_p - l_p$.

For each $p \in \pi \setminus \pi'$ it is $h_p^{g/H}(g + H) \leq h_p^{g/K}(g + K) \leq h_p^{g/S}(g + S) = h_p^{g/H}(g + H)$ and for each prime $p \in \pi'$ the preceding parts give

$h_p^g(g) < h_p^g(K)$, so $h_p^{g/K}(g + K) = h_p^g(g) = h_p^{g/H}(g + H) < h_p^{g/S}(g + S)$, and $h_p^g(g) \geq h_p^g(K)$. To finish the proof it suffices now to show that k is the smallest such integer and to apply Theorem 9.

By hypothesis and Theorem 9 there are infinitely many primes $p \in \pi'$ with $(n-1)h_p^g(g) < h_p^g(H) = s_p$. Suppose now that for such a prime p it is $(k-1)h_p^g(g) \geq h_p^g(K)$. Then

$$l_p + (k-1)h_p^g(g) \geq s_p > (n-1)h_p^g(g),$$

hence $l_p > (n-k)h_p^g(g)$ and consequently

$$l_p - 1 + kh_p^g(g) \geq (n-k)h_p^g(g) + kh_p^g(g) = nh_p^g(g) \geq h_p^g(H) = s_p$$

which contradicts the choice of l_p .

DEFINITION 12: The factor-splitting length, $\text{fsl}(G)$, of a torsion-free group G is defined to be $\sup \{\text{sl}(G/H) \mid H \text{ a subgroup of } G\}$.

COROLLARY 13: *If a torsionfree group G of rank two has the factor-splitting length $n < \infty$ then for each $k \in \mathbb{N}$, $k \leq n$, there is a homomorphic image of G having the splitting length k .*

PROOF: By the definition there is a (rank one) subgroup H of G with $\text{sl}(G/H) = n$ and it suffices to use Theorem 11.

REMARK 14: If G is a torsionfree group of rank two then it follows from Corollary 10 that $\text{fsl}(G) = \sup \{\text{sl}(G/\langle g \rangle) \mid 0 \neq g \in G\}$.

THEOREM 15: *A torsionfree group G of rank two has the factor-splitting length $n \geq 1$ if and only if n is the smallest integer such that for each basis $\{g, h\}$ of G and for almost all primes p with $h_p^g(g) \neq h_p^g(h)$ and $(\langle g \rangle_\pi^g \oplus \langle h \rangle_\pi^g) \otimes \mathbb{Z}_p \subsetneq G \otimes \mathbb{Z}_p$ it is $1/n < h_p^g(h)/h_p^g(g) < n$.*

PROOF: Assume first that $\text{fsl}(G) = n$. If $\{g, h\}$ is an arbitrary basis of G we put

$$\pi_1 = \{p \in \pi \mid h_p^g(g) < h_p^g(h), (\langle g \rangle_\pi^g \oplus \langle h \rangle_\pi^g) \otimes \mathbb{Z}_p \subsetneq G \otimes \mathbb{Z}_p\}$$

and

$$\pi_2 = \{p \in \pi \mid h_p^g(h) < h_p^g(g), (\langle g \rangle_\pi^g \oplus \langle h \rangle_\pi^g) \otimes \mathbb{Z}_p \subsetneq G \otimes \mathbb{Z}_p\}.$$

By the hypothesis we have $\text{sl}(G/\langle h \rangle) = k \leq n$. For each prime $p \in \pi_1$ Lemma 6 yields the inequality $h_p^{g/H}(g + H) < h_p^{g/S}(g + S)$, where $H = \langle h \rangle$ and $S = \langle h \rangle_\pi^g$, and consequently Theorem 9 gives the inequality $1 < h_p^g(h)/h_p^g(g) \leq k \leq n$ for almost all primes $p \in \pi_1$: Considering the factor-group $G/\langle g \rangle$ we similarly obtain the inequality $1/n \leq h_p^g(h)/h_p^g(g) < 1$ for almost all primes $p \in \pi_2$.

Conversely, assume that the condition of Theorem is satisfied. Let $0 \neq h \in G$ be an arbitrary element and $g \in G \setminus \langle h \rangle_\pi^g$ be arbitrary. Denote $H = \langle h \rangle$, $S = \langle h \rangle_\pi^g$ and set

$$\pi' = \{p \in \pi \mid h_p^{g/H}(g + H) < h_p^{g/S}(g + S)\}.$$

Then for each $p \in \pi'$ $h_p^g(g) < h_p^g(H)$ by Lemma 4 and

$$\langle g \rangle_\pi^g \oplus \langle h \rangle_\pi^g \otimes \mathbb{Z}_p \subsetneq G \otimes \mathbb{Z}_p$$

by Lemma 6. By hypothesis, for almost all primes $p \in \pi'$ it is $nh_p^g(g) \geq h_p^g(H)$ and, consequently, $\text{sl}(G/\langle h \rangle) \leq n$ by Theorem 9. So with respect to Remark 14 we have $\text{fsl}(G) \leq n$ and it remains to show the existence of $h \in G$ with $\text{sl}(G/\langle h \rangle) = n$.

By hypothesis, there is a basis $\{g, h\}$ of G such that for no $k < n$, $k \in \mathbb{N}$, the inequalities $1/k \leq h_p^g(h)/h_p^g(g) \leq k$ hold for almost all primes

$$p \in \pi_1 \cup \pi_2,$$

$$\pi_1 = \{p \in \pi \mid h_p^g(g) < h_p^g(h), \langle g \rangle_\pi^g \oplus \langle h \rangle_\pi^g \otimes \mathbb{Z}_p \subsetneq G \otimes \mathbb{Z}_p\},$$

$$\pi_2 = \{p \in \pi \mid h_p^g(h) < h_p^g(g), \langle g \rangle_\pi^g \oplus \langle h \rangle_\pi^g \otimes \mathbb{Z}_p \subsetneq G \otimes \mathbb{Z}_p\}.$$

Then either π_1 is infinite and n is the exact upper bound of $h_p^g(h)/h_p^g(g)$ for almost all primes $p \in \pi_1$, or π_2 is infinite and $1/n$ is the exact lower bound of $h_p^g(h)/h_p^g(g)$ for almost all primes $p \in \pi_2$. In the first case we easily get from Lemma 6, Lemma 4 and Theorem 9 that $\text{sl}(G/\langle h \rangle) = n$ and the same arguments in the second case yields $\text{sl}(G/\langle g \rangle) = n$.

EXAMPLE 16: Consider the group G generated by the elements $a, b, a_n, b_n, c_n, p \in \pi$, with respect to the relations $pa_n = a, p^n b_n = b, pc_n = a_n + b_n, p \in \pi, n \geq 2$. The group G is torsionfree of rank two, the element b has the characteristic (n, n, \dots) and all the elements outside of $\langle b \rangle_\pi^g$ are of the type $(1, 1, \dots)$. Hence for each $h \in \langle b \rangle_\pi^g$,

$H = \langle h \rangle$, $S = \langle h \rangle_\pi^a$, we have $h_p^{a/H}(a + H) < h_p^{a/S}(a + S)$, $nh_p^a(a) = h_p^a(H)$ for almost all primes p and $\text{sl}(G/\langle h \rangle) = n$ by Theorem 9. Moreover, for each $h \in G \setminus \langle b \rangle_\pi^a$ Proposition 8 yields the splitting of $G/\langle h \rangle$ and G has the factor-splitting length n .

EXAMPLE 17. Consider the group G generated by the elements $a, b, b_p, c_p, p \in \pi$, with respect to the relations $pb_p = b$, $pc_p = a + b_p$, $p \in \pi$. The group G is torsionfree of rank two, the element b has the characteristic $(1, 1, \dots)$ and all the elements outside of $\langle b \rangle_\pi^a$ are of the type Z (the additive group of integers). For each $h \in G \setminus \langle b \rangle_\pi^a$ Proposition 8 yields the splitting of $G/\langle h \rangle$. Consider now the subgroup H of G with $\langle b \rangle \subseteq H \subseteq \langle b \rangle_\pi^a = S$. If $\hat{t}(H) = \hat{t}^a(b)$ then the torsion part of G/H is finite and G/H therefore splits. In the opposite case there is an infinite set of primes with $h_p^a(H) = 1$. For all these primes we have $h_p^{a/H}(g + H) < h_p^{a/S}(g + S)$ but there is no $n \in \mathbb{N}$ with $nh_p^a(a) \geq h_p^a(H)$. Hence, by Theorem 9, G/H has no finite splitting length. From these observations it immediately follows that $\text{fsl}(G) = \infty$ and for each factor-group G/H of G it is either $\text{sl}(G/H) = \infty$ or $\text{sl}(G/H) = 1$.

THEOREM 18: (i) *For each positive integer n there exists a torsion-free group G of rank two having the factor-splitting length n ;*

(ii) *There exists a torsionfree group G of rank two and of infinite factor-splitting length such that G has a homomorphic image of splitting length n for each $n \in \mathbb{N} \cup \{\infty\}$;*

(iii) *For each $n \in \mathbb{N}$ there exists a torsionfree group G of rank two and of infinite factor-splitting length such that the splitting length of any homomorphic image of G is either infinite or at most n .*

PROOF: (i) See Example 16.

(ii) Decompose the set π into disjoint infinite subset π_1, π_2, \dots and consider the group G (« composition » of groups from Example 16 for all $n \geq 2$) generated by the elements $a, b, a_{p_i}, b_{p_i}, c_{p_i}, p_i \in \pi_i, i = 2, 3, \dots$, with respect to the relations

$$p_i a_{p_i} = a, \quad p_i^i b_{p_i} = b, \quad p_i c_{p_i} = a_{p_i} + b_{p_i}, \quad p_i \in \pi_i, \quad i = 2, 3, \dots$$

For $h = b$, $H = \langle h \rangle$, $S = \langle h \rangle_\pi^a$ we obviously have $h_p^{a/H}(a + H) < h_p^{a/S}(a + S)$ for each prime $p \in \pi$ and $h_p^a(H) = ih_p^a(a)$ for each $p \in \pi_i$.

Hence $\text{sl}(G/H) = \infty$ by Theorem 9. Further, for $H = \langle b \rangle_{\pi}^g \setminus \pi_n$ we have $h_p^g(H) = n$ for each $p \in \pi_n$ and, consequently, $\text{sl}(G/H) = n$.

(iii) For $n = 1$ see Example 17. If $n \geq 2$ then decompose the set π into two disjoint infinite parts π_1 and π_2 and consider the group G (« composition » of groups from Examples 16 and 17) generated by the elements $a, b, a_p, b_p, c_p, b_a, c_a, p \in \pi_1, q \in \pi_2$, with respect to the relations $pa_p = a, p^nb_p = b, pc_p = a_p + b_p, qb_a = b, qc_a = a + b_a, p \in \pi_1, q \in \pi_2$. Using similar arguments as in Examples 16 and 17 we see that $G/\langle h \rangle$ splits for each $h \in G \setminus \langle b \rangle_{\pi}^g$. Further, for $\langle b \rangle \subseteq H \subseteq \langle b \rangle_{\pi}^g$ it is $\text{sl}(G/H = \infty)$ whenever $|\{p \in \pi_2 | h_p^g(H) = 1\}| = \infty$. In the opposite case, using Corollary 10, we easily get $\text{sl}(G/H) = k \leq n$. Especially, $\text{sl}(G/\langle b \rangle_{\pi_2}^g) = n$.

REMARK 19: Theorem 15 and Example 16, 17, as well as the proof of Theorem 18 show that the factor-splitting length of a torsionfree group G of rank two depends on the type set $\hat{\tau}(G)$ of G , only. It is not too hard to show that the factor-splitting length of G can be found in the following way: Let $M = \{\hat{\tau}, \hat{\sigma} | \hat{\tau} \neq \hat{\sigma}, \hat{\tau}, \hat{\sigma} \in \hat{\tau}(G)\}$ and let $g, h \in G$ be such that $\hat{\tau}^g(g) = \hat{\tau}, \hat{\tau}^g(h) = \hat{\sigma}$. Set $\pi_M = \{p \in \pi | \tau(p) \neq \sigma(p)\}$ are finite and $\langle g \rangle_{\pi}^g \oplus \langle h \rangle_{\pi}^g$ is not p -pure in G and

$$\varrho_M = \limsup_{p \in \pi_M} \sigma(p)/\varrho(p), \quad \sigma_M = \liminf_{p \in \pi_M} \sigma(p)/\varrho(p),$$

where we put $m/0 = \infty$. If $\varrho = \sup \varrho_M, \sigma = \inf \sigma_M$, where M ranges over all pairs of different types from $\hat{\tau}(G)$, then $\text{fsl}(G) = \infty$ if either $\varrho = \infty$ or $\sigma = 0$ and $\text{fsl}(G) = n$ if n is the smallest integer with $1/n \leq \sigma$ and $\varrho \leq n$.

REFERENCES

- [1] L. BICAN, *Mixed abelian groups of torsionfree rank one*, Czech. Math. J., **20** (1970), pp. 232-242.
- [2] L. BICAN, *Factor-splitting abelian groups of rank two*, Comment. Math. Univ. Carolinae, **11** (1970), pp. 1-8.
- [3] L. BICAN, *Factor-splitting abelian groups of finite rank*, Comment. Math. Univ. Carolinae, **17** (1976), pp. 473-480.
- [4] L. BICAN, *Factor-splitting abelian groups of arbitrary rank*, Comment. Math. Univ. Carolinae, **19** (1978), pp. 653-672.

- [5] L. BICAN: *The splitting length of mixed abelian groups of rank one*, Czech. Math. J., **27** (1977), pp. 144-154.
- [6] L. BICAN: *The splitting of the tensor product of two mixed abelian groups of rank one*, to appear.
- [7] L. FUCHS, *Abelian groups*, Budapest, 1958.
- [8] L. FUCHS, *Infinite abelian groups I*, Academic Press, New York and London, 1970.
- [9] L. FUCHS, *Infinite abelian groups II*, Academic Press, New York and London, 1973.
- [10] I. M. IRWIN - S. A. KHABBAZ - G. RAYNA, *The rôle of the tensor product in the splitting of abelian groups*, J. Algebra, **14** (1970), pp. 423-442.
- [11] L. PROCHÁZKA, *O rasčepļajemosti faktorgrupp abeleвых grupp bez kručenija konečnogo ranga (Russian)*, Czech. Math. J., **11** (1961), pp. 521-557.
- [12] L. PROCHÁZKA, *Zametka o faktorno rasčepļajemych abeleвых gruppach (Russian)*, Čas. pěst. mat., **12** (1962), pp. 404-414.
- [13] A. I. STRATTON, *A splitting theorem for mixed abelian groups*, Symposia Mathematica, Vol. XIII, Academic Press, London, 1974, pp. 109-125.

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