

RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

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**On a synonymy relation for extensional 1st
order theories. Part II. A sufficient criterion
for non synonymy. Applications**

Rendiconti del Seminario Matematico della Università di Padova,
tome 70 (1983), p. 13-19

http://www.numdam.org/item?id=RSMUP_1983__70__13_0

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On a Synonymy Relation for Extensional 1st Order Theories.

PART II

A Sufficient Criterion for Non Synonymy. Applications.

C. BONOTTO - A. BRESSAN (*)

7. Admissible generalized interpretations for the extension $\overline{\mathcal{T}}$ of $\mathcal{T}$ having primitive implication and equivalence ⁽¹⁾.

Let $\overline{\mathcal{L}}$ be the language obtained from $\mathcal{L}$ by adding the new logical symbols $\supset_p$ and $\equiv_p$, to be called primitive implication and equivalence (signs) respectively. Obviously $\overline{\mathcal{L}}$'s formation rules are those of $\mathcal{L}$ —see [1], § 2—and the following

- (i) *if $\mathcal{A}_1$ and $\mathcal{A}_2$ are wffs of $\overline{\mathcal{L}}$, then $\mathcal{A}_1 \equiv_p \mathcal{A}_2$ and $\mathcal{A}_1 \supset_p \mathcal{A}_2$ also are.*

In connection with the above theory $\mathcal{T}$ —see [1], § 6—we denote $D'_v \equiv_p D''_v$ by D^p_v and the wff (or wff-scheme) obtained from A3.*r* by replacing the occurrences of $\supset$ [$\equiv$] with $\supset_p$ [$\equiv_p$], by $A^p3.r$ ($r = 7, 8$).

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Lavoro eseguito nell'ambito dell'attività dei Gruppi di Ricerca Matematica del C.N.R., negli anni accademici 1979-80 e 1980-81.

⁽¹⁾ The present paper is the second part of a work whose first part is [1]. Therefore the numbering of its sections follows the one for [1].

Furthermore we consider the (barred) extension

$$(7.1) \quad \overline{\mathcal{T}} = (S \cup \{\equiv_p, \supset_p\}, \text{wfe}_{\overline{\mathcal{T}}}, LA, \overline{PA}, R, \{D_r^p\}_{0 < r < \omega})$$

of $\mathcal{T}$, where $\text{wfe}_{\overline{\mathcal{T}}}$ is the class of wfes of $\overline{\mathcal{L}}$ in which only symbols in $S \cup \{\equiv_p, \supset_p\}$ occur and

$$(7.2) \quad \overline{PA} = PA \cup \{A^p3.7, A^p3.8\} - \{A3.7, A3.8\}.$$

Let $\mathcal{J} = (\mathcal{D}, \mathcal{J}, \alpha)$ be an interpretation of $\mathcal{T}$. Then the v -valuations (of $\mathcal{T}$) on $\mathcal{D}$, or $\mathcal{J}$ -valuations, are called the v -valuations of $\overline{\mathcal{T}}$ on $\mathcal{D}$.

DEFINITION 7.1. *We say that $\overline{\mathcal{J}} = (\mathcal{D}, \overline{\mathcal{J}}, \alpha)$ is a generalized ⁽²⁾ interpretation of $\overline{\mathcal{T}}$ if $\mathcal{D}$ is the non-empty set and $\overline{\mathcal{J}}$ is a function defined on the constants of $\overline{\mathcal{T}}$ (which are those of $\mathcal{T}$) and on $\sim, \supset$, such that, first, the restriction of $\overline{\mathcal{J}}$ on the constants of $\overline{\mathcal{T}}$ is a c -valuation of $\mathcal{T}$ (on $\mathcal{D}$), and second,*

$$(7.3) \quad \sim^* = \overline{\mathcal{J}}(\sim) \in \{0, 1\}^{\{0,1\}}, \quad \supset^* = \overline{\mathcal{J}}(\supset) \in \{0, 1\}^{\{0,1\}^2}.$$

Let us now fix a generalized interpretation $\overline{\mathcal{J}} = (\mathcal{D}, \overline{\mathcal{J}}, \alpha)$ of $\overline{\mathcal{T}}$ and a v -valuation V on $\mathcal{D}$ —to be called $\mathcal{J}$ -valuation. Then the (generalized) designatum $\Delta^* = \text{des}_{\overline{\mathcal{J}}, V}(\Delta)$ of the wfe Δ (of $\overline{\mathcal{T}}$) at $\overline{\mathcal{J}}$ and V , and the function $\Psi_{\mathcal{A}; v_1, \dots, v_n; \overline{\mathcal{J}}, V}$ (where $\overline{\mathcal{J}}$ and perhaps also V can be dropped) associated with the wff $\mathcal{A}$ and the n variables y_1 to y_n (with respect to $\overline{\mathcal{J}}$ and V) are defined recursively and simultaneously by clauses (1) to (9) below, where n and i run over Z^+ and $\mathcal{A}$ and $\mathcal{B}$ are arbitrary wffs of $\overline{\mathcal{T}}$.

(1) If Δ is $x_i [c_i]$, then Δ^* is $V_i [\mathcal{J}(c_i)]$.

(2) [(3)] If τ_1 to τ_n are terms and Δ is $f_i^n(\tau_1, \dots, \tau_n) [R_i^n(\tau_1, \dots, \tau_n)]$, then Δ^* is $f_i^{n*}(\tau_1^*, \dots, \tau_n^*)$ [0 or 1 according to whether or not $(\tau_1^*, \dots, \tau_n^*) \in R_i^{n*}$], where $f_i^{n*} = \overline{\mathcal{J}}(f_i^n)$ and $R_i^{n*} = \overline{\mathcal{J}}(R_i^n)$.

(4) If Δ is $\mathcal{A} \equiv_p \mathcal{B}$ and $\mathcal{A}^* = [\neq] \mathcal{B}^*$, then $\Delta^* = 0$ [1].

⁽²⁾ Ordinary interpretations are special generalized interpretations.

(5) If Δ is $\mathcal{A} \supset_p \mathcal{B}$ and $\mathcal{A}^* = 0$ or $\mathcal{B}^* = 1$, then $\Delta^* = 1$; otherwise $\Delta^* = 0$.

(6) $\Psi_{\mathcal{A}; \bar{v}_1, \dots, \bar{v}_n; \bar{\mathcal{F}}, V}$ is the function $g \in \{0, 1\}^{\mathcal{D}^n}$ such that

$$(7.4) \quad g(\xi_1, \dots, \xi_n) = \text{des}_{\bar{\mathcal{F}}, W}(\mathcal{A}) \quad \text{where } W = \begin{pmatrix} y_1 & \dots & y_n \\ \xi_1 & \dots & \xi_n \end{pmatrix} V$$

$$(\forall \xi_1, \dots, \xi_n \in \mathcal{D}).$$

(7) [(8)] If Δ is $\sim \mathcal{A} [\mathcal{A} \supset \mathcal{B}]$, then Δ^* is $\sim^*(\mathcal{A}^*) [\supset^*(\mathcal{A}^*, \mathcal{B}^*)]$ —see (7.3).

(9) If Δ is $(x_i) \mathcal{A}$, then Δ^* is 0 if $\Psi_{\mathcal{A}; x_i; \bar{\mathcal{F}}, V}(\xi) = 0 \forall \xi \in \mathcal{D}$, and 1 otherwise.

DEFINITION 7.2. We shall say that the generalized interpretation $\bar{\mathcal{F}} = (\mathcal{D}, \bar{\mathcal{F}}, \alpha)$ of $\bar{\mathcal{T}}$ is admissible if $\bar{\mathcal{F}}$ satisfies D_v^p ($v = 1, 2, \dots$), A^p 3.7-8, and $A3.6$.

8. A criterium for non-synonymy. An application of it to logic.

THEOREM. 8.1. If Δ_1 and Δ_2 are wfes of $\mathcal{T}$ and $\Delta_1 \succ \Delta_2$, then

$$(8.1) \quad \text{des}_{\bar{\mathcal{F}}, V} \Delta_1 = \text{des}_{\bar{\mathcal{F}}, V} \Delta_2$$

for every admissible generalized interpretation $\bar{\mathcal{F}}$ of $\bar{\mathcal{T}}$ and all $\mathcal{F}$ -valuations V .

Note that admissible interpretations, unlike models (of $\bar{\mathcal{T}}$) are considered in the theorem above.

PROOF OF THEOR. 8.1. Let $\mathcal{S}$ be the equivalence relation among wfes of $\mathcal{T}$ such that $\Delta_1 \mathcal{S} \Delta_2$ iff $\text{des}_{\bar{\mathcal{F}}, V} \Delta_1 = \text{des}_{\bar{\mathcal{F}}, V} \Delta_2$ for every admissible generalized interpretation $\bar{\mathcal{F}}$ of (the barred extension) $\bar{\mathcal{T}}$ (of $\mathcal{T}$) and every $\mathcal{F}$ -valuation V .

We now show that $\mathcal{S}$ fulfils conditions $C_1)$ to $C_7)$, which define $\succ$ in [1]. To this end we consider an arbitrary choice of $\mathcal{F}$ and V above.

1) Since $\bar{\mathcal{F}}$ is admissible $\text{des}_{\bar{\mathcal{F}}, V} (D'_v \equiv_v D''_v) = \text{des}_{\bar{\mathcal{F}}, V} D_v^p = 0$ ($v = 1, 2, \dots$). Hence, by clause (4) in § 7, $\text{des}_{\bar{\mathcal{F}}, V} D'_v = \text{des}_{\bar{\mathcal{F}}, V} D''_v$. Then, (by the above arbitrariness of $\bar{\mathcal{F}}$ and V) $D'_v \mathcal{S} D''_v$.

2) Assume that $f \mathcal{S} f'$ and $\Delta_i \mathcal{S} \Delta'_i$, where f and f' are some f_i^n 's while Δ_i and Δ'_i are terms; hence $f^* = f'^*$ and $\Delta_i^* = \Delta'^*_i$ ($i = 1, 2, \dots, n$). Then, by clause (2) in § 7, $\text{des}_{\overline{\mathcal{S}}, V}(f(\Delta_1, \dots, \Delta_n)) = \text{des}_{\overline{\mathcal{S}}, V}(f'(\Delta'_1, \dots, \Delta'_n))$. Then $f(\Delta_1, \dots, \Delta_n) \mathcal{S} f'(\Delta'_1, \dots, \Delta'_n)$.

3) Assume $R \mathcal{S} R'$ and $\Delta_i \mathcal{S} \Delta'_i$, where R and R' are some R_i^n 's while Δ_i and Δ'_i are terms, so that $R^* = R'^*$ and $\Delta_i^* = \Delta'^*_i$ ($i = 1, \dots, n$). Then, by clause (3) in § 7, $\text{des}_{\overline{\mathcal{S}}, V}(R(\Delta_1, \dots, \Delta_n)) = \text{des}_{\overline{\mathcal{S}}, V}(R'(\Delta'_1, \dots, \Delta'_n))$. Hence $R(\Delta_1, \dots, \Delta_n) \mathcal{S} R'(\Delta'_1, \dots, \Delta'_n)$.

4)—6) Assume $p \mathcal{S} p'$ and $q \mathcal{S} q'$, where p, p', q , and q' are wffs. Then $p^* = p'^*$ and $q^* = q'^*$ for every $\mathcal{S}$ -valuation V . Hence, by clauses (7) and (8) in § 7, $\text{des}_{\overline{\mathcal{S}}, V}(\sim p) = \sim^*(p^*) = \sim^*(p'^*) = \text{des}_{\overline{\mathcal{S}}, V}(\sim p')$ and (similarly) $\text{des}_{\overline{\mathcal{S}}, V}(p \supset q) = \text{des}_{\overline{\mathcal{S}}, V}(p' \supset q')$. Lastly, by the above arbitrariness of V ,

$$\Psi_{p; x_i; \overline{\mathcal{S}}, V}(\xi) = \Psi_{p'; x_i; \overline{\mathcal{S}}, V}(\xi) \quad (\forall \xi \in \mathcal{D}).$$

Then, by clause (9) in § 7, $\text{des}_{\overline{\mathcal{S}}, V}((x_i)p) = \text{des}_{\overline{\mathcal{S}}, V}((x_i)p')$.

7) Let $\mathcal{A}(x_i)$ and $\mathcal{A}(x_j)$ be (x_i, x_j) -similar wffs. Then, by induction one can prove (the same way as in connection with ordinary interpretations) that

$$\text{des}_{\overline{\mathcal{S}}, V}((x_i)\mathcal{A}(x_i)) = \text{des}_{\overline{\mathcal{S}}, V}((x_j)\mathcal{A}(x_j)).$$

Hence

$$(x_i)\mathcal{A}(x_i) \mathcal{S} (x_j)\mathcal{A}(x_j).$$

We have shown that $\mathcal{S}$ is a relation that fulfils conditions $C_1)$ to $C_7)$. Since $\succsim$ is the smallest among these relations, $\succsim \subseteq \mathcal{S}$ q.e.d.

Note that Theor. 8.1 affords a criterium to recognize when two wffs Δ and Δ' of $\mathcal{S}$ are not synonymous: *it suffices to find an admissible generalized interpretation $\overline{\mathcal{S}}$ of $\overline{\mathcal{S}}$ and an $\mathcal{S}$ -valuation V for which $\text{des}_{\overline{\mathcal{S}}, V}(\Delta) \neq \text{des}_{\overline{\mathcal{S}}, V}(\Delta')$.*

As an example of application of the criterium above we show (8.2)₁ below

$$(8.2) \quad p \not\mathcal{S} \sim \sim p, \quad \sim p \mathcal{S} \sim \sim \sim p, \quad \sim p \not\mathcal{S} \sim \sim \sim p,$$

where p is any atomic wff ⁽³⁾.

⁽³⁾ Relation $\mathcal{S}$ is defined at the outset of the proof of Theor. 8.1.

To this end, we assume that the atomic wff p has the truth value 0 ($p^* = 0$) in the admissible interpretation $\overline{\mathcal{I}}$. In fact it is independent of the choice of $\sim^*$. By stipulating that $\sim^*(0) = 1$ and $\sim^*(1) = 1$, we have $p^* = 0$ and $(\sim \sim p)^* = 1$. Hence the relation $\mathcal{S}$ fails to hold between p and $\sim \sim p$. If $p^* = 1$ the same conclusion is reached by interchanging the roles of 0 and 1 in the reasoning above. Thus (8.2)₁ holds.

REMARK. Relations (8.2)_{2,3} imply that Theor. 8.1 affords a condition that is sufficient for two wfes of $\mathcal{T}$ to be non-synonymous, but is not necessary for this.

We now prove (8.2)₂; (8.2)₃ will be proved in [2], § 14. Assume that $\overline{\mathcal{I}}$ is any generalized interpretation of $\overline{\mathcal{T}}$, V is any $\mathcal{I}$ -valuation, and $f = \sim^* = \overline{\mathcal{I}}(\sim)$. Then $f \in A = \{0, 1\}^{\{0,1\}} = \{I, \underline{0}, \underline{1}, \underline{1} - I\}$ where $I(x) = x$, $\underline{0}(x) = 0$, $\underline{1}(x) = 1$ ($x = 0, 1$). Then $f \circ f \circ f = f$ ($\forall f \in A$). Hence, by clause (7) in § 7, $\text{des}_{\overline{\mathcal{I}}, V}(\sim p) = \text{des}_{\overline{\mathcal{I}}, V}(\sim \sim \sim p)$ q.e.d.

9. Application of the preceding criterion of non-synonymy to an example with arithmetics.

The 1st order theory S introduced in [3], Chap. 3, to treat natural numbers (using our notations) has, besides the logical symbols and the equality attribute R_1^2 , the individual constant c_2 and the function letters f_1^1, f_1^2 , and f_2^2 to denote zero, successor, sum, and product respectively. In order to construct a variant, Σ , of S , fit for our purposes we add S with the attribute R_1^1 and the function letters f_3^2 and f_4^2 , to express natural numbers, exponentiation and logarithm respectively. Furthermore, partly in harmony with [3], we write: 0 for c_2 , $t = s$ for $R_1^2(t, s)$, $t \in \mathcal{N}$ —to be read as « t is a natural number»—for $R_1^1(t)$, t' for $f_1^1(t)$, $t + s$, $t \cdot s$, t^s , and $\lg_t s$ for $f_r^2(t, s)$ with $r = 1$ to 4 respectively, and x to z for x_1 to x_3 respectively.

We define the numeral $\bar{n}$ recursively: $\bar{0} = c_2$, $\overline{n+1} = \bar{n}'$.

The non-logical axioms of Σ are axioms Σ_1 to Σ_{13} below. Among them $\Sigma_{1,2}$ —i.e. Σ_1 to Σ_2 —concern identity, $\Sigma_{3,6}$ and Σ_{13} are Peano's axioms (in a weak version), and $\Sigma_{7,8}$, $\Sigma_{9,10}$, and $\Sigma_{11,12}$ afford the inductive definitions of sum, product and exponentiation respectively, where e.g. $(\forall x, y \in \mathcal{N})p$ means $(x)(y) (x \in \mathcal{N} \wedge y \in \mathcal{N} \supset p)$.

$$\begin{aligned}
\Sigma_1 & (\forall x, y, z \in \mathcal{N}) \wedge x = y \supset (x = z \supset y = z) \\
\Sigma_2 & (\forall x, y \in \mathcal{N}) \supset x = y \supset x' = y' \\
\Sigma_{3.4} & 0 \in \mathcal{N}, \quad x \in \mathcal{N} \supset x' \in \mathcal{N} \\
\Sigma_{5.6} & x \in \mathcal{N} \supset 0 \neq x', \quad (\forall x, y \in \mathcal{N}) \supset x' = y' \supset x = y \\
\Sigma_{7.8} & x \in \mathcal{N} \supset x + 0 = x, \quad (\forall x, y \in \mathcal{N}) \supset x + y' = (x + y)' \\
\Sigma_{9.10} & x \in \mathcal{N} \supset x \cdot 0 = 0, \quad (\forall x, y \in \mathcal{N}) \supset x \cdot y' = x \cdot y + x \\
\Sigma_{11.12} & x \in \mathcal{N} \supset x^0 = \bar{1}, \quad (\forall x, y \in \mathcal{N}) \supset x^{y'} = x^y \cdot x \\
\Sigma_{13} & \mathcal{A}(0) \wedge (\forall x \in \mathcal{N}) [\mathcal{A}(x) \supset \mathcal{A}(x')] \supset (\forall x \in \mathcal{N}) \mathcal{A}(x) \text{ for every } fbf \\
& \mathcal{A}(x) \text{ of } \Sigma.
\end{aligned}$$

The definition system $\{D_\alpha\}_{0 < \alpha < \omega}$ of Σ contains only the following (non-recursive) definition

$$(9.1) \quad y = \lg_x z \equiv_D x^y = z \wedge (E_1 y) x^y = z \vee y = c_1 \wedge \sim (E_1 y) x^y = z.$$

Incidentally, for any wff $\mathcal{A}$ of the above-mentioned theory S —see [3]—, let $\mathcal{A}^\mathcal{N}$ be the wff of Σ obtained from the universal closure of $\mathcal{A}$ by replacing every quantifier (x_i) with its restriction to $\mathcal{N}$, i.e. $(\forall x_i \in \mathcal{N})$.

The axioms (S_1) to (S_9) of S are practically included in axioms Σ_1 to Σ_{13} , in that so are $(S_1)^\mathcal{N}$ to $(S_9)^\mathcal{N}$; and the axioms of Σ that have no counterparts in S are only $\Sigma_{3.4}$ and $\Sigma_{11.12}$. Let us incidentally add that

$$(9.2) \quad \vdash_S \mathcal{A} \Leftrightarrow \vdash_\Sigma \mathcal{A}^\mathcal{N} \quad \text{for every wff } \mathcal{A} \text{ of } S.$$

We now consider the barred extension $\bar{\Sigma}$ of Σ —see (7.1)—and the following ordinary interpretation $\mathcal{J} = (\mathcal{D}, \mathcal{J}, \alpha)$ of it, which describes the case when only 4 natural numbers exist and hence only 3 proper (or existing) individuals exist.

It is assumed that, for some $\alpha \notin \mathcal{N}$, $\mathcal{D} = \{0, 1, 2, 3, \alpha\}$, and that $\mathcal{J}$ is the c -valuation that fulfils conditions $\mathcal{C}_{1.9}$ below where $\Delta^* = \mathcal{J}(\Delta)$ for every wfe Δ of Σ , and where ξ and η are arbitrary elements of $\mathcal{D}$.

$$\begin{aligned}
\mathcal{C}_{1.4} \quad c_1^* &= \alpha, (c_2^*) 0^* = 0, R_1^{1*} = \mathcal{N}^* = \{0, 1, 2, 3\}, \text{ and } R_1^{2*} = \\
&= (=^*) = \text{identity in } \mathcal{D}.
\end{aligned}$$

$$\mathcal{C}_5 \quad f_1^1(\xi) \text{ is } \xi + 1 \text{ if } \xi \in \{0, 1, 2\}, \text{ and } \alpha \text{ otherwise.}$$

$\mathcal{C}_{6.9}$. If both $\xi, \eta \in \{0, 1, 2, 3\}$ and some (unique) number n in $\{0, 1, 2, 3\}$ fulfils the r -th of the equalities $n = \xi + \eta$, $n = \xi \cdot \eta$, $n = \xi^n$, and $\xi^n = \eta$, then $f_r^2(\xi, \eta) = n$; otherwise $f_r^2(\xi, \eta) = \alpha$ ($r = 1, \dots, 4$).

The interpretation $\mathcal{J}$ is admissible in that it satisfies definition (9.1). Incidentally condition $\mathcal{C}_9$ is a consequence of $(\mathcal{C}_{1.8})$ and the requirement that (9.1) should be true in $\mathcal{J}$; furthermore by $\mathcal{C}_{5.9}$ f_1^1 and f_1^2 to f_4^2 express proper functions.

Let us add that $\mathcal{J}$ is not a model of Σ only in that it fails to satisfy axioms Σ_4 and Σ_6 , which are essential to assert that natural numbers are infinitely many. In particular $\mathcal{J}$ satisfies $\Sigma_{7.12}$, which can be regarded as inductive definitions. Thus $\mathcal{J}$ can be considered as admissible in the strong sense, so that the application below of our criterion of non-synonymy can be accepted also when inductive definitions are required to have the same role—in connection with synonymy—as the other definitions.

We are now ready to show that in Σ

$$(9.3) \quad \lg_2 \bar{8} \not\prec \bar{3}, \quad y = \lg_2 \bar{8} \not\prec y = \bar{3}.$$

Indeed, referring to $\mathcal{J}$, $\bar{2}^* = 2$, $\bar{3}^* = 3$, $\bar{8}^* = \alpha$; hence, by $\mathcal{C}_9$, $\lg_2 \bar{8} = \alpha \neq 3 = \bar{3}^*$. Thus, by Theor. 8.1, $(9.3)_1$ holds. At this point it is clear that for $V(y) = 3$

$$\text{des}_{\mathcal{J}, V}(y = \bar{3}) = 0 \neq 1 = \text{des}_{\mathcal{J}, V}(y = \lg_2 \bar{8}).$$

Hence $(9.3)_2$ holds.

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Manoscritto pervenuto in redazione il 4 novembre 1981.