RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Contributions to foundations of probability calculus on the basis of the modal logical calculus MC^{ν} or MC_{*}^{ν}

Rendiconti del Seminario Matematico della Università di Padova, tome 70 (1983), p. 1-11
<http://www.numdam.org/item?id=RSMUP_1983__70__1_0>

Rend. Sem. Mat. Univ. Padova, Vol. 70 (1983)

Contributions to Foundations of Probability Calculus on the Basis of the Modal Logical Calculus MC^ν or MC^ν_*.

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Part III

An Analysis of the Notions of Random Variables and Probability Spaces, Based on Modal Logic.

12. Introduction.

The present paper (Part 3) is concerned with foundations of the probability calculus. We want, first, to analyze the notion of casual or random variables, which is usually dealt with as a primitive, on the basis of MC^ν_* [MC^ν] (through TP_* [TP]) and in particular by use of absolute concepts and their extensionalizations [N. 3]—cf. [3]. More precisely two versions of this intuitive notion are widely used in the literature; so to say, one is physical (or natural) and the other is purely mathematical. We emphasize and analyze this distinction by defining *physical* (or *casual*) *random variable* [N. 13] and *absolute random variable* [N. 14] rigorously, within TP_* [TP].

Second, within the object language itself we define the (standard) physical notion of a probability space relative to an assertion α, in a natural way connected with MC^ν_* (or MC^ν) [N. 15]. Among these

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spaces there are the maximal ones. These may be infinite, finite, or the trivial one, which is substantially formed by the ranges of α, $\sim\alpha$, $\alpha\wedge\sim\alpha$, and $\alpha\vee\sim\alpha$, and exists for every choice of α. In N. 17 any set formed with propositions having a probability relative to α and closed under conjunction is proved to belong to a probability space relative to α. Incidentally this theorem constitutes a bridge between, so to say, the geometrical theories of probability and the existential ones, we mean between those that start with (a rather mathematical notion of) probability spaces (and define probability as a measure on them) and the probability theories such as von Mises's, or Reichenbach's (or our theories TP_* and TP) that start with existence axiom.

In the afore-mentioned treatment of probability based on the theory TP_* [TP], probability is a primitive notion that has the form of a function of propositions [proposition ranges]: hence probability cannot be reintroduced as a (primitive) measure on a (probability) space. Instead one has to define when a measure on a probability space relative to the trial α is a probability measure.

Let us note that our theory TP_* or TP happens to agree with Freudenthal and De Finetti's views about the approach to probability calculus—cf. [10], [8]—, in that e.g. they just criticize the usual systems of probability axioms which substantially are definitions of particular purely mathematical measure spaces, and among other things they consider these systems to say «little if anything about probability»—cf. [10] p. 261.

It must be added that unlike [10] and [8] the present paper has been written with no didactical purposes. However it can also be regarded as a theoretical support to Freudenthal and De Finetti's views.

13. A first notion of random variable.

In the literature two notions of random variables are substantially used: so to say, a physical one and a mathematical notion. The first one, with which this section is concerned, is defined e.g. by Castelnuovo in [6] p. 30 as follows: we shall call (physical or casual) «*random variable* a variable quantity x that can take various real values $x_1, x_2, \dots, x_n$ according as one of the incompatible events $E_1, E_2, \dots, E_n$ having the known probabilities $p_1, p_2, \dots, p_n$ of sum $p_1 + p_2 + \dots + p_n = 1$ occurs». He obviously means that x takes value x_i whenever E_i occurs ($i = 1, \dots, n$).

It is obvious that « variable quantity » is used here and, for instance, in the assertion « $x = \sin\omega t$ is a variable quantity » (which refers to a harmonic motion of period $2\pi/\omega$) in very different senses. The first is essentially modal in that x can assume (physically or casually) various real values $x_1, x_2, \dots, x_n$, that is, x can happen to equal the privileged absolute real numbers x_1 to x_n ($x_i \in \mathbb{R}$, i.e. x_1 is a modally fixed real number for $i = 1, \dots, n$); thus the variable real number x is what we simply call a real number: $x \in \mathbb{R}^{(e)}$—cf. (3.4). Instead in the last assertion both t and x are used as (modally) fixed real numbers ($t, x \in \mathbb{R}$); x varies only in that various values (in $\mathbb{R}$) are naturally assigned to the variable t.

Incidentally, remembering the semantics of ML^{ν} or ML^{ν}_{*}, from [3] or [5] respectively, we see that the casual variable x varies in connection with one given value-assignment $\mathfrak{V}$ to variables, in that $\mathfrak{V}$ assigns x an intension or quasi-intension, and the extension $\bar{x}(\gamma)$ of x in the elementary (possible) case γ varies when γ describes the class Γ of these cases. In the second case, which can be dealt with e.g. the extensional semantics presented in [11], the values of $x = (\sin\omega t)$ varies in that it takes different values in corrispondence with different choices of $\mathfrak{V}$ that assign t values in $\mathbb{R}$ (that exhaust $\mathbb{R}$) and a same value to ω. Let us add that in the first case γ varies in the same way as t and x do in the second one; but γ is a metalinguistic variable, whereas t and x belong to the object language.

In order to define (physical or) random variables in MC^{ν}_{*} we first consider a particular case: let the proposition (trial) α necessarily imply exactly one among the propositions (events) α_1 to α_n; and let α_i have the probability p_i with respect to α ($i = 1, \dots, n$). In this case the events α_1 to α_n will be said to *determine a probability space relative to the trial* α; and this will be expressed in MC^{ν}_{*} by $\mathrm{DPrS}(\alpha, \alpha_1, \dots, \alpha_n)$:

(13.1) $$\mathrm{DPrS}(\alpha, \alpha_1, \dots, \alpha_n) \equiv_D$$
$$\equiv_D \Big(\alpha \supset^{\cap} \bigvee_{i=1}^{n} \alpha_i\Big) \bigwedge_{i=1}^{n-1} \bigwedge_{j=i+1}^{n} (\alpha\alpha_i \supset^{\cap} \sim \alpha_j) \bigwedge_{i=1}^{n} \Diamond\, \alpha\alpha_i (\alpha \ni \alpha_i)\,.$$

Now we can define (*real*) *physical* (*or casual*) *random variables associated with the events* α_1 *to* α_n *and trial* α ($\mathfrak{PhRV}$):

(13.2) $$\Delta \in \mathfrak{PhRV}_{\alpha, \alpha_1, \dots, \alpha_n} \equiv_D$$
$$\equiv_D \mathrm{DPrS}(\alpha, \alpha_1, \dots, \alpha_n)(\alpha \supset^{\cap} \Delta \in \mathbb{R}^{(e)}) \bigwedge_{i=1}^{n} (\exists x_i \in \mathbb{R})(\alpha_i \supset^{\cap} \Delta = x_i)$$

where x_1 *to* x_n *are distinct variables that do not occur in* Δ.

Obviously (13.2) is a definition scheme; and on its basis we cannot define physical random variables but only those of them that are capable of at most n values ($\mathfrak{I}\mathrm{hRV}_n$, $n = 1, 2, \ldots$).

$$(13.3) \qquad \Delta \in \mathfrak{I}\mathrm{hRV}_n \equiv_D (\exists \alpha, \alpha_1, \ldots, \alpha_n)\, \Delta \in \mathfrak{I}\mathrm{hRV}_{\alpha, \alpha_1, \ldots, \alpha_n} .$$

The considerations above are tightly connected with ordinary (particular) definitions of random variables such as Castelnuovo's. Now, to generalize $\mathfrak{I}\mathrm{hRV}$ (into $\mathfrak{I}\mathrm{h}\mathfrak{R}\mathfrak{V}$ below), we define a general notion of *physical random variables relative to the trial* α ($\mathfrak{I}\mathrm{h}\mathfrak{R}\mathfrak{V}_\alpha$):

$$(13.4) \qquad \Delta \in \mathfrak{I}\mathrm{h}\mathfrak{R}\mathfrak{V}_\alpha \equiv_D$$

$$\equiv_D \Diamond \alpha \wedge (\alpha \supset^{\cap} \Delta \in \mathbf{R}^{(e)})(\forall a \in \mathbf{R})(\alpha \in \Delta > a) \qquad [> =^{\cap} >^{(e)}]$$

that is, $\Delta \in \mathfrak{I}\mathrm{h}\mathfrak{R}\mathfrak{V}_\alpha$ iff α happen, Δ is a variable real number and for every (modally) fixed real number a the probability of the event $\Delta > a$ relative to α exists. *Physical random variables* ($\mathfrak{I}\mathrm{h}\mathfrak{R}\mathfrak{V}$) are defined by

$$(13.5) \qquad \Delta \in \mathfrak{I}\mathrm{h}\mathfrak{R}\mathfrak{V} \equiv_D (\exists \alpha)\, \Delta \in \mathfrak{I}\mathrm{h}\mathfrak{R}\mathfrak{V}_\alpha .$$

14. A second notion of random variable.

For instance Daboni says—cf. [7], p. 54—that a random number —« numero aleatorio »—x is a P-measurable real-valued function defined on a partition Ω on which a probability distribution P has been assigned.

In connection with the propositions α and α_1 to α_n considered in N. 13 we can intuitively regard the set $\mathcal{N}_n = \{1, \ldots, n\}$ formed with the indices in α_1 to α_n, as representing a probability space to be associated with (α and) α_1 to α_n. For $A \subseteq \mathcal{N}_n$, i.e. for $A \in \mathcal{A}$ where $\mathcal{A}$ is the class $S\mathcal{N}_n$ of the subsets of $\mathcal{N}_n$, let us identify $P(A)$ with the sum of the p_i's with $i \in A$. Thus an instance of the afore-mentioned probability distribution is obtained.

Now let f be any real valued function defined on $\mathcal{N}_n$, hence it is P-measurable. It is a random variable according to e.g. Daboni's definition above. We can say that f is a (*non standard*) *absolute random variable relative to the trial* α *and the representation* $r \to \alpha_r$ *of the events* α_1

to α_n *on* $\mathcal{N}_n$. Obviously $n!$ different such representation exist. Setting $x_i = f(i)$, we have a variable number called a random variable by Dore (¹).

Remark that the mathematical function f corresponds naturally to—or represents—the physical random variable x relative to the trial α, that equals $f(i)$ in case α_i occurs, for $i = 1, \ldots, n$; formally

$$(14.1) \qquad x =_D (\imath f \in \mathbf{R}^{\mathcal{N}_n})(i)\,[i \leqslant_{\mathbf{N}} n \wedge \alpha_i \supset^{\cap} x = f(i)]\,.$$

Obviously, for instance, any permutation $s \to r_s$, of $\mathcal{N}_n$ changes $r \to \alpha_r$ into another representation $s \to \alpha_{r_s}$, of α_1 to α_n on $\mathcal{N}_n$; and it changes f in the absolute random variable f': $f'(s) = f(r_s)$ (f' is relative to α and the representation $s \to \alpha_{r_s}$ of α_1 to α_n on $\mathcal{N}_n$). Both f and f' naturally correspond to the same physical random variable x. Thus f and f' may be regarded as equivalent. Even in the simple use above we have many equivalent absolute random variables—more than $n!$, cf. the last part of footnote (¹). This multiplicity depends on the analogous multiplicity for what can be taken to be the probability space in the same case.

The multiplicity above increases when the probability space can be identified with S_n, the set of n-typles of real numbers. Therefore it is useful to fix once for all a unique standard probability space in every situation of the preceding kind. When one accepts to base the theory on nature being dealt with on the (modal) logical calculus MC_*^ν or MC^ν, a certain choice of the probability space is natural. For instance, in the aforementioned use connected with α and α_1 to α_n, we can replace $\mathcal{N}_n$ with the indicator $\iota\alpha$ of α—cf. (4.9)—and Ω with the family of subsets of $\iota\alpha$ generated by $\iota\alpha_1$ to $\iota\alpha_n$—cf. [7], p. 52-4.

(¹) In [9] pp. 88-89 Dore says: « Il numero variabile x_i che contrassegna l'evento A_i ed è suscettibile di assumere uno degli r valori costituenti l'insieme inerente ai fatti aleatori A_i, si suol designare come variabile aleatoria (o stocastica o casuale); la qualità di aleatorietà essendo appunto caratterizzata dal fatto che ad ognuno dei valori di cui essa variabile è suscettibile corrisponde un valore P_i della probabilità tale che $\sum P_i = 1$.

L'insieme delle P_i può considerarsi come una « distribuzione » di probabilità tra i valori x_i della variabile ».

Obviously x_i is a variable number in that i is variable. Furthermore his probability distribution to the value x_1 to x_n (which are fixed real numbers) shows that Dore gives $\{x_1, \ldots, x_n\}$ the role of our $\mathcal{N}_n$.

Then the above function f has to be replaced by the mapping $\bar{f}$ of $\iota\alpha$ onto $\mathcal{N}_n$ such that $\bar{f}(u) = f(r)$ for $u \in \iota\alpha_r$ $(r = 1, \ldots, n)$.

On the basis of the intuitive considerations above we can define in MC_*^ν or MC^ν *(standard) absolute random variables* relative to the trial α (ARV_α), before defining (standard) probability space relative to α, as follows:

$$f \in \mathrm{ARV}_\alpha \equiv_D \Diamond\, \alpha \wedge f \in \mathbb{R}^{\iota\alpha} \wedge (\forall a \in \mathbb{R}).\, \alpha \ni (\exists u)(\,|_u \wedge f(u) > a)\,, \tag{14.2}$$

where

$$f \in \mathbb{R}^{\iota\alpha} \equiv_D (u).(u \in \iota\alpha \supset f(u) \in \mathbb{R})(u \notin \iota\alpha \supset f(u) =^\cap a^*)\,. \tag{14.3}$$

Now we can define *absolute random variables* (ARV) by

$$f \in \mathrm{ARV} \equiv_D (\exists\alpha) f \in \mathrm{ARV}_\alpha\,. \tag{14.4}$$

15. A first notion of probability spaces.

If $\mathrm{DPrS}_{\alpha,\alpha_1,\ldots,\alpha_n}$ holds—cf. (13.1)—then according to the intuitive considerations on Ω made in N. 14, in any $\mathcal{L}$ of the calculi MC_*^ν and MC^ν we can define (formally) the standard (probability) space $\mathcal{P}\mathrm{rS}_{\alpha,\alpha_1,\ldots,\alpha_n}$ associated with the trial α and the events α_1 to α_n after [7], by

$$\mathcal{P}\mathrm{rS}_{\alpha,\alpha_1,\ldots,\alpha_n} =_D \{\iota\alpha_1, \ldots, \iota\alpha_n\}^{(i)} \tag{15.1}$$

$$\left(= (\lambda x) \bigvee_{i=1}^{n} x =^\cap \iota\alpha_i \text{—cf. [3], p. 68}\right).$$

Incidentally it is useful to mean trial simply as a proposition that can be true. However we think it convenient to define notions involving a trial α in such a way that they are meaningful (but uninteresting) also in case α cannot occur. The analogue holds with other definitions such as (15.3) below.

In order to generalize this notion into a general notion of *standard probability space associated with the trial or proposition* α ($\mathcal{P}\mathrm{rSp}_\alpha$) we first remark that

$$\vdash_{\mathcal{L}} \mathcal{U} =^\cap \iota\alpha \supset \alpha \equiv^\cap \mathrm{prop}_{\mathcal{U}}\,, \tag{15.2}$$

where

$$\text{prop}_{\mathcal{U}} \equiv_D \text{prop}(\mathcal{U}) \equiv_D (\exists u).u \in \mathcal{U} \wedge |_u\,, \tag{15.3}$$

which, among other things, shows how variables restricted to El can be used also in MC^v, as propositional variables. Then we introduce (in $\mathfrak{L}$) the class PrS_α of *probability α-subranges*, which are the ranges for propositions that have a probability relative to the proposition (trial) α:

$$\text{PrS}_\alpha =_D (\lambda\mathcal{U})(\mathcal{U} \subseteq \text{El})(\alpha \ni \text{prop}_{\mathcal{U}})\,. \tag{15.4}$$

Now we can define in $\mathfrak{L}$, in the usual extensional way, when a family $\mathcal{A}$ of subsets of a set $\mathcal{S}$ is an (extensional) ***Boolean Algebra*** [or σ-Algebra] on $\mathcal{S} = \bigcup \mathcal{A}$ (Boolalg [Bool σ-Alg]). In case $\mathcal{S}$, $\mathcal{A}$, and the elements of $\mathcal{A}$ are absolute, we call $\mathcal{A}$ ***absolute*** ($\boldsymbol{A}$ BoolAlg [$\boldsymbol{A}$ Bool σ-Alg]). Incidentally, by Theor. 41.1 (V) in [3], $\vdash F \subseteq G \wedge F \in \text{MConst} \wedge \wedge\, G \in \text{Abs} \supset F \in \text{Abs}$, we can write

$$\mathcal{A} \in \boldsymbol{A}\ \text{BoolAlg} \equiv_D \emptyset \in \mathcal{A} \wedge \cup\, \mathcal{A} \in \mathcal{A} \wedge (\mathcal{A}, \bigcup \mathcal{A} \in \text{Abs}) \wedge (\forall F, G). \tag{15.5}$$
$$.(F, G \in \mathcal{A}) \supset F \cap G \in \mathcal{A} \wedge (\bigcup \mathcal{A}) - F \in \mathcal{A} \wedge F \in \text{MConst}\,.$$

$$\mathcal{A} \in \boldsymbol{A}\ \text{Bool}\ \sigma\text{-Alg} \equiv_D \mathcal{A} \in \boldsymbol{A}\ \text{BoolAlg} \wedge .(\forall f \in \mathcal{A}^{\mathbf{N}}) \supset \tag{15.6}$$
$$\supset \bigcup_{n \in \mathbf{N}} f(n) \in \mathcal{A}\,.$$

In connection with MC^v_* or MC^v it is natural to define the *standard probability spaces associated with a proposition* α to be the absolute Boolean σ-Algebras on PrS_α

$$\Sigma \in \mathfrak{P}\text{rSp}_\alpha \equiv_D \Sigma \in \boldsymbol{A}\ \text{BoolAlg} \wedge \Sigma \subseteq \boldsymbol{S}^{(\mathrm{mc})}\, \text{PrS}_\alpha \tag{15.7}$$

where $\boldsymbol{S}^{(\mathrm{mc})}$, *modally constant subset*, is defined—cf. [4] p. 62—by

$$\boldsymbol{S}^{(\mathrm{mc})} F \equiv_D (\lambda G).G \subseteq F \wedge G \in \text{MConst}\,. \tag{15.8}$$

Remark that for every proposition α—e.g. $\text{prop}_{\mathcal{U}}$ see (15.3), (where $\mathcal{U} \subseteq \text{El}$)—we have

$$\vdash \mathcal{A}_0 = {}^{\cap}\{\iota\alpha, \emptyset\}^{(\iota)} \supset \mathcal{A}_0 \in \mathfrak{P}\text{rSp}_\alpha \tag{15.9}$$

(and $\iota\alpha$ is $\emptyset$ in case α cannot occur).

Indeed $\vdash \mathcal{T}_{\alpha,\alpha} = 1$, and $\mathcal{T}_{\alpha,\alpha\wedge\sim\alpha} = 0$ by (6.1) and (10.8), and $\vdash \mathcal{A}_0 \in \mathrm{Bool}\,\sigma\text{-Alg}$. Thus, *for every* α, $\mathcal{T}\mathrm{rSp}_\alpha$ *is non-empty*. However $\mathcal{T}\mathrm{rSp}_\alpha$ might contain only the *trivial probability* space $\mathcal{A}_0$ above. It is logically possible that this holds for every α. However in the next sections we prove that some non-trivial probability spaces exist in connection with proposition α for which PrS_α is non trivial in a suitable sense—see below (16.1).

16. Towards an existence theorem for probability spaces.

In the next section we prove a non-trivial existence theorem for probability spaces, which, so to say, is a bridge between geometrical theories of probability and existential ones—cf. N. 12. In this section we state some preliminaries for the goal above. First we introduce the class $\boldsymbol{F}\mathrm{PrS}_\alpha^\cap$ of $\cap$*-closed families of probability* α*-subranges* of PrS_α:

$$(16.1)\qquad \boldsymbol{F}\mathrm{PrS}_\alpha^\cap \equiv_D (\lambda \mathcal{G}).\mathcal{G} \in \boldsymbol{S}^{(\mathrm{mc})}\mathrm{PrS}_\alpha(\forall \mathcal{U}, \mathcal{V} \in \mathcal{G})\,\mathcal{U} \cap \mathcal{V} \in \mathcal{G} \quad \text{— cf. (15.8)}\,.$$

Hence if $\mathcal{G} \in \boldsymbol{F}\mathrm{PrS}_\alpha^\cap$ and $\iota\alpha$, $\iota\beta \in \mathcal{G}$, then $\iota(\alpha \wedge \beta) \in \mathcal{G}$, which justifies the notation $\boldsymbol{F}\mathrm{PrS}_\alpha^\cap$. We can say that PrS_α is *trivial* when $F \in \boldsymbol{F}\mathrm{PrS}_\alpha^\cap$ yields $F = \iota\alpha$ or $F = \emptyset$.

The Boolean Algebra $\overline{\mathcal{G}}$ generated by $\mathcal{G}$ can be easily defined in $\mathfrak{L}$. For the sake of brevity we don't make this definition explicit. However we write the theorem in MC_*^ν:

$$(16.2)\qquad \vdash \mathcal{G} \in \boldsymbol{F}\mathrm{PrS}_\alpha^\cap \wedge (\iota\beta, \iota\gamma \in \overline{\mathcal{G}}) \supset [\iota\mathcal{F}(\beta, \gamma) \in \overline{\mathcal{G}}]$$

where $\mathcal{F}(\beta, \gamma)$ is any wff constructed by means of β, γ, $\sim$, $\wedge$, and parentheses. Its easy proof is based on (4.11)—cf. [1]. On the other hand we have

$$(16.3)\qquad \vdash \mathcal{G} \in \boldsymbol{F}\mathrm{PrS}_\alpha^\cap \supset \overline{\mathcal{G}} \in \boldsymbol{F}\mathrm{PrS}_\alpha^\cap\,.$$

Indeed let $\mathcal{G} \in \boldsymbol{F}\mathrm{PrS}_\alpha^\cap$ and $F \in \overline{\mathcal{G}}$. We prove that $F \in \mathrm{PrS}_\alpha$, so that $\overline{\mathcal{G}} \in \boldsymbol{F}\mathrm{PrS}_\alpha^\cap$ (being $\overline{\mathcal{G}}$ $\cap$-closed). If $F \in \overline{\mathcal{G}}$, then for some sequence $H_1, \ldots, H_n$ we have $H_n = F$ and, for $i = 1$ to n, either (*a*) $H_i \in \mathcal{G}$ or (*b*) $H_i = \iota\alpha - H_j$ for some $j < i$, or (*c*) $H_i = H_r \cap H_s$ for some $r < i$ and $s < i$.

Let $\mathcal{G}'$ be the set of the H_i's that fulfill (a). Hence $F \in \overline{\mathcal{G}}'$ and $\mathcal{G}'$ is finite, say $\mathcal{G}' = {}^{\cap}\{B_1, \ldots, B_m\}^{(i)}$. Define $\beta_r \equiv_D \mathrm{prop}_{B_r}$ $(r = 1, \ldots, m)$ —see (15.3). Since $B_r \in \mathcal{G}(\in \boldsymbol{F}\,\mathrm{PrS}^{\cap}_{\alpha})$, $\mathcal{P}_{\alpha, \beta_{i_1} \wedge \ldots \wedge \beta_{i_l}}$ exists for any $l \in$

$\in \mathcal{N}_m = \{1, \ldots, m\}$ and any injection $s \to i_s$ of N_l into $\mathcal{N}_m$; let it equal $\overline{p}_{i_1, \ldots, i_l}$ us set

$$\overline{p}_{i_1, \ldots, i_l} =_D \mathcal{P}_{\alpha, \beta_{i_1} \wedge \ldots \wedge \beta_{i_l}} \,. \tag{16.4}$$

Furthermore let β'_s be either β_s or $\sim \beta_s$ $(s = 1, \ldots, r)$. If the probability $\mathcal{P}_{\alpha, \beta'_1 \wedge \ldots \wedge \beta'_r}$ exists, its value p' is a well known function of the $\overline{p}_{i_1, \ldots, i_l}$ that can be calculated on the basis of A5.4 and A5.6.

In particular $\overline{p}'$ is independent of the propositions $\alpha, \beta_1, \ldots, \beta_n$ that fulfil (16.4). Hence by the existence rule 10.1—cf. [2]—, (d) $\mathcal{P}_{\alpha, \beta'_1 \wedge \ldots \wedge \beta'_r}$ exists $(r = 1, \ldots, m)$.

As is well known, $F(\in \overline{\mathcal{G}}')$ has the form $C_1 \cup \ldots \cup C_h$ where C_1 to C_h are (different and hence) disjoint sets of the form $B'_1 \cap \ldots \cap B'_m$, where B'_r $(= \iota\beta'_r)$ is B_r or $\iota\alpha - B_r$ $(r = 1, \ldots, m)$. Therefore F is the range of $\gamma_1 \vee \ldots \vee \gamma_h$, where $\gamma_s = \mathrm{prop}(C_s)$ $(s = 1, \ldots, h)$. Hence $\vdash \alpha \wedge (\gamma_1 \wedge \ldots \wedge \gamma_{s-1}) \supset \sim \gamma_s$ $(s = 2, \ldots, h)$ so that, by A5.6 $\mathcal{P}_{\alpha, \gamma_1 \vee \ldots \vee \gamma_s}$ exists $(s = 1, \ldots, h)$. For $s = h$ this implies that $F \in \mathrm{PrS}_{\alpha}$. We conclude that (16.3) holds. q.e.d.

17. A non-trivial existence theorem for probability spaces.

We now briefly prove the following non-trivial existence theorem in TP^* (or TP)

$$\vdash \mathcal{A} \in \boldsymbol{F}\,\mathrm{PrS}^{\cap}_{\alpha} \supset (\exists \mathcal{S}).\mathcal{S} \in \mathcal{P}\mathrm{rSp}_{\alpha} \wedge \mathcal{A} \subseteq \mathcal{S} \text{—cf. (15.7)} \,. \tag{17.1}$$

It is an immediate consequence of the assertions

(α) *any* $\mathcal{A} \in \boldsymbol{F}\,\mathrm{PrS}^{\cap}_{\alpha}$ *belongs to a maximal element of the family* $\boldsymbol{F}\,\mathrm{PrS}^{\cap}_{\alpha}$ (*ordered* by set inclusion).

(β) *any maximal element of* $\boldsymbol{F}\,\mathrm{PrS}^{\cap}_{\alpha}$ *is a Boolean σ-Algebra.*

In order to prove them briefly, let us remember that the relation $\subseteq$ induces a partial order $\leqslant$ on the family $\boldsymbol{F}_{\alpha} =_D \boldsymbol{F}\,\mathrm{PrS}^{\cap}_{\alpha}$. Let Γ be any subchain, i.e. a subset of $\boldsymbol{F}_{\alpha}$ simply ordered and non-empty; and set $\mathcal{K} = \bigcup \Gamma$. Since $\mathcal{F} \in \Gamma$ implies $\mathcal{F} \subseteq \mathrm{PrS}_\alpha$, we have $\mathcal{K} \in \mathrm{PrS}_\alpha^\cap$ ($\mathcal{K} \subseteq \mathrm{PrS}_\alpha$).

Furthermore, let $F, G \in \mathcal{K}$. Then for some elements $\mathcal{F}$ and $\mathcal{G}$ of Γ, $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Since Γ is a chain, it is not restrictive to assume $\mathcal{F} \subseteq \mathcal{G}$ ($\in \boldsymbol{F}\mathrm{PrS}_\alpha^\cap$). Hence $F \cap G \in \mathcal{G}$ ($\subseteq \mathcal{K}$). We conclude that $\mathcal{K} \in \boldsymbol{F}\mathrm{PrS}_\alpha^\cap$. By this result the lattice $\langle \boldsymbol{F}\mathrm{PrS}_\alpha^\cap, \leqslant \rangle$ can be said to be inductive. Then by Zorn's lemma any $\mathcal{A} \in \boldsymbol{F}\mathrm{PrS}_\alpha^\cap$ belongs to some maximal element $\mathcal{G}$ of $\boldsymbol{F}\mathrm{PrS}_\alpha^\cap$. Hence ($\alpha$) holds.

To prove (β) let $\mathcal{G}$ be any maximal element of $\boldsymbol{F}\mathrm{PrS}_\alpha^\cap$, so that $\overline{\mathcal{G}} \in \boldsymbol{F}\mathrm{PrS}_\alpha^\cap$ by (16.3) and, since $\mathcal{G} \subseteq \overline{\mathcal{G}}$, $\mathcal{G}(= \overline{\mathcal{G}})$ is a Boolean algebra.

Now we can assume $B_i \in \mathcal{G}$ ($i = 1, 2, \dots$) (2). We also set briefly

$$(17.2) \qquad C_1 =_D B_1, \qquad C_r =_D B_r \sim \bigcup_{s=1}^{r-1} B_s \quad (r = 2, 3, \dots), \qquad C_\infty =_D \bigcup_r C_r,$$

whence $\vdash C_\infty = \bigcup_r B_r$, and (3)

$$(17.3) \qquad \gamma_r \equiv_D \mathrm{prop}(C_r), \qquad \gamma_\infty =_D (\exists r \in \mathrm{N})\gamma_r .$$

Then

$$(17.4) \qquad \vdash \iota\gamma_r = C_r, \qquad \vdash \iota\gamma_\infty = C_\infty, \qquad \vdash (\forall r, s). r <_{\mathrm{N}} s \wedge \alpha\gamma_r \supset^\cap \sim \gamma_s .$$

Since $\mathcal{G}$ is a Boolean algebra, $C_r \in \mathcal{G}$ ($\subseteq \boldsymbol{F}\mathrm{PrS}_\alpha^\cap$) for every $r \in \mathrm{N}$, so that by $(17.4)_1$ and (16.1) $\alpha \ni_{p_r} \gamma_r$ for some p_r. Then by (17.3), $(17.4)_3$, (15.6), and A5.9 $\alpha \ni_q \gamma_\infty$ for $q = \sum_{r=1}^n p_r$. Hence by $(17.4)_2$, $C_\infty \in \boldsymbol{F}\mathrm{PrS}_\alpha^\cap$.

Now let $D \in \mathcal{G}$ and set $\delta =_D \mathrm{prop}(D)$. Then—cf. $(17.3)_2$

$$(17.5) \qquad D \cap C_\infty = \iota(\delta \wedge \gamma_\infty) = \iota(\exists r \in \mathrm{N})(\delta \wedge \gamma_r) .$$

(2) To translate (17.2-3) into $\mathcal{L}$ rigorously, one can replace B_r with $B(r)$, C_r with $C(r)$, and (17.2) with $C = (\imath g)\, g(1) =^\cap B(1) \wedge (\forall r \geqslant_{\mathrm{N}} 2)\, g(r) =^\cap B^{1,2}(r) \sim \sim \bigcup_{s=1}^{r-1} B(s) \wedge (x)\; x \notin \mathrm{N}\; g(x) =^\cap a^*$. Now it is clear that γ_r and γ_∞ are two wffs of $\mathcal{L}$.

(3) Of course « B_i » stands for $B(i)$.

Likewise « C » in (17.2) has to be regarded as a functor, while « γ » in (17.3-4) acts as on attribute.

Since $D, C_r \in \mathcal{G}$ $(\in \boldsymbol{F}\,\mathrm{PrS}_\alpha^\cap)$, $D \cap C_r \in \mathcal{G}$ $(r = 1, 2, \ldots)$, so that $\alpha \ni_{p_r}$ $\ni_{p_r} \delta \wedge \gamma_r$ for some p_r. Furthermore by $(17.4)_3$, we have $(\forall r, s)$, $r <_{\mathrm{N}} s \wedge \alpha\delta\gamma_r \supset^\cap \sim (\delta\gamma_s)$. Then by A5.9 $\alpha \ni_q (\exists r \in \mathrm{N})(\delta \wedge \gamma_r)$ for some q. Then by (17.5) $D \cap C_\infty \in \mathrm{PrS}_\alpha$. We conclude that $\mathcal{G}^* \in \boldsymbol{F}\,\mathrm{PrS}_\alpha^\cap$ for $\mathcal{G}^* = {}^\cap\,\mathcal{G} \cup \{C_\infty\}^{(3)}$. Since $\mathcal{G}$ is a maximal element of $\boldsymbol{F}\,\mathrm{PrS}_\alpha^\cap$, we have $C_\infty \in \mathcal{G}$. Thus (β) holds. q.e.d.

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Manoscritto pervenuto in redazione il 12 dicembre 1980.