RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Solutions of lower semicontinuous differential inclusions on closed sets

Rendiconti del Seminario Matematico della Università di Padova, tome 69 (1983), p. 99-107

<http://www.numdam.org/item?id=RSMUP_1983__69__99_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 69 (1983)

Solutions of Lower Semicontinuous Differential Inclusions on Closed Sets.

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SUNTO - Si dimostra un teorema di esistenza di soluzioni per la relazione differenziale $\dot{x} \in F(x)$, a secondo membro inferiormente semicontinuo, su di un compatto D soddisfacente opportune ipotesi di tangenza alla Nagumo.

1. Introduction.

Let F be a mapping defined on a Banach space E, taking values in the family of closed and bounded subsets of E. By a solution of the autonomous system

$$\dot{x} \in F(x) \tag{1}$$

$$x(0) = x_0 , \tag{2}$$

we mean an absolutely continuous mapping $x: [0, \tau) \to E$, with $x(0) = x_0$ and $\dot{x} \in F(x)$ almost everywhere (a.e.) on some interval $[0, \tau)$. If F is Hausdorff continuous and convex valued, the evolution system (1)-(2) behaves very much like an ordinary differential equation, and existence results are comparatively easy to obtain. In the case where F is continuous but $F(x)$ is not necessarily convex, existence of solutions was first proven by A. Filippov [3]. Lojasiewicz [5] and the author [2]

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(**) Supported by CNR, Gruppo Nazionale per l'Analisi Funzionale ed Applicazioni.

solved the lower semicontinuous case, using the selection technique developed in [1]. Apparently less attention has received the problem of finding solutions of (1)-(2) taking values inside a closed set $D \subseteq E$. In the present paper we construct a solution lying on a compact set D, provided that the following condition of Nagumo type holds:

$$\forall x \in D,\ \forall y \in F(x), \qquad \lim_{h\to 0+} d(x + hy, D)\, h^{-1} = 0 \,. \tag{3}$$

The proof given here relies on the careful construction of a family of piecewise linear approximate solutions, and shows that the original technique used by A. Filippov can be adapted to the lower semi-continuous case as well. We also show how our result yields a general method for deriving the existence of solutions of (1)-(2) from the properties of the convex-valued orientor field

$$\dot{x} \in \overline{\mathrm{co}}\, F(x) \,. \tag{1*}$$

2. Notations and statement of the main result.

To fix the ideas, we assume that D is a locally closed subset of a Banach space E and F maps D into $\mathcal{K}$, where $\mathcal{K}$ is the family of nonempty, closed but not necessarily convex subsets of the ball $\{x \in E\colon \|x\| \leqslant M\}$, for some $M > 0$. We use the symbol $h(\cdot, \cdot)$ for the Hausdorff distance on $\mathcal{K}$. $d(x, A)$ stands for the distance $\inf\{\|x - a\|\ a \in A\}$ from the point x to the set A, ΔA for the diameter of A, $B[A, \varepsilon]$ for the closed ball $\{x\colon d(x, A) \leqslant \varepsilon\}$ of radius $\varepsilon > 0$ about the set A. The closure and the convex closure of A are denoted by $\bar{A}$ and $\overline{\mathrm{co}}\, A$ respectively. The map F is lower semicontinuous iff

$$\forall x_0 \in D,\ \forall \varepsilon > 0, \quad \exists \delta > 0\colon x \in D, \quad \|x - x_0\| \leqslant \delta \Rightarrow F(x_0) \subseteq B[F(x), \varepsilon] \,.$$

We denote by $\mathbb{N}$ the set of natural numbers and by $\mathcal{L}^1$ the set of Lebesgue integrable mappings from $[0, 1]$ into E. Our main result is the following

THEOREM. *Let D be compact and $F\colon D \to \mathcal{K}$ be a lower semicontinuous orientor field for which* (3) *holds. Then, for any $x_0 \in D$, the system* (1)-(2) *has a solution x defined on* $[0, +\infty)$ *with values in D.*

3. A set of integrable functions.

It clearly suffices to prove the existence of a solution on the interval $[0, 1]$. To do this, we begin by choosing a set of integrable functions that will be used in the construction of approximate solutions. For each integer $n \geqslant 1$, define inductively a finite set of points $A_n = \{a_{n1}, \ldots, a_{nk}\}$ and open neighborhoods $V_{n1}, \ldots, V_{nk}$ with the following properties:

$$a_{ni} \in V_{ni}, \tag{4}$$

$$\bigcup_i V_{ni} \supseteq D, \tag{5}$$

$$F(a_{ni}) \subseteq B[F(x), 2^{-n}] \qquad \forall x \in D \cap V_{ni}, \tag{6}$$

$$\Delta V_{ni} \leqslant \lambda_{n-1}/2 \qquad (\text{if } n \geqslant 2), \tag{7}$$

where λ_{n-1} is a Lebesgue number for the open covering $\{V_{(n-1),i}\}$, i.e. any closed ball with center at some point $x \in D$ and radius λ_{n-1} is entirely contained in some $V_{(n-1),i}$. Note that all this can be done because of the compactness of D and of the lower semicontinuity of F. From now on, if $a \in A_n$, we shall write $V(a)$ for the neighborhood of a that has been selected in the present construction Next define $B_n \subseteq \subseteq A_1 \times A_2 \times \ldots \times A_n$ to be the set of all sequences $(a_1, \ldots, a_n)$ with $a_i \in A_i$ and $V(a_i) \subseteq V(a_{i-1})$. By (5) and (7), for every $a_n \in A_n$ there are $a_i \in A_i$ $(i = 1, \ldots, n-1)$ such that $(a_1, \ldots, a_n) \in B_n$. To each $b \in B_n$ assign, in an inductive way, a point $y(b) \in E$ satisfying

$$y(b) = y(a_1, \ldots, a_n) \in F(a_n), \tag{8}$$

and, if $n > 1$,

$$\|y(b) - y(a_1, \ldots, a_{n-1})\| \leqslant 2^{-n+1}. \tag{9}$$

A suitable $y(b)$ exists because $a_n \in V(a_{n-1})$, hence by (6)

$$y(a_1, \ldots, a_{n-1}) \in B[F(a_n), 2^{-n+1}].$$

We notice that from (9) by induction follows that for any m, $j \geqslant 1$,

$(a_1, \ldots, a_{m+j}) \in B_{m+j}$,

$$\|y(a_1, \ldots, a_{m+j}) - y(a_1, \ldots, a_m)\| \leqslant 2^{-m+1} . \tag{10}$$

Take now a sequence of numbers $\{h_n\}$, decreasing to zero, with

$$1/h_n , \quad h_n/h_{n+1} \in \mathbf{N} , \quad (M+2)\cdot h_n \leqslant \lambda_n/2 \qquad (n = 1, 2, \ldots) \tag{11}$$

Denote by C_n the set of all functions $c\colon [0,1] \to B_n$ such that if $c(t) = (\alpha_1(t), \ldots, \alpha_n(t))$ then α_i is constant on intervals of the type $[sh_i, (s+1)h_i)$ for any integer $s = 0, 1, \ldots, 1/h_i - 1$. We can now define the sets

$$W_n = \{u\colon [0,1] \to E\colon u(t) = y(c(t)) \text{ for some } c \in C_n\} . \tag{12}$$

Every W_n is then a finite set of piecewise constant functions.

4. A basic lemma.

The building blocks for the construction of approximate solutions are now provided.

LEMMA. *Let n be a fixed positive integer, $z_0 \in D$, $B(z_0, \lambda_n) \subseteq V(a)$ for some $a \in A_n$. Then for any $y \in F(a)$ there exists a continuous, piecewise linear mapping $z\colon [0, h_n] \to E$ such that*

$$z(0) = z_0 , \quad z(h_n) \in D , \tag{13}$$

$$\|\dot z(t) - y\| \leqslant 2^{-n+2} \quad \text{a.e. in } [0, h_n] . \tag{14}$$

PROOF. Fix $\varepsilon > 0$. Let Γ be the set of mappings $v\colon [0, \tau] \to E$ for which some $t_0, t_1, \ldots, t_k \in [0, 1]$ exist and satisfy

i) $t_0 = 0$, $t_k = \tau$, $0 < t_i - t_{i-1} \leqslant \varepsilon$,

ii) $v(0) = z_0$, $v(t_i) \in D$, v is linear on $[t_{i-1}, t_i]$ $\qquad (i = 1, \ldots, k)$,

iii) $\|\dot v(t) - y\| \leqslant 2^{-n+1}$ a.e. in $[0, \tau]$,

and let $\Omega = \{\tau\colon \exists v \in \Gamma,\ v(\tau) \in D\}$.

We claim that $T = \sup \Omega > h_n$. Indeed, if $T \leqslant h_n$, let $\{\tau_m\}$ be a sequence in Ω tending to T and let $v_m(\tau_m)$ be the corresponding sequence of points in D. By compactness, we have

$$\lim_{m' \to +\infty} v_{m'}(\tau_{m'}) = \bar{x}$$

for some subsequence $v_{m'}$. Choose $\bar{y} \in F(\bar{x})$ for which $\|\bar{y} - y\| \leqslant 2^{-n}$. Note that this is possible because, by iii), $\|\dot{v}_{m'}\| \leqslant M + 2$ a.e., hence $\|\bar{x} - z_0\| \leqslant (M + 2) h_n \leqslant \lambda_n/2$ and $\bar{x} \in V(a)$. From (3) now follows that there exist $\delta > 0$, $x \in D$ with $\delta < \varepsilon$ and $\|\delta^{-1}(x - \bar{x}) - \bar{y}\| \leqslant 2^{-n-1}$. Therefore, for the same x,

$$\|\delta^{-1}(x - v_{\bar{m}})\tau_{\bar{m}})) - \bar{y}\| \leqslant 2^{-n} \quad \text{and} \quad \sigma = \tau_{\bar{m}} + \delta > T$$

for some $\bar{m}$ large enough. The map $v_{\bar{m}}$ can therefore be extended to $\tilde{v} \in \Gamma$ on the interval $[0, \sigma]$ defining $\tilde{v}(t) = v_{\bar{m}}(t)$ on $[0, \tau_{\bar{m}}]$, $\tilde{v}(\sigma) = x$, $\tilde{v}$ linear on $[\tau_{\bar{m}}, \sigma]$. This contradiction shows that $\sup \Omega \geqslant h_n$.

Let now $v \in \Gamma$ be defined on $[0, \tau]$ with $\tau \geqslant h_n$. By i) and ii) $v(\bar{\tau}) \in D$ for some $\bar{\tau} \in (h_n - \varepsilon, h_n]$. If ε was chosen to be $\leqslant 2^{-n} \cdot h_n/(M + 2)$, one checks that the map z defined by

$$z(t) = v\big((\bar{\tau}/h_n) \cdot t\big) \qquad t \in [0, h_n]$$

satisfies both (13) and (14).

5. Approximate solutions.

Fix any positive integer n. We now consider an approximate solution x_n with the following properties:

i) x_n is continuous and piecewise linear on $[0, 1]$

(15) $$x_n(0) = x_0\,, \qquad x_n(sh_n) \in D \qquad (s = 1, 2, \ldots, 1/h_n)$$

ii) there exists some $c_n \in C_n$, $c_n(t) = \big(\alpha_1(t), \ldots, \alpha_n(t)\big)$ such that

(16) $$B[x_n(t), \lambda_j/2] \subseteq V\big(\alpha_j(t)\big)\,, \qquad j = 1, \ldots, n, \; t \in [0, 1],$$

(17) $$\|\dot{x}_n(t) - y\big(c_n(t)\big)\| \leqslant 2^{-n+2} \qquad \text{a.e. in } [0, 1].$$

To construct x_n, we proceed by induction on s.

Assume that x_n and c_n have been defined for $t \in [0, sh_n]$ for some s, $0 \leqslant s < 1/h_n$. Set

$$I = \{i: 1 \leqslant i \leqslant n,\ sh_n = ph_i \text{ for some integer } p\}\,.$$

From (11) it follows that $I = \{i: k \leqslant i \leqslant n\}$ for a suitable integer k. For $t \in [sh_n, (s+1)h_n)$ we put

$$\text{(18)} \qquad \begin{cases} \alpha_i(t) = \alpha_i\big((s-1)h_n\big) & \text{if } i \notin I \\ \alpha_i(t) = a_i \in A_i & \text{if } i \in I \end{cases}$$

where the new constants a_i satisfy

$$\text{(19)} \qquad B[x_n(sh_n), \lambda_i] \subseteq V(a_i)\,, \qquad \forall i \in I.$$

We need to show that $c_n(t) = \big(\alpha_1(t), \ldots, \alpha_n(t)\big) \in B_n$, i.e. that

$$\text{(20)} \qquad V\big(\alpha_i(t)\big) \subseteq V\big(\alpha_{i-1}(t)\big) \qquad (i = 2, \ldots, n)\,.$$

If $i < k$, then (20) is true by inductive hypothesis because the corresponding a_i did not change. If $i > k$, by (7)

$$V\big(\alpha_i(t)\big) \subseteq B[x_n(sh_n), \lambda_{i-1}] \subseteq V\big(\alpha_{i-1}(t)\big)\,.$$

For $i = k$, denoting by $\bar{q}$ the largest integer q for which $qh_{k-1} \leqslant sh_n$, we have $sh_n - \bar{q}h_{k-1} < h_{k-1}$. Therefore (8), (11), (17) and (19) yield

$$\|x_n(sh_n) - x_n(\bar{q}h_{k-1})\| \leqslant (M+2)\,h_{k-1} \leqslant \lambda_{k-1}/2\,,$$

$$V(a_k) \subseteq B[x_n(sh_n), \lambda_{k-1}/2] \subseteq B[x_n(\bar{q}h_{k-1}), \lambda_{k-1}] \subseteq V(a_{k-1})\,,$$

proving (20). Using the Lemma, we can now define x_n on $[sh_n, (s+1)h_n]$ to be any function assuming the previously assigned value $x_n(sh_n)$ at sh_n and satisfying

$$x_n\big((s+1)h_n\big) \in D\,,$$

$$\|\dot{x}_n(t) - y\big(c(t)\big)\| \leqslant 2^{-n+2} \qquad \text{a.e.}$$

When $x_n(t)$ has been defined for all $t \in [0, 1]$ following the above procedure, it is clear that each function $\alpha_i(t)$, $i = 1, \dots, n$, is constant on every interval of the type $[sh_i, (s+1)h_i)$, hence $c_n = (\alpha_1, \dots, \alpha_n) \in C_n$. If $t \in [0, 1]$, $1 \leqslant i \leqslant n$, denote by $\bar{q}$ the largest integer q for which $qh_i \leqslant t$. Then

$$\|x_n(t) - x_n(\bar{q}h_i)\| \leqslant h_i(M + 2) \leqslant \lambda_i/2 ,$$

$$B[x_n(t), \lambda_i/2] \subseteq B[x_n(\bar{q}h_i), \lambda_i] \subseteq V(\alpha_i(\bar{q}h_i)) = V(\alpha_i(t)) .$$

This proves (16) and completes our construction of approximate solutions.

6. Completion of the proof.

Let $\{x_n\}$ be a sequence of approximate solutions defined according to i)-ii) in the previous section. Then the sequence of derivatives $\Sigma = \{\dot{x}_n \colon n \geqslant 1\}$ is relatively compact in $\mathfrak{L}^1$. Indeed, using (10) and (17). one checks that for any $m \geqslant 1$ the finite set of balls

$$\{B[\dot{x}_n, 2^{-m+2}], n \leqslant m\} \cup \{B[u, 2^{-m+2}], u \in W_m\}$$

covers Σ.

By compactness of $\bar{\Sigma}$ there exists some $\eta \in \mathfrak{L}^1$ and a subsequence $\{x_{n'}\}$ such that

$$\|\dot{x}_{n'} - \eta\|_{\mathfrak{L}^1} \to 0 \qquad \text{and} \qquad \dot{x}_{n'}(t) \to \eta(t) \qquad \text{a.e.}$$

This implies that

$$x_{n'}(t) \to x_0 + \int_0^t \eta(s)\, ds = x(t)$$

uniformly on $[0, 1]$. We claim that x is a solution of (1)-(2) on D. Note that $x_n(sh_i) \in D$ for all integers $s = 1, \dots, 1/h_i$, $i \leqslant n$, therefore $x(t) \in D$ on the dense subset $\{t \in [0, 1],\ t = sh_i,\ 0 \leqslant s \leqslant 1/h_i,\ i \geqslant 1\}$. By continuity, $x(t) \in D$ for every $t \in [0, 1]$.

To show that (1) holds, we consider the sequence of functions $\{c_n\}$: $c_n(t) = (\alpha_{n1}(t), \dots, \alpha_{nn}(t))$ corresponding to the X_n in (16)-(17). For a fixed m, the set of mappings $\{t \to (\alpha_{n1}(t), \ldots, \alpha_{nm}(t)), n \geq 1\} \subseteq C_m$ is finite. Taking a subsequence of $\{x_{n'}\}$, call it $\{x_\nu\}$, we can therefore assume that $(\alpha_{\nu 1}(\cdot), \ldots, \alpha_{\nu m}(\cdot))$ is the same mapping $c_m = (\alpha_1(\cdot), \ldots, \alpha_m(\cdot))$ for every ν. By (16)

$$B[x_\nu(t), \lambda_m/2] \subseteq V(\alpha_m(t)) .$$

Taking the limit as $\nu \to \infty$ we get

$$x(t) \in V(\alpha_m(t)) \qquad \forall t \in [0, 1] . \tag{21}$$

This yields the inequality

$$\begin{aligned} d\big(\dot{x}(t), F(x(t))\big) \leqslant \|\dot{x}(t) - \dot{x}_\nu(t)\| + \|\dot{x}_\nu(t) - y(c_\nu(t))\| + \\ + \|y(c_\nu(t)) - y(c_m(t))\| + d\big(y(c_m(t)), F(x(t))\big) , \quad \text{a.e. in } [0, 1]. \end{aligned} \tag{22}$$

Letting $\nu \to \infty$, the first two terms on the right hand side of (22) tend to zero a.e.. Using (10), (21) and (16) we can now estimate the last two terms and get

$$d\big(x(t), F(x(t))\big) \leqslant 2^{-m+1} + 2^{-m} \qquad \text{a.e. on } [0, 1]$$

Since m was arbitrary, our theorem is proved.

7. Concluding remarks.

It is worth noticing that (3) cannot be replaced by the weaker condition

$$\forall x \in D, \ \exists y \in F(x) , \qquad \lim_{h \to 0+} d(x + hy, D)\, h^{-1} = 0 . \tag{23}$$

To see this, consider the set $D = \Gamma \cup \{0\} \subseteq \mathbb{R}^2$, where Γ is the spiral defined in polar coordinates by

$$\Gamma = \{(\varrho, \theta) \colon \varrho = 2\pi \cdot \theta^{-1}, \ \theta \geqslant 2\pi\} .$$

Take $x_0 = 0$, $F(x) = \{y \in \mathbb{R}^2 \colon \|y\| = 1\}$.

Then F satisfies (23), but the system (1)-(2) has no solution because any path on D connecting the origin with any other point of D has infinite length.

We remark that from our result one can derive a technique for solving some existence problems for a general orientor field (1) simply by means of the properties of the corresponding system (1*).

PROPOSITION. *Let $D \subseteq E$ and let $F: D \to \mathcal{K}$ be lower semicontinuous. Assume that the orientor field* (1*) *has the following properties*:

a) The set Λ of the solutions of (1*)-(2) *on* $[0, 1]$ *with values in D is nonempty and compact,*

b) For each $z_0 \in \mathcal{F}$, $y \in F(z_0)$, there exist $z \in \Lambda$, $t_0 \in [0, 1)$ for which $\lim_{h \to 0+} \big(z(t_0 + h) - z_0\big) h^{-1} = y$,

where $\mathcal{F}$ is the funnel $\{x \in D: \exists z \in \Lambda,\ \exists t \in [0, 1),\ z(t) = x\}$.

Then the system (1)-(2) *has a solution on D.*

Indeed by *a*) $\overline{\mathcal{F}}$ is a compact subset of D and by *b*) the tangential condition (3) holds for each $x \in \Psi$. The same kind of arguments used in the proof of our theorem now provide the existence of a solution of (1)-(2) taking values in $\overline{\mathcal{F}}$.

The author wishes to thank prof. A. Cellina and E. Schechter for their helpful suggestions and comments.

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Manoscritto pervenuto in redazione il 25 novembre 1981.