Rendiconti del Seminario Matematico della Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 69 (1983), p. 63-76

<http://www.numdam.org/item?id=RSMUP_1983__69__63_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 69 (1983)

On a Synonymy Relation for Extensional 1st Order Theories.

PART 1

A Notion of Synonymy.

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1. Introduction (**).

The expressions of a language are generally assigned, besides a designatum, a sense. The study of this notion, which is strictly related to synonymy, is central in the development of the theory of intensions. This notion is something finer and deeper than both extensions of ordinary mathematical logic, and the intensions of modal logic. These intensions, unlike senses, are insufficient to treat belief-sentences, which have already been considered in Carnap's well known booklet [7]. There this treatment is based on the equiformity of wffs (well formed

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Lavoro eseguito nell'ambito dell'attività dei Gruppi di Ricerca Matematica del C.N.R., negli anni accademici 1979-80 e 1980-81.

(**) Bonotto's contribution concerns especially the choice of a criterion sufficient for non-synonymy, Theor. 8.1, and some parts, worked out by her completely, such as Theor. 5.1, Theor. 11.7, Theor. 12.1 and § 14. Furthermore she revised and improved a first draft of the whole work.

formulas). These ideas have been developed by other authors, such as J. Hintikka and R. Montague, and their schools. The topic has been studied in connection with natural languages.

The present work concerns synonymy in connection with an extensional (scientific) formal language, which expresses a typical axiomatic theory $\mathscr{T}$; and it is made from the following point of view.

E.g. in plane geometry the *axis of the segment* AB ($A \neq B$) is often defined to be the class $\{P: |AP| = |BP|\}$ ($|PQ|$ = distance between P and Q). Then the assertion (α) (*for* $A \neq B$) *the axis of the segment* AB *is* $\{P: |AP| = |BP|\}$, or (β) (*for every* P) $|AP| = |BP|$ *iff* P *belongs to the axis of the segment* AB, has no mathematical content, besides no factual content, as well as all mathematical theorems. In fact (α) expresses the identity of two individual notions which have the same sense (besides the same extension), i.e. are synonymous; and (β) is synonymous with a usual definition of $\{P: |AP| = |BP|\}$.

Briefly speaking, our construction of couples formed with synonymous expressions—see § 6—starts from definitions, and then goes on by replacing one or several wfes (well formed expressions) Δ_1 to Δ_n occurring in a wfe Δ with some expressions Δ_1' to Δ_n' that are respectively synonymous with the former.

Especially by the fact above, the definition of a synonymy relation—see Part 1, § 6—is not of the usual inductive type. This may give rise to difficulties, e.g., in proving that some given expressions are not synonymous. Hence it is natural to search for criteria useful for these proofs. To find some of them is one of the aims of the present paper. In particular a relatively simple condition for non-synonymy is given in Part 2, Theor. 8.1, and it is applied to two examples belonging one to logic in § 8 and the other to arithmetics in § 9.

In Part 3 we state two necessary and sufficient conditions for two wfes of $\mathscr{T}$ to be synonymous—see Theors 12.1 and 13.1. To reach this aim, we introduce an auxiliary theory $\dot{\mathscr{T}}$, which is substantially capable to speak of the ($\mathscr{I}$, V)-senses of the wfes of $\mathscr{T}$, i.e. the senses of these wfes relative to the interpretation $\mathscr{I}$ of $\mathscr{T}$ (which need not be a model of $\mathscr{T}$)—see § 10—and to an $\mathscr{I}$-valuation V (which assigns $\mathscr{T}$'s variables with value belonging to the domain of $\mathscr{I}$).

Since $\dot{\mathscr{T}}$ contains term-term operators binding variables (which can be applied to terms and produce terms), in Part 1, § 2, the first order language L, substantially used in [15], is extendet into another, $\mathscr{L}$, which includes the above term-term operators. Furthermore—see § 3—we recall the completeness theorem for theories based on $\mathscr{L}$, proved in [1], because its application to $\dot{\mathscr{T}}$ is essential in the proof of a main theorem, Theor. 13.1 (1).

Let us add that the « non-existing object » substantially introduced by Frege in the semantics of extensional languages, is used also in the semantics considered in § 2; and, in connection with it, *proper* (and *improper*) *functions* and *relations* are considered, in § 4, together with some examples, such as the notion of the *proper vanishing extension* of a function.

Lastly, in § 14 [Part 3] $\sim p$ and $\sim\sim\sim p$, with p atomic, are proved not to be synonymous. By the result $(8.2)_3$ in Part 2, this shows that the rather simple condition proved in Part 2 to be sufficient for non-synonymy, is not necessary for this.

* * *

As far as the notion of synonymy is concerned, e.g. Carnap decides in [8] that $p \wedge q$ and $q \wedge p$ are synonymous. From the contest it is clear that, although this decision is motivated by a use valid in justice courts, it has a certain arbitrariness. If, e.g., one remembers the proofs of the theorem $p \wedge q \equiv q \wedge p$ given in [15] and [17], since they are (and especially the latter is) very complex, it is natural to refuse the synonymy of $p \wedge q$ and $q \wedge p$. This is one among the examples fit to show that it is reasonable to consider various synonymy relations, to be used in correspondence with different situations and different aims.

We consider a synonymy relation, $\asymp$, associated with an extensional 1^{st} order theory, completely formalized and endowed with a definition system. This relation takes the facts above into account and is narrower than any other synonymy relation (considered by us). Furthermore $\asymp$ differs from other synonymy relations dealt with so far—see [1], [2], [18]—in that it is strictly connected with a definition system (in a certain way).

Let us add that any theory $\mathscr{T}_1$ having two synonymous primitive terms Δ and Δ', is not explicitly considered here, because this case can

(1) The paper [3] deals with 1^{st} order extensional theories including general (wfe-wfe) operators binding variables. It has been done as a necessary preliminary for the present work. In fact, so far only wff-term operators had been treated—see [10], [11], [12]. In [10] only the semantic point of view is considered, but [11] and [12] are made from a syntactical point of view and completeness theorems are proved there.

immediately be dealt with by means of a suitable theory $\mathcal{T}_2$, of the kind treated in the present work, that e.g.

(i) contains only the first of those terms as primitive, and

(ii) contains the definition $\Delta' = \Delta$.

2. Semiotics and semantics for a natural generalization $\mathcal{L}$ of a well known 1st order language, that contains term-term operators.

We consider the extension $\mathcal{L}$ of the 1st order language L dealt with in [15], obtained from L by addition of term-term operators Ω_i^n binding variables—see (c) below. The symbols of $\mathcal{L}$ are the connectives $\sim$ and $\supset$, open and closed parentheses, comma, the *individual variables* x_i, *individual constants* c_i, *predicative constants* (or *letters*) R_i^n, *functional ones* f_i^n, *and term-term operator constants* (or *letters*) (*binding variables*) Ω_i^n ($n, i = 1, 2, \ldots$). Incidentally c_1 will be used to express the «non-existing object» α.

The *terms* of $\mathcal{L}$ are defined recursively by conditions (a) to (c) below, where n, $i = 1, 2, \ldots$:

(a) *x_i and c_i are terms,*

(b) *if τ_1 to τ_n are terms, then $f_i^n(\tau_1, \ldots, \tau_n)$ is a term,*

(c) *if τ is a term and y_1 to y_n are n (distinct) variables, then $(\Omega_i^n y_1, \ldots, y_n)(\tau)$ is a term.*

Well formed formulas, briefly wffs, are defined by means of $\sim$, $\supset$ and the universal quantifiers (x_i) in the usual way—see e.g. [15], pp. 46-47. If Δ is a term or a wff, it will be said to be a *well formed expression*, briefly wfe. Furthermore, if y_1 to y_n are n variables, we shall say that $(\Omega_i^n\, y_1, \ldots, y_n)$ is an *operator*, that it *binds* or *acts* on y_1 to y_n, and that the term τ is its *scope* in $(\Omega_i^n\, y_1, \ldots, y_n)(\tau)$ or any occurrence of this expression in any wfe.

An occurrence of x_i in a wfe Δ will be said to be *bound* or *free* according to whether or not it belongs to the scope of a quantifier acting on it (and possibly on other variables). The term τ is said to be *free for x_i* in the wfe Δ if no occurence of x_i in Δ belongs to the scope of a quantifier — $(\Omega_i^n\, y_1, \ldots, y_n)$ or (y_1)—acting on some variable free in Δ.

Convention 2.1. a) Assume that (i) Δ is a wfe of L, or $\mathcal{L}$, without any of the operators Ω_i^n, (ii) X is a set of variables including those free in $\varDelta$, (iii) $\varDelta$ has q occurrences ω_1 to ω_q of universal quantifiers (z_1) to (z_q), (iv) u_j is the first variable outside $X \cup \{z_1, \ldots, z_q\}$ and different from u_1 to u_{j-1}, and (v) $\varDelta_0$ is $\varDelta$, while $\varDelta_j$ is obtained from $\varDelta_{j-1}$ by replacing z_j with u_j in ω_j and at the free occurrences of z_j in the scope of ω_j $(j = 1, \ldots, q)$. Then $\varDelta_q$ will be denoted by $\varDelta^{(X)}$.

(*b*) Assume that $\mathscr{T}$ is based on $\mathscr{L}$ and has operators. Then the obvious analogue of (*a*) holds.

Note that if $\bar{\omega}_j$ is the j-th occurrence in $\varDelta^{(X)}$ of a term-term operator $(\Omega_i^n\, v_1, \ldots, v_n)$ or a universal operator (v_1) (from the left to the right), then for $i \neq j$ the variables in $\bar{\omega}_i$ are different from all variables in $\bar{\omega}_j$, (and from the free variables in $\varDelta$).

CONVENTION 2.2. If, first, the wfe $\varDelta$ is denoted by $\hat{\varDelta}(y_1, \ldots, y_n)$, where y_1 to y_n are n variables, and then the expression $\hat{\varDelta}(\tau_1, \ldots, \tau_n)$ is used, where τ_1 to τ_n are terms, this denotes the result of replacing the free occurrences of y_1 to y_n in $\varDelta^{(X)}$ with τ_1 to τ_n respectively, where X is the set of the variables free in any of the wfes τ_1 to τ_n and $\varDelta$—so that τ_1 to τ_n are free for y_1 to y_n respectively in $\varDelta^{(X)}$.

As far as the semantics for $\mathscr{L}$ is concerned, by extending the § 2 of Chap. 2 in [15], p. 49, we say that $\mathscr{I} = (\mathscr{D}, \mathscr{I}, \alpha)$ is an interpretation of $\mathscr{L}$, if $\mathscr{D}$ is a non-empty set (the *domain* of $\mathscr{I}$), $\alpha \in \mathscr{D}$ (α is to be used as the « non-existing object », see (4.3-5) below), $\mathscr{I}$ is a function called a *c-valuation* in that it is defined (only) on the constants of $\mathscr{L}$ and (it evaluates them in the sense that), for n, $i = 1, 2, \ldots$, we have $c_i^* = \mathscr{I}(c_i) \in \mathscr{D}$ with $c_1^* = \alpha$, $R_i^{n*} = \mathscr{I}(R_i^n) \subseteq \mathscr{D}^n$ (n-th cartesian power of $\mathscr{D}$), $f_i^{n*} = \mathscr{I}(f_i^n) \in \mathscr{D}^{\mathscr{D}^n}$ (set of the mappings of $\mathscr{D}^n$ into $\mathscr{D}$), and lastly $\Omega_i^{n*} = \mathscr{I}(\Omega_i^n)$ is a functional, and precisely a mapping of $\mathscr{D}^{\mathscr{D}^n}$ into $\mathscr{D}$.

A denumerable sequence $V = (V_1, V_2, \ldots)$ of elements of $\mathscr{D}$ $(V \in \mathscr{D}^\omega)$ will be said to be an *$\mathscr{I}$-valuation* or a *v-valuation* looking forward to considering V_i as the value of the variable x_i $(i = 1, 2, \ldots)$.

If $V \in \mathscr{D}^\omega$, i_1 to i_n are in Z^+—i.e. are positive integers—, and $\xi_1, \ldots, \xi_n \in \mathscr{D}$, then as a convention we stipulate that

$$(2.2) \qquad \begin{cases} W = \begin{pmatrix} x_{i_1} \ldots x_{i_n} \\ \xi_1 \ \ldots \ \xi_n \end{pmatrix} V \Leftrightarrow W \in \mathscr{D}^\omega \text{ and } W_{i_s} = \xi_s \\ (s = 1, \ldots, n) \quad \text{and } W_l = V_l \ \left(l \in Z^+ - \{i_1, \ldots, i_n\}\right). \end{cases}$$

Let us fix an interpretation $\mathscr{I}$ and an $\mathscr{I}$-valuation V. Then, by simultaneous recursion, clauses (1) to (4) below define the *designatum*

$\tau^* = V^*(\tau) = \mathrm{des}_{\mathscr{I},V}(\tau)$ of the term τ (of $\mathscr{L}$) at $\mathscr{I}$ and V, and the *function* $\varphi_{\tau;y_1,\ldots,y_n;\mathscr{I},V}$ *associated* to τ and the n variables y_1 to y_n, with respect to $\mathscr{I}$ and V.

(1) *If τ is x_i $[c_i]$, then τ^* is V_i $[\mathscr{I}(c_i)]$.*

(2) *If τ_1 to τ_n are terms and τ is $f_i^n(\tau_1, \ldots, \tau_n)$, then $\tau^* = f_i^{n*}(\tau_1^*, \ldots, \tau_n^*)$ where f_i^{n*} is $\mathscr{I}(f_i^n)$.*

(3) *$\varphi_{\tau;y_1,\ldots,y_n;\mathscr{I},V}$ is the function $f \in \mathscr{D}^{\mathscr{D}^n}$ such that*

$$\begin{cases} \varphi(\xi_1, \ldots, \xi_n) = \mathrm{des}_{\mathscr{I},W}(\tau) \qquad \textit{where } W = \begin{pmatrix} y_1 \ldots y_n \\ \xi_1 \ldots \xi_n \end{pmatrix} V \\ (\forall \xi_1 \ldots \xi_n \in \mathscr{D}) . \end{cases} \tag{2.3}$$

(4) *If τ is the term $(\Omega_i^n\, y_1, \ldots, y_n)(\Delta)$, so that Δ is a term, then $\tau^* = \Omega_i^{n*}(\varphi_{\Delta;y_1,\ldots,y_n;\mathscr{I},V})$.*

Now the metalinguistic expression (i) *the wff $\mathscr{A}$ is satisfied (in $\mathscr{I}$) by the v-valuation V*, and (ii) *$\mathscr{A}$ is true in $\mathscr{I}$* and (iii) *$\mathscr{I}$ is a model of the class Γ of wffs of $\mathscr{L}$* can substantially be defined as in [15], p. 51.

3. Statement of completeness theorems for a 1st order extensional theory $\mathscr{T}$ with term-term operators.

Briefly, we can identify a theory $\mathscr{T}$ with a 6-tuple (S, wfe, LA, PA, R, DS) where S is a non-empty set (formed with $\mathscr{T}$'s *symbols*), wfe is a set of finite sequences of elements of S (to be called *wfes* of $\mathscr{T}$), LA and PA are subsets of wffs (formed by the *logical axioms* and *proper axioms*), DS (*the definition system* in $\mathscr{T}$) is a well ordered set of wffs of $\mathscr{T}$, and R is a set of functions from sequences of wffs to wffs (to be called *inference rules*). We shall say that $\mathscr{T}$ is *based* on $\mathscr{L}$ if

(i) $\mathscr{T}$'s symbols are $\sim$, $\supset$, open and closed parentheses, comma, the x_i's, some (possibly none or all of the) c_i's, some f_i^n's, some Ω_i^n's, and at least one R_i^n.

(ii) $\mathscr{T}$'s wfes are the wfes of $\mathscr{L}$ that are formed with symbols of $\mathscr{T}$, and

(iii) $\mathscr{T}$'s logical axioms are the wffs of $\mathscr{T}$ having the forms of the logical axioms for the predicate calculus in [15], p. 57, to be denoted here by A3.1 to A3.5.

(iv) $\mathscr{T}$'s proper axioms are (arbitrary) wffs of $\mathscr{T}$, which hovewer include axioms AA3.6-8 below for the non-existing object c_1, identity, and operators.

(v) $\mathscr{T}$'s inference rules are modus ponens (MP) and the generalization rule (Gen),

(vi) the definition system DS, possibly empty, has the form specified in § 6.

The axioms for identity considered in [15] are substantially equivalent with the following two:

A3.6-7 $$x_i = x_i\,, \qquad x_i = x_j \wedge \mathscr{A}(x_i) \supset \mathscr{A}(x_j)\,.$$

Since $\mathscr{T}$ also has operators, here identity must be required to fulfil also axiom A3.8 below.

DEF. 3.1. (*a*) *Let y_1 to y_n be n variables as well as z_1 to z_n. We say that the wfes $\hat{\Delta}_1 = \hat{\Delta}_1(y_1, \ldots, y_n)$ and $\hat{\Delta}_2 = \hat{\Delta}_2(z_1, \ldots, z_n)$ are $(y_1, z_1, \ldots, y_n, z_n)$-similar iff for some variables v_1 to v_n free in $\Delta_1 [\Delta_2]$ for y_1 to y_n $[z_1$ to $z_n]$, the wfes $\hat{\Delta}_2(v_1, \ldots, v_n)$ and $\hat{\Delta}_1(v_1, \ldots, v_n)$ coincide up to a bijection of the bound variables of the former onto those of the latter.* (*b*) *We shall say that Δ_1 and Δ_2 are similar if they are $(y_1, z_1, \ldots, y_n, z_n)$-similar for some variables $y_1, z_1, \ldots, y_n, z_n$.*

Obviously the above similarity is an equivalence relation.

A3.8 $$(y_1) \ldots (y_n)(\tau = \tau_1) \supset (\Omega_i^n y_1, \ldots, y_n)\,\tau = (\Omega_i^n z_1, \ldots, z_n)\,\tau_2$$

where τ_1 and τ_2 are $(y_1, z_1, \ldots, y_n, z_n)$-similar terms.

By definition, an *interpretation* $\mathscr{I}' = (\mathscr{D}, \mathscr{I}', \alpha)$ of $\mathscr{T}$ is what one obtains from an interpretation $\mathscr{I} = (\mathscr{D}, \mathscr{I}, \alpha)$ of $\mathscr{L}$ by identifying $\mathscr{I}'$ with the restriction $\mathscr{I}_{\mathscr{T}}$ of $\mathscr{I}$ to the constants of $\mathscr{T}$. We obviously say that $\mathscr{I}_{\mathscr{T}}$ is a *model* of $\mathscr{T}$, if it is a model of $\mathscr{T}$'s (proper) axioms. Furthermore this model is called *normal* if the designatum $=^*$ of $=$ in it is the identity on $\mathscr{D}$. Theor. 3.1 below can be regarded as an analogue for $\mathscr{L}$ of Proposition 2.12 in [15], p. 65, combined with the contractibility theorem of any model into a normal one—see [15], p. 80—and with Proposition 2.27 in [15], p. 80, on the existence of countable normal models for theories based on L.

THEOR. 3.1. *Every consistent theory* (2) $\mathcal{T}$ *based on* $\mathcal{L}$ *has a countable normal model* (3).

Then Theors 3.2-3 below are briefly deduced from Theor. 3.1, substantially the same way as their analogues for L—see Corollaries 2.14 and 2.15 (a) in [15], p. 68—are deduced from Prop. 2.12 in [15].

THEOR. 3.2. *Every logically valid wff* $\mathcal{A}$ *of a theory* $\mathcal{T}$ *based on* $\mathcal{L}$, *is a theorem of* $\mathcal{T}$.

THEOR. 3.3 [3.4] (*of completeness*). *A wff* $\mathcal{A}$ *of the theory* $\mathcal{T}$ *based on* $\mathcal{L}$, *is true in every* (*countable*) *model* [*every normal model*] *of* $\mathcal{T}$ *iff it is a theorem of* $\mathcal{T}$.

Since every model of $\mathcal{L}$ can be contracted in an equivalent normal model, Theor. 3.4 is a straightforward consequence of Theor. 3.3.

4. Proper functions and relations.

If DS is non-empty, $\mathcal{T}$ must have the constant c_1 to denote the non-existing object α, and the predicate letter R_1^2 to be expressed by «$=$» (as well as for the theories with identity considered in [15]). The wff $f_i^n(\tau_1, \ldots, \tau_n) = c_1$ will be considered to express that $\langle \tau_1^*, \ldots, \tau_n^* \rangle$ is outside the domain of f_i^{n*}. Therefore e.g. the wff $f_i^n(c_1, \ldots, c_n) \neq c_1$ seems unacceptable. If this wff is true, it is natural to say that the function f_i^{n*} is not *proper*. Formally *proper n-ary functions* and *relations* (PrF_n, PrR_n) can be defined as follows

$$(4.1) \qquad f_i^n \in \mathrm{PrF}_n \equiv_D x_1 = c_1 \vee \ldots \vee x_n = c_1 \supset f_i^n(x_1, \ldots, x_n) = c_1 ,$$

$$(4.2) \qquad R_i^n \in \mathrm{PrR}_n \equiv_D x_1 = c_1 \vee \ldots \vee x_n = c_1 \supset \sim R_i^n(x_1, \ldots, x_n) .$$

Practically all primitive notions in mathematical theories—e.g.

(2) A theory is said to be *consistent* if the set of its axioms has a model.

(3) This theorem is a special case of Theor. 7.1 in [3], proved there in connection with more general operators—see footnote (1). Its proof is rather complex, apparently because it refers to a first order theory (without descriptions), where Zermelo Fraenkel axioms for sets are disregarded. In fact the treatment of a general operators made in [4], for the modal calculus $\mathcal{MC}^\nu$ based on a type system with infinitely many levels and containing descriptions, is very simple.

natural number or successor—are proper. However words such as « inexistence » and « inexistent » are used e.g. as follows:

a) l is inexistent, where $l = \lim_{x \to 0} \sin x^{-1}$, *and*

b) the inexistence of l is easy to prove.

Formally we can translate *a*) by Inexist(l) where

$$\text{(4.3)} \qquad Inexist\,(x) \equiv_D x = c_1 .$$

Assertion *b*) is not extensional (and e.g. not (causally) modal); in it the sense of l's inexistence is essential. Therefore we prefer to give an example of another improper function. The following functions Φ and Ψ are useful to define *the improper* and *proper vanishing extensions* $f = f_\mu$ and $g = g_\mu$ respectively of a mapping μ of a part of $\mathbb{R}$ into $\mathbb{R}$ —e.g. $\mu(x) = \sqrt{x}$ ($\geqslant 0$):

$$\text{(4.3)} \qquad \begin{cases} \Phi(y) &= (\imath z)(y \neq c_1 \wedge z = y \vee y = c_1 \wedge z = 0) \\ \Psi(y, x) &= (\imath z)[x \neq c_1 \wedge z = \Phi(y) \vee x = c_1 \wedge z = c_1] \end{cases}$$

Then, setting e.g.

$$\text{(4.4)} \qquad f(x) = \Phi(\sqrt{x}) , \qquad g(x) = \Psi(\sqrt{x}, x) ,$$

we have

$$\text{(4.5)} \qquad f(x) = g(x) = \begin{cases} \sqrt{x} & \text{for } x \geqslant 0 \\ 0 & \text{for } x < 0 \end{cases} ; \qquad \begin{cases} f(x) = 0 \\ g(x) = c_1 \end{cases} \quad \text{for } x \notin R .$$

The domain of f is the whole set of individuals, hence $f(c_1) = 0$, while the one of g is $\mathbb{R}$. The function Φ and Ψ are improper as well as f and unlike g.

5. On normal interpretations. Independence of the additional axiom for identity with operators.

THEOR. 5.1. *a) Every normal interpretation* $\mathscr{I} = (\mathscr{D}, \mathscr{I}, \alpha)$ *of a theory* $\mathscr{T}$ *based on* $\mathscr{L}$ *satisfies* A3.8.

b) There exists an interpretation of $\mathscr{T}$ in which AA3.6-7 *hold and* A3.8 *fails to hold.*

Indeed let $\mathscr{I}$ be any normal interpretation of $\mathscr{T}$. Then it obviously satisfies AA3.6-7. Assume that (i) V is any $\mathscr{I}$-valuation, (ii) y_1 to y_n, as well as z_1 to z_n are n variables, while τ, τ_1, and τ_2 are terms, (iii) τ_1 and τ_2 are $(y_1, z_1, \ldots, y_n, z_n)$-similar, and (iv) $\mathrm{des}_{\mathscr{I},V}[(y_1) \ldots (y_n)\, \tau = \tau_1] = 0$. By (iv), for all $\xi_1, \ldots, \xi_n \in \mathscr{D}$, $\mathrm{des}_{\mathscr{I},W}(\tau = \tau_1) = 0$, i.e.

$$(5.1) \qquad \begin{cases} \mathrm{des}_{\mathscr{I},W}(\tau) = \mathrm{des}_{\mathscr{I},W}(\tau_1)\,, \qquad \text{where } W = \begin{pmatrix} y_1 \ldots y_n \\ \xi_1 \ldots \xi_n \end{pmatrix} V \\ (\forall \xi_1, \ldots, \xi_n \in \mathscr{D})\,. \end{cases}$$

On the other hand, by (iii)

$$(5.2) \qquad \begin{cases} \mathrm{des}_{\mathscr{I},W}(\tau_1) = \mathrm{des}_{\mathscr{I},W'}(\tau_2)\,, \qquad \text{where } W' = \begin{pmatrix} z_1 \ldots z_n \\ \xi_1 \ldots \xi_n \end{pmatrix} V \\ (\forall \xi_1, \ldots, \xi_n \in \mathscr{D})\,. \end{cases}$$

By (5.1-2) and clause (3) in § 2, $\varphi_{\tau;\, y_1,\ldots,y_n;\, \mathscr{I},V} = \varphi_{\tau_2;\, z_1,\ldots,z_n;\, \mathscr{I},V}$. Then by clause (4) in § 2,

$$\mathrm{des}_{\mathscr{I},V}[(\Omega_i^n y_1, \ldots, y_n)\, \tau] = \mathrm{des}_{\mathscr{I},V}[(\Omega_i^n z_1, \ldots, z_n)\, \tau_2]\,,$$

so that $\mathrm{des}_{\mathscr{I},V}(C_{3.8}) = 0$, where $C_{3.8}$ is the consequent of the implication in A3.8. Hence A3.8 is true in $\mathscr{I}$, so that *a*) holds.

In order to prove (*b*), we consider a theory $\mathscr{T}$ based on $\mathscr{L}$ (with $DS = \emptyset$) that has only one individual constant, c_2, one predicative letter, $=$, two functional letters f_1^1 and f_2^1, and one operator Ω^1. Let p ($\in \mathrm{N}$, the set of natural numbers) be a prime. We consider an interpretation $\mathscr{I} = (\mathscr{D}, \mathscr{I}, \alpha)$ of $\mathscr{T}$ for which $\mathscr{D} = \mathrm{N}$,

$$(5.3) \qquad c_2^* = 0\,, \quad f_1^{1*}(\xi) = 0\,, \quad f_2^{1*}(\xi) = p \quad (\forall \xi \in \mathrm{N})\,,$$

and $=^*$ is the congruence $\bmod\, p$ on N. Furthermore

$$(5.4) \qquad \Omega^{1*}(\varphi) = \begin{cases} 1 & \text{if } \varphi = f_1^{1*}\,, \\ 2 & \text{otherwise}\,. \end{cases}$$

On the one hand, $(x_i)\, f_1^1(x_i) = f_2^1(x_i)$ is true in $\mathscr{I}$. On the other hand

$(\Omega^1 x_i) f_1^1(x_i) = (\Omega^1 x_i) f_2^1(x_i)$ is false in $\mathscr{I}$, because $\varphi_{f_1^1(x_i); x_i, V}(\xi) = 0$ and $\varphi_{f_2^1(x_i); x_i, V}(\xi) = p$ $(\forall \xi \in \mathrm{N})$, so that

$$\Omega^{1*}(\varphi_{f_1^1(x_i); x_i, V}) = 1 \neq 2 = \Omega^{1*}(\varphi_{f_2^1(x_i); x_i, V}) .$$

We conclude that A3.8 is *independent of the usual axioms* AA3.6-7 *on identity.*

6. On the synonymy relation for a 1st order theory $\mathscr{T}$ based on L.

Let $\mathscr{T}_0 = (S_{\mathscr{T}_0}, \ldots, R_{\mathscr{T}_0}, \emptyset)$ be an ordinary 1st order theory with identity (such as those considered in [15]), hence without operators and with $R_{\mathscr{T}_0} = \{\mathrm{MP}, \mathrm{Gen}\}$. Furthermore let $\mathscr{T}_0$ be a theory with the non-existing object, to be denoted by c_1.

We say that $\{D_\nu\}_{0<\nu<\omega}$ is an *admissible definition system* for $\mathscr{T}_0$ if

(i) D_ν is a wff of the form $D_\nu^{\mathrm{I}} \equiv D_\nu^{\mathrm{II}}$ (D_ν^{I} [D_ν^{II}] is called the *definiendum* [*definiens*] of D_ν) $(\nu = 1, 2, \ldots)$ and

(ii) denoting by $\mathscr{T}_{\nu-1}$ the theory obtained from $\mathscr{T}_0$ by adding D_1 to $D_{\nu-1}$ as proper axioms for $\nu = 1, 2, \ldots$, i.e. $\mathscr{T}_{\nu-1} = (S_{\mathscr{T}_0} \cup \mathscr{C}_{\nu-1}, \ldots, R_{\mathscr{T}_0}, \{D_n\}_{0<n\leqslant\nu-1})$ where $\mathscr{C}_{\nu-1}$ is formed by the constants in $D_1^{\mathrm{I}} \wedge \ldots \wedge D_{\nu-1}^{\mathrm{I}}$, one of alternatives (a) to (c) below holds.

(*a*) D_ν^{I} *is* $R_i^n(x_1, \ldots, x_n)$ $(n > 0)$, *where* $R_i^n \notin S_{\mathscr{T}_{\nu-1}}$—*see* § 3—*and the variables free in* D_ν^{II} *are included in the list* x_1 *to* x_n;

(*b*) D_ν^{I} *has the form* $x_{n+1} = f_i^n(x_1, \ldots, x_n)$ $(n > 0)$, *where* $f_i^n \notin S_{\mathscr{T}_{\nu-1}}$ —*see* § 3—*and the free variables of* D_ν^{II} *are in the list* x_1 *to* x_{n+1}, *and* D_ν^{II} *has the form*

$$D_\nu^{\mathrm{II}} \equiv \left(D_\nu \wedge (E_1 x_{n+1}) D_\nu^{\mathrm{III}}\right) \vee \left(x_{n+1} = c_1 \wedge \sim (E_1 x_{n+1}) D_\nu^{\mathrm{III}}\right) ; \tag{6.1}$$

(*c*) D_ν^{I} *is* $x_1 = c_r$ *for some constant* c_r *not in* $S_{\mathscr{T}_{\nu-1}}$, D_ν^{II} *includes (at most) the free variable* x_1, *and has the form*

$$D_\nu^{\mathrm{II}} \equiv D_\nu^{\mathrm{III}} \wedge (E_1 x_1) D_\nu^{\mathrm{III}} \vee x_1 = c_1 \wedge \sim (E_1 x_1) D_\nu^{\mathrm{III}} . \tag{6.2}$$

Let us add that a function f_i^n is often introduced by a definition in the following way. Under the assumption $D^{\mathrm{III}} \wedge D^{\mathrm{IV}}$ one proves that $(E_1 x_{n+1}) D^{\mathrm{III}}$; and as a definition one asserts that, if $D^{\mathrm{III}} \wedge D^{\mathrm{IV}}$, then one says that D^{I} (i.e. $x_{n+1} = f_i^n(x_1, \dots, x_n)$) holds. Furthermore in these cases, generally, $D^{\mathrm{III}} \wedge \sim D^{\mathrm{IV}}$ implies $\sim (E_1 x_{n+1}) D^{\mathrm{III}}$ (4). The above situation is obviously mirrored by contextual definition of the form D_ν, i.e. $D_\nu^{\mathrm{I}} \equiv D_\nu^{\mathrm{III}}$ where (6.1) holds, or where as a special case ($n = 0$) so does (6.2). In fact (c) is substantially condition (b) for $n = 0$.

Let us set

$$(6.3) \qquad \mathscr{T} = (S_{\mathscr{T}_0} \cup \mathscr{C}_\omega, \dots, R_{\mathscr{T}_0}, \{D_\nu\}_{0<\nu<\omega}),$$

where $\mathscr{C}_\omega = \bigcup_{0<i<\omega} \mathscr{C}_i$.

We say that $\asymp$ is the *synonymy relation* for $\mathscr{T}$ if $\asymp$ is the smallest equivalence between wfes of $\mathscr{T}$ that fulfils conditions C_1) to C_7) below, where (i) *f and f' [R and R'] are arbitrary functional [predicative] constants of $\mathscr{T}$*, (ii) *Δ_1 to Δ_n and Δ_1' to Δ_n' are arbitrary terms of $\mathscr{T}$*, (iii) *p, q, p' and q' are arbitrary wffs of $\mathscr{T}$ and* (iv) *ν and n are arbitrary positive integers.*

C_1) $D_\nu^{\mathrm{I}} \asymp D_\nu^{\mathrm{II}}$

C_2) $f \asymp f'$ *and* $\Delta_i \asymp \Delta_i'$ $(i = 1, \dots, n) \Rightarrow f(\Delta_1, \dots, \Delta_n) \asymp f'(\Delta_1', \dots, \Delta_n')$

C_3) $R \asymp R'$ *and* $\Delta_i \asymp \Delta_i'$ $(i = 1, \dots, n) \Rightarrow$
$\Rightarrow R(\Delta_1, \dots, \Delta_n) \asymp R'(\Delta_1', \dots, \Delta_n')$

C_4) $p \asymp p'$ *and* $q \asymp q' \Rightarrow p \supset q \asymp p' \supset q'$

C_5) $p \asymp p' \Rightarrow \sim p \asymp \sim p'$

C_6) $p \asymp p' \Rightarrow (x_i)p \asymp (x_i)p'$

C_7) $(x_i)A(x_i) \asymp (x_j)A(x_j)$

if $A(x_i)$ and $A(x_j)$ are (x_i, x_j)-similar—see § 3.

By induction on the length of Δ one can easily prove the following

(4) For instance, let us consider D_ν to be the definition of *axis of the segment AB* in plane Eucledean geometry (so that (b) holds for $n = 2$). Then D_ν^{I}, D_ν^{III} and D_ν^{IV} can be chosen as follows:

$D_\nu^{\mathrm{I}} \equiv_D$ « *r is the axis of the segment AB* »

$D_\nu^{\mathrm{III}} \equiv_D$ « $P \in r$ iff *P is a point having equal distances from A and B* », and

$D_\nu^{\mathrm{IV}} \equiv_D$ « *A and B are distinct points* ».

THEOR. 6.1. *Assume that* (i) $\mathcal{T}^{(p)}$ *is the theory obtained from* $\mathcal{T}$ *by adding the propositional letters* $p_1, p_2, \ldots$ *and the corresponding axioms of the first order predicate calculus,* (ii) Δ *is a wfe of* $\mathcal{T}^{(p)}$, (iii) $\mathcal{F}(\tau_1, \ldots, \tau_n; \mathcal{A}_1, \ldots, \mathcal{A}_m)$ *is the wfe of* $\mathcal{T}$ *obtained from* Δ *by replacing the free occurrences of* x_i *in* Δ *with the term* τ_i $(i = 1, \ldots, n)$ *and* p_s *with the wff* $\mathcal{A}_s$ $(s = 1, \ldots, m)$, *and* (iv) τ_i *and* τ_i' *are terms and* $\tau_i \asymp \tau_i'$ $(i = 1, \ldots, n)$, *while* $\mathcal{A}_s$ *and* $\mathcal{A}_s'$ *are wffs of* $\mathcal{T}$ *and* $\mathcal{A}_s \asymp \mathcal{A}_s'$ $(s = 1, \ldots, m)$. *Then* $\mathcal{F}(\tau_1, \ldots, \tau_n; \mathcal{A}_1, \ldots, \mathcal{A}_m) \asymp \mathcal{F}(\tau_1', \ldots, \tau_n'; \mathcal{A}_1', \ldots, \mathcal{A}_m')$.

Note that in clause (iii) τ_i need not be supposed to be free for x_i in Δ.

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Manoscritto pervenuto in redazione il 4 novembre 1981.