

RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova,
tome 69 (1983), p. 37-40

http://www.numdam.org/item?id=RSMUP_1983__69__37_0

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A Remark on Certain Overdetermined Systems of Partial Differential Equations.

LAMBERTO CATTABRIGA (*)

SUMMARY - A connection between the surjectivity of a differential polynomial on a Gevrey space and the solvability of certain overdetermined systems is indicated.

Let $P(D)$, $D = (D_1, \dots, D_n)$, $D_j = -i \partial x_j$, $j = 1, \dots, n$, be a linear differential operator on R^n with constant coefficients and let $d_j \geq 1$ be rational numbers, $d = (d_1, \dots, d_n)$. Denote by $\Gamma^d(\Omega)$, Ω an open set of R^n , the set of all C^∞ complex valued functions f on Ω such that for every compact subset K of Ω there exists a positive constant c , depending on K and f , such that

$$\sup_{\alpha \in \mathbb{Z}_+^n} \sup_{x \in K} c^{-|\alpha|} |\Gamma(\langle d, \alpha \rangle + 1)^{-1} |D^\alpha f(x)| < \infty.$$

Here $\mathbb{Z}_+$ is the set of all non negative integers, Γ is the Euler gamma function and $\langle d, \alpha \rangle = \sum_{j=1}^n d_j \alpha_j$.

A connection had been pointed out in [1] between the surjectivity of $P(D)$ on the space $\Gamma^1(\Omega) = A(\Omega)$ of all the real analytic functions

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on Ω and the solvability of the system

$$P(D)v = w \quad \left(\sum_{j=1}^n D_j^2 + D_t^2 \right) v = 0, \quad D_t = -i \partial_t,$$

in an open neighborhood of Ω in R^{n+1} . Here we show that the same type of result holds for any $I^q(\Omega)$.

Let

$$(1) \quad Q(D, D_t) = \sum_{\langle d, \alpha \rangle + h \leq m} c_{\alpha, h} D^\alpha D_t^h,$$

be a linear differential operator on R^{n+1} with constant coefficients $c_{\alpha, h}$, $(\alpha, h) \in \mathbb{Z}_+^{n+1}$, and assume that $c_{0, m} \neq 0$. If f_h , $h = 0, \dots, m-1$, are given functions in $I^q(\Omega)$, then according to a theorem by G. Talenti [4], there exist an open neighborhood U of Ω in R^{n+1} and one and only one function $u \in I^{(d, 1)}(U)$ such that

$$(2) \quad \begin{aligned} Q(D, D_t)u &= 0 \quad \text{in } U, \\ \partial_t^h u(x, 0) &= f_h(x), \quad x \in \Omega, \quad h = 0, \dots, m-1. \end{aligned}$$

Consider the problem of finding a solution v of the system

$$(3) \quad P(D)v = w, \quad Q(D, D_t)v = 0$$

in an open neighborhood V of Ω in R^{n+1} , assuming that $Q(D, D_t)$, of the form (1), be $(d, 1)$ -hypoelliptic on R^{n+1} ⁽¹⁾. From this assumption it follows that every distribution solution of (3) on V is in $I^{(d, 1)}(V)$. Note also that a necessary condition for the solvability of (3) in V is that

$$(4) \quad Q(D, D_t)w = 0 \quad \text{in } V.$$

Hence $w \in I^{(d, 1)}(V)$.

⁽¹⁾ This implies that $c_{0, m} \neq 0$, if m is the order of Q . If $d_j = r/s_j$, $j = 1, \dots, n$; r, s_j positive integers, an example of operator of the form (1) which is $(d, 1)$ -hypoelliptic on R^{n+1} is given by

$$Q(D, D_t) = \sum_{j=1}^n D_j^{2s_j} + D_t^{2r}.$$

Suppose now that the equality $P(D)\Gamma^a(\Omega) = \Gamma^a(\Omega)$ holds for the given $P(D)$ and $\Omega \subset R^n$ and that w is a given function on V satisfying (4). Put

$$f_h(x) = \partial_i^h w(x, 0), \quad x \in \Omega, \quad h = 0, \dots, m-1,$$

and let $u_h \in \Gamma^a(\Omega)$ be such that $P(D)u_h = f_h$ in Ω . By the theorem quoted above there exist an open neighborhood $U \subset V$ of Ω in R^{n+1} and one and only one function $v \in \Gamma^{(a,1)}(U)$ such that

$$Q(D, D_i)v = 0 \quad \text{in } U,$$

$$\partial_i^h v(x, 0) = u_h(x), \quad x \in \Omega, \quad h = 0, \dots, m-1.$$

This implies that

$$Q(D, D_i)(P(D)v) = 0 \quad \text{in } U,$$

$$\partial_i^h (P(D)v)(x, 0) = f_h(x), \quad x \in \Omega, \quad h = 0, \dots, m-1.$$

Hence $P(D)v = w$ in U by the uniqueness of the solution of the problem (2) in $\Gamma^{(a,1)}(U)$. Thus we conclude that v is a solution of the system (3) in U .

On the other hand, given $f \in \Gamma^a(\Omega)$, there exists a solution $u \in \Gamma^{(a,1)}(U)$ of the problem (2) when $f_0 = f$, $f_h = 0$, $h = 1, \dots, m-1$. If the system (3), with $w = u$, has a solution v in an open neighborhood V of Ω in R^{n+1} , then the function $z(x) = v(x, 0) \in \Gamma^a(\Omega)$ is a solution of the equation $P(D)z = f$.

So we have proved the following result.

THEOREM. Let $P(D)$ be a linear differential operator with constant coefficients and let $d_j \geq 1$, $j = 1, \dots, n$, be rational numbers, $d = (d_1, \dots, d_n)$.

Then for every open set Ω contained in R^n and every $(d, 1)$ -hypoelliptic differential operator of the form (1), the following statements are equivalent:

- i) $P(D)\Gamma^a(\Omega) = \Gamma^a(\Omega)$,
- ii) for every w satisfying (4) there exists a solution of the system (3) in an open neighborhood of Ω in R^{n+1} .

When $\Omega = \mathbb{R}^n$, sufficient conditions on $P(D)$ for i) to hold are proved in [3]. For the case when all the d_j 's are equal and $\Omega = \mathbb{R}^n$, see [2].

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Manoscritto pervenuto in redazione il 30 dicembre 1981.