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On a Characterization of Reflexive Banach Spaces.

EMILIA PERRI (*)

0. Introduction.

Let X be a Banach space. Consider the Cauchy problem

$$(1) \quad \dot{x} = f(t, x), \quad x(t_0) = x_0$$

where $x, x_0 \in X$ and $f: \mathbb{R} \times X \rightarrow X$ (here the symbol « $\cdot$ » denotes the strong derivative).

Let C be the space of all strongly continuous functions from $\mathbb{R} \times X$ into X , with supremum norm. It is well known that, when X is finite dimensional, (1) has a solution for every $f \in C$ and $x_0 \in X$.

Dieudonné ([3]) remarked that in the case $X = \mathbb{C}_0$ the existence of solutions is not guaranteed for every continuous function f .

Recent results assure that in the infinite dimensional case the set of all $f \in C$ for which problem (1) has no solution is a non-empty, dense, of first category subset of C (see [7], [5], [8], [4]).

In this paper we are interested in nonexistence of weak solutions of problem (1). Let $\mathfrak{C}$ be the set of all continuous functions from $\mathbb{R} \times (X, \tau)$ into (X, τ) , where τ denotes the weak topology of X . For $f \in \mathfrak{C}$ and $(t_0, x_0) \in \mathbb{R} \times X$ denote by $[f; t_0, x_0]$ the weak version of problem (1). Let $\mathfrak{C}$ the set of all $f \in \mathfrak{C}$ for which the problem $[f; t_0, x_0]$ has no weak solution.

It is well known that in reflexive Banach spaces the problem $[f; t_0, x_0]$ has a weak solution for every $f \in \mathfrak{C}$ ([9]). Moreover, it has

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been shown that for every non reflexive retractive Banach space the set $\mathcal{C}$ is non empty.

In this paper we state that the non reflexivity is in itself sufficient to imply $\mathcal{C} \neq \emptyset$, and hence that the existence of a weak solution for every $f \in \mathfrak{T}$ is a characterization of the reflexive Banach spaces. Furthermore we prove that $\mathcal{C}$ and $\mathfrak{T} \setminus \mathcal{C}$ are dense in $\mathfrak{T}$.

1. Definitions.

Let X be a Banach space, τ the weak topology of X and Ω an open subset of $\mathbb{R} \times (X, \tau)$. We recall some definitions which we shall use in the following.

DEFINITION 1. We shall call τ -neighbourhood of $\bar{x} \in X$ the set

$$U = \{x \in X: |x^*(x - \bar{x})| < \varepsilon, x^* \in Y^*\}$$

where $\varepsilon > 0$ and Y^* is a finite subset of the dual space X^* of X .

DEFINITION 2. The map $f: \mathbb{R} \times X \rightarrow X$ is continuous from Ω into (X, τ) if for every $(t', x') \in \Omega$ and arbitrary τ -neighbourhood U of the point $f(t', x')$, there exist $\delta > 0$ and a τ -neighbourhood V of x' such that $f(t, x) \in U$ if $x \in V$, $|t - t'| < \delta$ and $(t, x) \in \Omega$.

Denote by $\mathfrak{T}$ the set of all continuous functions from $\mathbb{R} \times (X, \tau)$ into (X, τ) .

DEFINITION 3. $x: \mathbb{R} \rightarrow X$ is weakly continuous at t_0 if $t \rightarrow t_0$ implies

$$x^*(x(t)) \rightarrow x^*(x(t_0)) \quad \text{for every } x^* \in X^*.$$

DEFINITION 4. $x: \mathbb{R} \rightarrow X$ is weakly differentiable at t_0 if there exists $y \in X$ such that $t \rightarrow t_0$ implies

$$\frac{x^*(x(t) - x(t_0))}{t - t_0} \rightarrow x^*(y) \quad \text{for every } x^* \in X^*.$$

We shall say that y is the weak derivative of x at t_0 and we shall denote it by $\dot{x}(t_0)$.

DEFINITION 5. $x: \mathbf{R} \rightarrow X$ is weakly integrable in $[a, b]$ if the function $x^*x: \mathbf{R} \rightarrow \mathbf{R}$, defined by $x^*x(t) = x^*(x(t))$, is Riemann-integrable in $[a, b]$ for every $x^* \in X^*$ and there exists $\bar{x} \in X$ such that

$$x^*(\bar{x}) = \int_a^b x^*(x(t)) dt \quad \text{for every } x^* \in X^*.$$

DEFINITION 6. $x: \mathbf{R} \rightarrow X$ is a weak solution of $[f; t_0, x_0]$ if there exists $\delta > 0$ such that for every $t \in [t_0, t_0 + \delta]$

- a) $(t, x(t)) \in \Omega$,
- b) x is weakly differentiable at t ,
- c) $\dot{x}(t) = f(t, x(t))$ in the sense of definition 4,
- d) $x(t_0) = x_0$.

Hence if x is a weak solution of $[f; t_0, x_0]$, this implies that, for every $x^* \in X^*$

$$(2) \quad x^*(x(t)) = x^*(x_0) + \int_{t_0}^t x^*(f(s, x(s))) ds, \quad t \in [t_0, t_0 + \delta].$$

2. Nonexistence of weak solutions.

THEOREM 1. Let X be a nonreflexive Banach space with norm $\|\cdot\|$. Given $a \in X$ and $(t_0, y_0) \in \mathbf{R} \times X$, there exists $f \in \mathcal{C}$ such that $f(t_0, y_0) = a$ and the problem $[f; t_0, y_0]$ has no τ -solution.

REMARK 1. In the case of X retractive, the result has been obtained in [10]: the additional hypothesis guarantees the crucial fact, required by the technique used there, that a continuous function defined on a subspace of (X, τ) can be extended to the whole space. The strategy here is, instead, the direct construction of a family of continuous functions defined on the whole space and, from this, of the f which satisfies the statement.

Proof of Theorem 1. Let B be the closed ball with center at y_0 and radius 1. Let S^* be the boundary of the unite ball of X^* . From

the James' characterization of reflexivity ([2]) there exists $v \in S^*$ such that $|v(x - y_0)| < 1$ for every $x \in B$.

If $v(y_0) = b$, by definition of the norm of v , we can get a sequence $\{x_n\}$ in the boundary of B , with $x_1 = y_0 + x/\|x\|$ ($x \in \ker v$), such that $v(x_n) < v(x_{n+1})$, $v(x_n) \rightarrow 1 + b$.

Let ϑ be the pseudonormed topology generated by v ($|x|_\vartheta = |v(x)|$). Let $B_\vartheta = \{x \in X: |x - y_0|_\vartheta < 1\}$ and $x_0 = 2y_0 - x_2$.

Note that the sets

$$O_n = \{x \in X: 2v(x_{n-1}) < v(x) + 1 + b < 2v(x_{n+1})\}, \quad n \in \mathbb{N}$$

are non-empty (for every $\bar{x}$, with $v(\bar{x}) = 1$, $x = 2x_n - y_0 - \bar{x} \in O_n$), ϑ -open, and their union is a point finite cover of B_ϑ . In fact it is not difficult to check that every point $x \in B_\vartheta$ belongs to at most two O_n and moreover that $B_\vartheta = \bigcup_{n \geq 2} O_n$.

Define

$$\varphi_0(x) = 1 - \frac{v(x) + 1 + b - 2v(x_0)}{2v(x_1 - x_0)},$$

and

$$\varphi_n(x) = \begin{cases} 0, & x \notin O_n; \\ 1 - \varphi_{n-1}(x), & x \in O_n \setminus O_{n+1}; \\ 1 - \frac{v(x) + 1 + b - 2v(x_n)}{2v(x_{n+1} - x_n)}, & x \in O_n \cap O_{n+1}. \end{cases}$$

It is easy to prove that the functions $\varphi_0, \varphi_n: X \rightarrow \mathbb{R}$ have the following properties:

- a) for every $n \in \mathbb{N}$ φ_n is continuous from (X, ϑ) into $[0, 1]$;
- b) φ_0 is continuous on (X, ϑ) and $\varphi_0(x) \in [0, 1)$ if $x \in O_1 \setminus O_2$;
- c) $\sum_{n=1}^{\infty} \varphi_n(x) \leq 1$ for every $x \in X$ and in particular $\sum_{n=1}^{\infty} \varphi_n(x) = 1$ if $x \in B_\vartheta$.

Consider the function $g: X \rightarrow X$ given by

$$g(x) = y_0 + \sum_{n=1}^{\infty} \frac{\varphi_n(x)}{2v(x_{n+1} - y_0)} (x_{n+1} - y_0)(v(x) + 1 - b).$$

g being continuous from (X, ϑ) into X , g is in fact continuous from (X, τ) into (X, τ) because the topology ϑ is weaker than the τ -topology.

For every $x \in X$, $g(x) \in B$. In fact:

if $x \notin \cup O_n$ then $g(x) = y_0$;

if $x \in O_1 \setminus B_\vartheta$ then

$$\|g(x) - y_0\| = \frac{\|\varphi_1(x)(x_2 - y_0)(v(x) + 1 - b)\|}{2v(x_2 - y_0)} < 1;$$

if $x \in B_\vartheta$ and n' is such that x belongs at most to $O_{n'-1}$ and $O_{n'}$, then

$$\begin{aligned} \|g(x) - y_0\| &= \varphi_{n'-1}(x)\|x_{n'} - y_0\| \frac{|v(x) + 1 - b|}{2v(x_{n'} - y_0)} + \\ &+ \varphi_{n'}(x)\|x_{n'+1} - y_0\| \frac{|v(x) + 1 - b|}{2v(x_{n'+1} - y_0)} < \varphi_{n'-1}(x) + \varphi_{n'}(x) = 1. \end{aligned}$$

Moreover, for every $x \in B_\vartheta$,

$$(3) \quad v(g(x)) = b + \frac{v(x) + 1 - b}{2} \sum_{n=1}^{\infty} \varphi_n(x) = \frac{v(x) + 1 + b}{2}.$$

Let

$$h(t, x) = \frac{x - y_0 - a(t - t_0)}{(t - t_0)^2}.$$

We claim that the function $f: \mathbb{R} \times X \rightarrow X$ given by

$$f(t, x) = \begin{cases} 2(t - t_0)g(h(t, x)) + a, & t \neq t_0; \\ a & t = t_0, \end{cases}$$

satisfies the statement of the theorem. Indeed, for every $t \neq t_0$, $g(h) \in \mathfrak{C}$, and for every $x^* \in S^*$ we have

$$|x^*[(f(t, x)) - a]| \leq 2|t - t_0| \|g(h(t, x))\| \leq 2|t - t_0|(1 + \|y_0\|).$$

So $f \in \mathfrak{C}$. Furthermore, if $y: \mathbb{R} \rightarrow X$ is a weak solution of $[f; t_0, y_0]$, then there exists $\delta > 0$ such that for any $t \in [t_0, t_0 + \delta]$, $x^* \in S^*$:

(see [2])

$$\begin{aligned} |x^*[h(t, y(t)) - y_0]| &\leq \frac{1}{(t - t_0)^2} \int_{t_0}^t |x^*[f(s, y(s)) - a - 2(s - t_0)y_0]| ds \leq \\ &\leq \frac{1}{(t - t_0)^2} \int_{t_0}^t 2(s - t_0) |x^*(g(h(s, y(s))) - y_0)| ds \leq \frac{1}{(t - t_0)^2} \int_{t_0}^t 2(s - t_0) ds = 1 \end{aligned}$$

hence $h(t, y(t)) \in B$. Consequently (see [3]),

$$v(g(h(t, y(t)))) = \frac{v(h(t, y(t))) + 1 + b}{2}$$

and so

$$\dot{v}(y(t)) = v(\dot{y}(t)) = v(f(t, y(t))) = \frac{v(y(t)) - b}{(t - t_0)} + (t - t_0)(1 + b).$$

Since the only solution of

$$\dot{\eta} = \frac{\eta - b}{(t - t_0)} + (t - t_0)(1 + b),$$

such that

$$\left| \frac{\eta - b - v(a)(t - t_0)}{(t - t_0)^2} - b \right| \leq 1$$

and $\eta(t_0) = b$, is given by

$$\eta(t) = (b + 1)(t - t_0)^2 + v(a)(t - t_0) + b,$$

then

$$v(h(t, y(t))) = \frac{1}{(t - t_0)^2} [\eta(t) - b - v(a)(t - t_0)] = b + 1.$$

But this is a contradiction as $h(t, x(t)) \in B$. This completes the proof.

3. Density result.

LEMMA 1. *Let X be a Banach space and U a τ -neighbourhood of the origin of X . If $F, F' \in \mathfrak{G}$ are such that $F(t_0, y_0) = F'(t_0, y_0)$, where (t_0, y_0) is any fixed point of $\mathbb{R} \times X$, then there exist $\alpha > 0$, $F'' \in \mathfrak{G}$ and*

a τ -neighbourhood V of y_0 such that

$$\begin{aligned} F''(t, x) &\in F'(t, x) + U, & (t, x) &\in \mathbb{R} \times X; \\ F''(t, x) &= F'(t, x), & x &\in V \text{ and } |t - t_0| \leq \alpha. \end{aligned}$$

PROOF. Let $U = \{x \in X: |x_i^*(x)| < \varepsilon, x_i^* \in X^*, i = 1, \dots, n\}$. Since $(F - F') \in \mathfrak{C}$, there exist $\delta > 0$, $\sigma > 0$ and a finite subset Y^* of X^* such that $F(t, x) - F'(t, x) \in U$ if $|t - t_0| < \delta$ and $|x^*(x - y_0)| < \sigma$ for every $x^* \in Y^*$.

Set $K(x) = \max_{x^* \in Y^*} |x^*(x - y_0)|$. Clearly K is a continuous function from (X, τ) into $\mathbb{R}$ and moreover, if $|t - t_0| < \delta$ and $K(x) < \sigma$, then $F(t, x) - F'(t, x) \in U$.

Let

$$\begin{aligned} I &= \left[t_0 - \frac{\delta}{2}, t_0 + \frac{\delta}{2} \right], \\ A &= \{(t, x): t \in I, K(x) < \sigma/2\}, \\ B &= \{(t, x): t \in I, \sigma/2 \leq K(x) \leq (2/3)\sigma\}, \\ C &= \{(t, x): t \in I, K(x) > (2/3)\sigma\}, \end{aligned}$$

and consider the function $G: I \times X \rightarrow X$ given by

$$G(t, x) = \begin{cases} F'(t, x), & (t, x) \in A; \\ F(t, x) + \frac{4\sigma - 6K(x)}{\sigma} [F'(t, x) - F(t, x)], & (t, x) \in B; \\ F(t, x) & (t, x) \in C. \end{cases}$$

We claim that $G(t, x) \in F(t, x) + U$ for every $(t, x) \in I \times X$. Indeed, if $(t, x) \in A \cup C$ it is obvious; if $(t, x) \in B$ then

$$|x_i^*(F(t, x) - G(t, x))| = \frac{|4\sigma - 6K(x)|}{\sigma} |x_i^*[F(t, x) - F'(t, x)]| < \varepsilon, \\ (i = 1, \dots, n).$$

Moreover G is continuous from $I \times (X, \tau)$ into (X, τ) .

Define a function $\gamma: I \times X \rightarrow U$ by

$$\gamma(t, x) = G(t, x) - F(t, x).$$

Let r be a continuous function from $\mathbb{R}$ into I such that $r(t) = t$ if $t \in I$.

The function $F'': \mathbb{R} \times X \rightarrow X$ given by

$$F''(t, x) = F(t, x) + \gamma(r(t), x)$$

is the required function, provided that $\alpha = \delta/2$ and $V = \{x \in X: |x^*(x - y_0)| < \sigma/2, x^* \in Y^*\}$. In fact, $F''(t, x) - F(t, x) = \gamma(r(t), x) \in U$ for every $(t, x) \in \mathbb{R} \times X$. In addition, if $(t, x) \in A$ then $\gamma(r(t), x) = \gamma(t, x)$ and so $F''(t, x) = G(t, x) = F'(t, x)$. This completes the proof.

DEFINITION 7. A subset $\mathcal{A}$ of $\mathfrak{C}$ is said to be τ -dense in $\mathfrak{C}$ if, for every $F \in \mathfrak{C}$ and for every τ -neighbourhood U of the origin of X , there exists $f \in \mathfrak{C}$ such that $f(t, x) \in F(t, x) + U$ for every $(t, x) \in \mathbb{R} \times X$.

Let $\mathcal{C} = \{f \in \mathfrak{C}: [f; t_0, y_0] \text{ has no weak solution}\}$.

REMARK 2. Given $a \in X$ and $(t_0, y_0) \in \mathbb{R} \times X$, there exists $\xi \in \mathfrak{C} \setminus \mathcal{C}$ such that $\xi(t_0, y_0) = a$; in fact the function $y(t) = a(e^{(t-t_0)} - 1) + y_0$ is a weak solution of the problem $[\xi; t_0, y_0]$ with $\xi(t, x) = x - y_0 + a$.

THEOREM 2. In nonreflexive Banach spaces, $\mathcal{C}$ is τ -dense in $\mathfrak{C}$.

PROOF. Given $F \in \mathfrak{C}$, $(t_0, y_0) \in \mathbb{R} \times X$ and an arbitrary τ -neighbourhood U of the origin of X , by Theorem 1, there exists $f \in \mathcal{C}$ such that $f(t_0, y_0) = F(t_0, y_0)$. Thence, by Lemma 1, there exist $\alpha > 0$, $F'' \in \mathfrak{C}$ and $V \subset X$ such that $F''(t, x) \in F(t, x) + U$ for $(t, x) \in \mathbb{R} \times X$ and $F''(t, x) = f(t, x)$ for $x \in V$ and $|t - t_0| < \alpha$.

Suppose $F'' \notin \mathcal{C}$. Then there exist $\delta' > 0$ and $y: \mathbb{R} \rightarrow X$ such that (see [2]) for every $x^* \in X^*$,

$$x^*(y(t)) = x^*(y_0) + \int_{t_0}^t x^*[F''(s, y(s))] ds, \quad t \in [t_0, t_0 + \delta'].$$

Since y is weakly continuous, there exists $\delta'' > 0$ such that $y(t) \in V$ for $|t - t_0| < \delta''$. Hence $F''(s, y(s)) = f(s, y(s))$ for $|s - t| < \min(\alpha, \delta', \delta'')$ and so $y(t)$ is a weak solution of $[f; t_0, y_0]$: a contradiction. The theorem is proved.

THEOREM 3. In non reflexive Banach spaces the set $\mathfrak{C} \setminus \mathcal{C}$ is τ -dense in $\mathfrak{C}$.

PROOF. Given $F \in \mathfrak{C}$, $(t_0, y_0) \in \mathbb{R} \times X$ and an arbitrary τ -neighbourhood U of the origin of X , by Remark 2 there exists $\xi \in \mathfrak{C} \setminus \mathcal{C}$

such that $\xi(t_0, y_0) = F(t_0, y_0)$. By Lemma 1 there exist $\alpha > 0$, $F'' \in \mathfrak{C}$ and $V \subset X$ such that $F''(t, x) \in F'(t, x) + U$ for $(t, x) \in \mathbb{R} \times X$ and $F''(t, x) = \xi(t, x)$ for $x \in V$ and $|t - t_0| < \alpha$. Since $\xi \in \mathfrak{C} \setminus \mathcal{C}$, there exist $\delta' > 0$ and a weak solution y of $[\xi; t_0, y_0]$, defined in $[t_0, t_0 + \delta']$, which is weakly continuous. Consequently there exists $\delta'' > 0$ such that $y(t) \in V$ if $|t - t_0| < \delta''$. Set $\delta = \min(\alpha, \delta', \delta'')$. Then, for every $x^* \in X^*$, we have (see [2])

$$x^*(y(t)) - x^*(y_0) = \int_{t_0}^t x^*[\xi(s, y(s))] ds = \int_{t_0}^t x^*[F''(s, y(s))] ds, \quad t \in [t_0, t_0 + \delta].$$

So $F'' \in \mathfrak{C} \setminus \mathcal{C}$ and the proof is complete.

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