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Continuous dependence and stability for non linear dispersive and dissipative waves

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Continuous Dependence and Stability For Non Linear Dispersive and Dissipative Waves.

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Introduction.

In this paper we give some continuous dependence and stability theorems for nonlinear dispersive and dissipative waves according to the Korteweg-de Vries-Burgers (K.d.V.B.) equation [1]. Our goal is to obtain the afore said theorems only by assumptions on the data and withouth assuming, a priori, on perturbations any kind of convergence at large spatial distance. To this end we use the weight function method [2], [3], [4] by which it is possible to remove from perturbation to the weight function the convergence conditions.

The paper is divided into four sections. In the first one—devoted to preliminaries—we obtain a weighted energy equality for the K.d.V.B. equation (lemma 1) and recall a pointwise estimates for functions with bounded first derivatives (lemma 2).

In section 2 we prove a L^2 energy inequality for solution u to the perturbed equation which, a priori, may grow polinomially at large spatial distance. As a conseguence of the afore said L^2 energy inequality in sections 3, 4 we give two continuous dependence theorems (sec. 3) and two stability theorem both in the L^2 and in the pointwise norm.

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1. Preliminaries.

The initial value problem (I.V.P.) associated to the K.d.V.B. equation, as is well known, is:

$$(1)\qquad \begin{cases} v_t + vv_x + \mu v_{xxx} = \varkappa v_{xx} + F & (x,t) \in R\times R^+ \\ v(x,0) = v_0(x) & x \in R \end{cases}$$

where $\mu \in R$ is the constant of dispersion and $\varkappa = \text{const} > 0$ is a coefficient of viscosity. The therm F wich appear in the right hand side may represent some kind of forcing action on the physical sistem and may also be considered as a controll therm wich represent—in some sense—the net error entailed in equation (1) with $F = 0$ as an approximate model. In the sequel we shall consider F ascribed and depending by x and t. Denoting by v and $v + u$ two classical solutions of the I.V.P. (1) and by $\{v_0, v_0 + u_0\}$, $\{F, F + f\}$ the initial data and the forces (the controlls), the perturbation u satisfies the following I.V.P.:

$$(2)\qquad \begin{cases} u_t = -(v+u)u_x - uv_x - \mu u_{xxx} + \varkappa u_{xx} + f \\ u(x,0) = u_0(x)\,. \end{cases}$$

Let $g(x,t) \geqslant 0$ be a generally differentiable « weight function » and denote by $\mathcal{E}^{\frac{1}{2}}$ the weighted $L^2(R)$ norm defined by

$$(3)\qquad \mathcal{E} = \|\sqrt{g}u\|^2 = \int_R gu^2\,dx\,.$$

Definition 1. *We shall say that a solution u to problem* (2) *is in the class Γ_V iff*

$$(4)\qquad \exists M, l > 0:\ |v|,\ |v_x|,\ |u|,\ |u_x|,\ |u_{xx}|,\ |u_0|,\ |f| \leqslant M|x|^K \qquad (K > 0,\ |x| > l)\,.$$

LEMMA 1. *If $u \in \Gamma_V$ then u verifies the weighted energy equality*

$$\frac{d\mathcal{E}}{dt} = \int_R \left\{ \left[g_t + g_x\left(v + \frac{2}{3}u\right) + \left(\varkappa g_{xx} + \mu g_{xxx}\right) - g v_x^2 \right] u^2 + \\ - \left(\varkappa g + \frac{3}{2}\mu g_x\right) u_x^2 + 2gfu \right\} dx \tag{5}$$

where

$$g = \exp\left[-\alpha(1 + \varepsilon x)(t + t_0)^\beta\right] \tag{6}$$

with

$$\begin{cases} \alpha > 0\,; \qquad t_0, \beta \geqslant 0 \\ \varepsilon = \begin{cases} 1 & x \geqslant 0 \\ -1 & x \leqslant 0\,. \end{cases} \end{cases} \tag{7}$$

PROOF. Multiplying (2_1) by $g \cdot u$ and integrating, we easily get (5) (1).
In the sequel we shall use the following lemma:

LEMMA 2. *Let N be a fixed positive constant and let $\mathfrak{I}_N$ be the class of functions*:

$$\varphi\colon R \to R\,, \qquad \varphi \in C^1(R),\ |\varphi'| \leqslant N\,.$$

Then $\forall x_0 \in R$ *and* $\forall h \in R^+$

$$\varphi \in \mathfrak{I}_N \Rightarrow |\varphi(x_0)| \leqslant K\left[\int_{x_0}^{x_0+h} \varphi^2\,dt + \left(\int_{x_0}^{x_0+h} \varphi^2\,dt\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \tag{8}$$

(1) Take into account that:

$$\begin{cases} g u u_t = \frac{1}{2}(gu^2)_t - \frac{1}{2} g_t u^2\,, \qquad g u u_x = \frac{1}{2}(gu^2)_x - \frac{1}{2} g_x u^2 \\ (v + u) g u u_x = \left(\frac{1}{2} v g u^2 + \frac{1}{3} g u^3\right)_x - \frac{1}{2} g_x\left(v + \frac{2}{3}u\right) u^2 - \frac{1}{2} g v_x u^2 \\ g u u_{xx} = \left(g u u_x - g_x \frac{u^2}{2}\right)_x + g_{xx}\frac{u^2}{2} - g u_x^2 \\ g u u_{xxx} = \left(g u u_{xx} - g_x u u_x - g\frac{u_x^2}{2} + g_{xx}\frac{u^2}{2}\right)_x - g_{xxx}\frac{u^2}{2} + \frac{3}{2} g_x u_x^2\,. \end{cases}$$

where K is a positive constant independent of x_0 [2].

2. L^2 energy inequality.

Let T be a positive constant. We have [3]:

THEOREM 1. *Let $u \in \Gamma_V$ with*

$$|v|,\ |u| \leqslant M(1 + |x|)\,, \qquad |v_x| \leqslant M\,. \tag{9}$$

Then

$$\begin{cases} u_0 \in L^2(R) \\ f \in L^2(R \times [0, T)] \end{cases} \Rightarrow \begin{cases} u \in L^2(R) \qquad \forall t > 0 \\ u_x \in L^2(R \times [0, T]) \end{cases} \tag{10}$$

and moreover u obeys the L^2 energy inequality $\forall t \leqslant T$, i.e.

$$\|u(t)\|^2 \leqslant \|u_0\|^2 - \int_0^t ds \int_R (V_x u^2 + 2\varkappa u_x^2 - 2fu)\, dx\,. \tag{11}$$

PROOF. Since

$$\begin{cases} g_t = -\alpha\beta(1 + |x|)(t + t_0)^{\beta-1} g\,, & g_x = -\alpha\varepsilon(t + t_0)^{\beta} g \\ g_{xx} = \alpha^2 (t + t_0)^{2\beta} g\,, & g_{xxx} = -\alpha^3 \varepsilon (t + t_0)^{3\beta} g \end{cases} \tag{12}$$

we have

$$\varkappa g_{xx} + \mu g_{xxx} \leqslant \alpha^2 (T + t_0)^2 [\varkappa + |\mu| \alpha (T + t_0)] g \tag{13}$$

(2) For the sake of completeness we shall sketch here the proof. We have:

$$\varphi^2(x_0) = \varphi^2(x) - \int_{x_0}^{x_0+h} \frac{d}{dt}\varphi^2\, dt \leqslant \varphi^2(x) + N h^{\frac{1}{2}} \left(\int_{x_0}^{x_0+h} \varphi^2\, dt \right)^{\frac{1}{2}}$$

Integrating over $(x_0, x_0 + h)$ we obtain

$$h\varphi^2(x_0) \leqslant \int_{x_0}^{x_0+h} \varphi^2(t)\, dt + 2Nh^{\frac{3}{2}} \left(\int_{x_0}^{x_0+h} \varphi^2\, dt \right)^{\frac{1}{2}}$$

(3) We denote by $\|\cdot\|$ the $L^2(R)$ norm, as we already made in relation (3).

and, for $\beta > \frac{5}{3} M(T + t_0)$

$$(14)\qquad g_t + \left(v + \frac{2}{3}u\right) g_x \leqslant \alpha g(1 + |x|)(t + t_0)^\beta \left(\frac{5}{3}M - \frac{\beta}{T + t_0}\right) < 0 \,.$$

Moreover, letting

$$(15)\qquad \bar{\varkappa} = \varkappa - \tfrac{3}{2}|\mu|\alpha(T + t_0)^\beta$$

for

$$(16)\qquad 0 < \alpha < \frac{2\varkappa}{3|\mu|(T + t_0)^\beta}$$

and for the assumption made on v_x we have

$$(17)\qquad \varkappa g + \tfrac{3}{2}\mu g_x \geqslant \bar{\varkappa} g \geqslant 0 \,.$$

Therefore, taking into account (13)-(17) and the Cauchy inequality $2fu \leqslant f^2 + u^2$, from (5) we obtain

$$(18)\qquad \frac{d\mathcal{E}}{dt} \leqslant A\mathcal{E} + \int_R g(f^2 - 2\bar{\varkappa} u_x^2)\, dx$$

with

$$(19)\qquad A = \alpha^2(T + t_0)^2[\varkappa + |\mu|\alpha(T + t_0)] + 1 + M \,.$$

Integrating (18) from 0 to $t \leqslant T$ we thus obtain (4)

$$(20)\qquad 2\bar{\varkappa}\int_0^t\int_R g u_x^2\, dx\, ds + \mathcal{E}(t) \leqslant \exp(AT)\left[\mathcal{E}(0) + \int_0^T ds \int_R g f^2\, dx\right] .$$

Therefore, since by assumptions (10) the right hand side of (20) con-

(4) Take into account the following generalization of the Gronwall's lemma [6]:

$$y(t) \leqslant K(t) + \int_0^t x(s)\, y(s)\, ds,\ t \geqslant 0 \Rightarrow$$

$$\Rightarrow y(t) \leqslant K(0) \exp\int_0^t x(s)\, ds + \exp\int_0^t x(s)\, ds \cdot \int_0^t K'(s) \exp\left[-\int_0^s y(\xi)\, d\xi\right] \cdot ds \,.$$

verges to a finite quantity as $\alpha \to 0$, by the monotone convergence theorem we deduce

$$\int_0^t ds \int_R u_x^2 \, dx + \int_R u^2 \, dx < \infty , \qquad \forall t \leqslant T . \tag{21}$$

Let us come back now to identity (5). Taking into account (14) we obtain

$$\mathcal{E}(t) \leqslant \mathcal{E}_0 + \int_0^t ds \int_R g[(\alpha^2 A_1 - v_x) u^2 - 2\bar{\varkappa} u_x^2 + 2fu] \, dx \tag{22}$$

with

$$A_1 = (T + t_0)^2 [\varkappa + |\mu| \alpha (T + t_0)] . \tag{23}$$

Letting $\alpha \to 0$ in (22), we obtain the inequality (11).

3. Continuous dependence theorems.

The inequality (20) allows to obtain immediately a continuous dependence theorem upon the data u_0 and f for solutions wich may grow spatially according (4)+(9).

THEOREM 2. *Let $u \in \Gamma_\gamma$ and let* (9) *holds. Then*

$$\|u_0\|^2 + \int_0^T \|f\|^2 \, ds < \delta \Rightarrow \int_0^T \|u_x\|^2 \, ds + \|u\|^2 \leqslant A^* \delta , \qquad \forall t \in [0, T] \tag{24}$$

where $A^* (> 0)$ is a constant indipendent of δ.

PROOF. From (20), taking into account (19) and letting $\alpha \to 0$, we obtain

$$2h \int_0^T \|u_x\|^2 \, ds + \|u\|^2 \leqslant \exp [(1 + M) T] - \left[\|u_0\|^2 + \int_0^T \|f\|^2 \, ds \right] , \qquad \forall t \in [0, T] \tag{25}$$

wich proves the theorem.

Starting from the theorem 2, wich in particular assures continuous dependence in the norm of L^2, it is possible to obtain continuous dependence in the pointwise norm.

THEOREM 3. *Let the assumptions of theorem 2 be satisfied with*

$$|u_x| \leqslant N \qquad (N = \text{const} > 0)\,. \tag{26}$$

Then

$$\|u_0\|^2 + \int_0^T \|f\|^2 ds < \delta \Rightarrow \sup_{R\times[0,T]} |u| < A_1^* \delta^p \qquad (p > 0) \tag{27}$$

with A_1^* constant indipendent of δ.

From theorem 2 follows

$$\|u\|^2 \leqslant A^* \delta\,, \qquad \forall t \in [0, T]$$

with A^* constant indipendent of δ. The theorem 3 follows easily then from the lemma 2.

REMARK (Uniqueness). *Let the assumptions of theorem 2 be satisfied with $f = 0$. Then*

$$u_0 = 0 \Rightarrow u = 0\,, \qquad \forall (x, t) \in R \times R^+\,. \tag{28}$$

4. Stability.

Starting from the L^2 energy inequality (11) and taking into account the lemma 2 it is possible to obtain a stability theorem in the L^2 norm and a stability theorem in the pointwise norm. ([5], n. 5).

Let

$$\varkappa^* = \frac{1}{2} \sup_{t\in[0,\infty)} \left[\sup_{w\in\Sigma} \frac{-\int_R v_x w^2\, dx}{\int_R w_x^2\, dx} \right] \tag{29}$$

where Σ is the set of one time differentiable functions in R. The following theorem holds:

THEOREM 4. *Let the hypotheses of theorem 2 be satisfied with* $f = 0$ *and* $\forall T > 0$. *Then, if*

$$\varkappa^* < \varkappa \tag{30}$$

the umperturbed solution v *is stable in the* L^2 *norm.*

PROOF. From (29) and from (11), we obtain $\forall t > 0$

$$\|u(t)\|^2 \leqslant \|u_0\|^2 + 2(r^* - r)\int_0^t ds \int_R u_x^2\, dx\,. \tag{31}$$

Therefore from (30)+(31) we deduce

$$\|u_0\|^2 < \delta \Rightarrow \|u(t)\|^2 \leqslant \delta\,, \qquad \forall t \geqslant 0\,. \tag{32}$$

THEOREM 5. *Let the hypotheses of theorem 4 be satisfied. Then, if* u_x *is uniformly bounded in* $R \times R^+$ *the solution* v *is pointwise stable.*

PROOF. Starting from inequality (8) in lemma 2 and taking into account (32) we thus obtain

$$|u(x, t)| \leqslant K(\delta + \delta^{\frac{1}{2}})^{\frac{1}{2}}\,, \qquad \forall t \geqslant 0 \tag{33}$$

from wich we deduce

$$\|u_0\|^2 < \delta \Rightarrow \sup_{R \times R^+} |u| < K\delta^p \qquad (p > 0) \tag{34}$$

wich proves the theorem.

REMARK 2. Since lemma 2 holds even in $C(R) \cap L_2(R)$, theorems 3 and 5 continue to hold substituting the hypothesis u_x bounded with $u_x \in L_2(R)$.

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