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Isoperimetric Distributions of Loads in Elastostatics.

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1. Introduction.

In the classical theory of elasticity it is customary to prescribe the shape and the elastic moduli of a body and determine the displacements produced by a given system of body forces and surface tractions. In some cases, however, it is necessary to solve the inverse problem: that is to find the forces able to engender a given state of displacements. Inverse boundary-value problems in elastostatics have long been appreciated and solved in the past, but only recently they have been studied systematically.

In this paper I consider the following inverse problem in elastostatics. A homogeneous and isotropic infinite solid is loaded by a constant distribution of body forces acting parallel to a given direction, which is taken to be the z-axis of a system of cartesian coordinates x, y, z. If the forces act on a region of prescribed volume, for what shape of the region is the elastic displacement w along the z-axis, evaluated at the origin, a maximum? It might be imagined that, if the domain of application of the forces is a figure of revolution about the z-axis and symmetric with respect the plane of x and y, the displacement of the origin is a maximum for a sphere, but the solution shows that this is not so.

The same problem can be formulated in plane elasticity. In a state of plain strain parallel to the x, y-plane, constant body forces are parallel to the x-axis and are applied at the points of a region of

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prescribed area. The shape of the region is sought such that the displacement of the origin along the x-axis is a maximum. Also in this case the solution shows that a load region having the form of a circle centered at the origin does not produce the greatest displacement of the origin.

It may be useful to observe that similar problems—rather unusual in elasticity—have been more widely studied within newtonian potential theory (cf. MacMillan [1, § 41]).

2. The three-dimensional case.

Consider an indefinite elastic medium, homogeneous and isotropic, characterized by Lamé moduli μ and λ, satisfying the classical inequalities

$$\mu > 0 , \quad 2\mu + 3\lambda > 0 . \tag{2.1}$$

We denote the coordinates of the point occupied by a particle, in the unstrained state, by x, y, z (Fig. 1), and the components of dis-

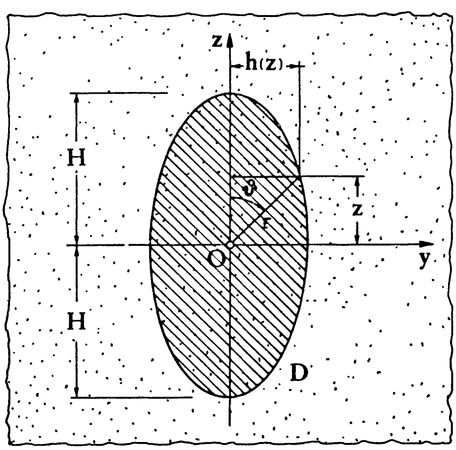

Figure 1

placement on the axes by u, v, w. The body is loaded by a distribution of body forces of the type

$$X = Y = 0 , \quad Z = \text{constant} , \tag{2.2}$$

acting on a bounded region D, of prescribed volume, so that they are statically equivalent to a single constant force acting in the direction of z. If $w(0, 0, 0)$ is the displacement of the origin O parallel to the z-axis, we want to determine the position and the shape of D in order that $w(0, 0, 0)$ is a maximum.

Though we have no preliminary information about the nature of D, it seems plausible to assume that D is a solid of revolution around the z-axis and symmetric with respect to the x, y-plane. The explicit solution will confirm that this conjecture is correct.

Under the above assumption it is easy to calculate che elastic displacement $w(0, 0, 0)$ due to the body forces $Z =$ constant acting in D. For this purpose consider a thin cross section of radius $h(z)$ and thickness dz at a distance z from the origin O. The body force applied at an element of the slice of volume $dx\,dy\,dz$ is $\varrho_0 Z\,dx\,dy\,dz$, where ϱ_0 is the density of the material. The displacement w of the origin produced by the body forces applied at an element $dx\,dy\,dz$ of coordinates x, y, z has the form (cf. Love [2, § 131])

$$dw(0, 0, 0) = A\left(\frac{z^2}{r^3} + \alpha\,\frac{1}{r}\right)\varrho_0 Z\,dx\,dy\,dz\,, \tag{2.3}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and A, α are constants defined by

$$A = \frac{\lambda + \mu}{8\pi\mu(2\mu + \lambda)}\,, \qquad \alpha = \frac{\lambda + 3\mu}{\lambda + \mu}\,. \tag{2.4}$$

It may be useful to observe that the conditions (2.1) imply $A > 0$, $\alpha > 1$. The total displacement produced by the forces at D is thus

$$w(0, 0, 0) = \varrho_0 Z A \iiint\limits_D \left(\frac{z^2}{r^3} + \alpha\,\frac{1}{r}\right) dx\,dy\,dz\,,$$

or, by the formula of reduction of triple integrals,

$$w(0, 0, 0) = 2\pi\varrho_0 Z A \int_{-H}^{H} dz \int_0^{h(z)} \left(\frac{z^2}{(z^2 + \varrho^2)^{\frac{3}{2}}} + \alpha\,\frac{1}{(z^2 + \varrho^2)^{\frac{1}{2}}}\right)\varrho\,d\varrho\,, \tag{2.5}$$

where $\varrho = \sqrt{x^2 + y^2}$, and H and $-H$ are the z coordinates of the points where the axis pierces the surface. By integrating with respect to ϱ and observing that $w(0, 0, 0)$ is an even function of z we obtain

$$(2.6)\qquad w(0,0,0) = 4\pi\varrho_0 ZA\int_0^H\left[-\frac{z^2}{(z^2+h^2)^{\frac12}} - (\alpha-1)z + \alpha(z^2+h^2)^{\frac12}\right]dz\,.$$

The problem is to find a function $h(z)$, the meridian section of the surface of D, which makes $w(0, 0, 0)$ a maximum for a given volume of D. This means that the condition

$$(2.7)\qquad \iiint_D dx\,dy\,dz = 2\pi\int_0^H h^2\,dz = V_0 = \text{constant}$$

must also be satisfied.

By requiring that the first variation of the functional $w - \lambda_0 V_0$, with lagrangean parameter λ_0, should vanish we easily derive the Euler equation of the variational problem, namely,

$$(2.8)\qquad \left[\left(\frac{z^2}{r^3}+\frac{\alpha}{r}\right)-\frac{\lambda_0}{2\varrho_0 ZA}\right]h = 0\,,$$

where $r = \sqrt{h^2 + z^2}$.

Disregarding the solution $h(z) = 0$ (which yields a minimum) we obtain

$$\frac{z^2}{r^3}+\frac{\alpha}{r} = \frac{\lambda_0}{2\varrho_0 ZA}\,,$$

or, since in spherical coordinates $z = r\cos\theta$,

$$(2.9)\qquad r = \frac{2\varrho_0 ZA}{\lambda_0}(\alpha+\cos^2\theta) \qquad \text{with } 0\leqslant\theta\leqslant\pi\,.$$

The parameter λ_0 can be determined by the condition (2.7). In spherical coordinates we have

$$(2.10)\qquad \iiint_D dx\,dy\,dz = \int_0^{2\pi} d\varphi\int_0^{\pi} d\theta\int_0^{r(\theta)}\varrho^2\sin\theta\,d\varrho =$$

$$= \frac{2\pi}{3}\left(\frac{2\varrho_0 ZA}{a_0}\right)^3\int_0^{\pi}(\alpha+\cos^2\theta)^3\sin\theta\,d\theta = V_0\,.$$

The integral on the right-hand side of (2.10) can be evaluated exactly, yielding

$$V_0 = \frac{4\pi}{3}\left(\frac{2\varrho_0 ZA}{\lambda_0}\right)^3 p(\alpha)\,, \tag{2.11}$$

where we have set $p(\alpha) = \alpha^3 + \alpha^2 + \frac{3}{5}\alpha + \frac{1}{7}$. It is easy to see that $p(\alpha) > \frac{8}{35}$.

Since equation (2.11) gives the value of λ_0, the final form of (2.9) becomes

$$r = \sqrt[3]{\frac{V_0}{\frac{4}{3}\pi p(\alpha)}}(\alpha + \cos^2\theta) \qquad \text{with } 0 \leqslant \theta \leqslant \pi\,, \tag{2.12}$$

and the meridian section of the solid is approximatively represented in Fig. 1. The radius of the sphere of equal volume is $r_0 = \sqrt[3]{V_0/\frac{4}{3}\pi}$.

Inspection of the sign of the second variation of (2.6) confirms that $w(0, 0, 0)$ attains a local maximum.

3. The plane problem.

In a state of plane strain parallel to the x, y-plane the displacement w vanishes, and the displacements u, v are functions of the coordinates x, y only. Assume the body forces have the form

$$X = \text{constant}\,, \qquad Y = 0\,,$$

and the former act inside a plane region D of prescribed area A_0. We wish to find the domain D with prescribed area for which u is a maximum at the origin.

We start from the hypothesis that D is symmetric with respect to the x and y-axes, and $y = h(x)$ is the equation of its boundary in the first quadrant. If $-H < x < H$ the condition of constant area becomes

$$\iint_D dx\,dy = 4\int_0^H h(x)\,dx = A_0 = \text{constant}\,. \tag{3.1}$$

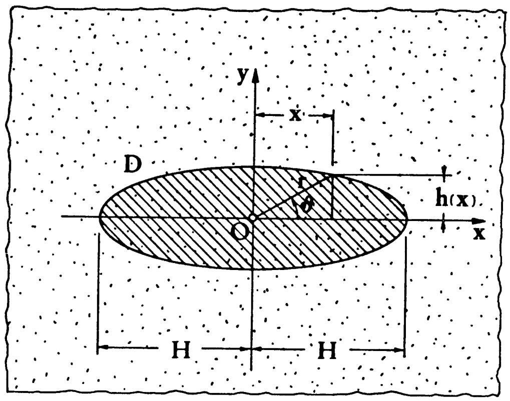

Figure 2

Since the displacement of the origin parallel to the x-axis produced by a unit force concentrated at the point (x, y) is known (cf. Love [2, § 148]), the total displacement $u(0, 0)$ due to the distribution of forces X in D is

$$(3.2)\qquad u(0,0) = -\varrho_0 X \iint_D \left(\beta \ln\sqrt{x^2+y^2} + \gamma \frac{y^2}{x^2+y^2}\right) dx\, dy\,,$$

where ϱ_0 is the density and β, γ are the constants

$$(3.3)\qquad \beta = \frac{\lambda + 3\mu}{4\pi\mu(\lambda+2\mu)}\,, \qquad \gamma = \frac{\lambda+\mu}{4\pi\mu(\lambda+2\mu)}\,.$$

It is useful to recall that the classical restrictions on λ and μ imply $\beta > \gamma > 0$.

Using the formula of reduction for double integrals (3.2) yields

$$(3.4)\qquad u(0,0) = -4\varrho_0 X \int_0^H dx \int_0^{h(x)} \left(\beta \ln\sqrt{x^2+y^2} + \gamma \frac{y^2}{x^2+y^2}\right) dy =$$

$$= -4\varrho_0 X \int_0^H \left[\beta h \ln\sqrt{x^2+h^2} - (\beta-\gamma)h + (\beta-\gamma)x \arctan\frac{h}{x}\right] dx\,.$$

If the first variation of $u - \lambda_0 A_0$ is required to vanish, the resulting Euler equation is

$$\beta \ln r + \beta \frac{h^2}{r^2} - (\beta - \gamma) + (\beta - \gamma) \frac{x^2}{r^2} + \frac{\lambda_0}{\varrho_0 X} = 0\,, \tag{3.5}$$

where $r = \sqrt{x^2 + h^2}$. Using polar coordinates $x = r\cos\theta$, $h = r\sin\theta$ this equation becomes

$$r = \exp\left[-\frac{\lambda_0}{\varrho_0 \beta X}\right] \exp\left[-\frac{\gamma}{\beta}\sin^2\theta\right] \quad \text{with } 0 \leqslant \theta \leqslant 2\pi\,. \tag{3.6}$$

The parameter λ_0 is determined by (3.1):

$$\begin{aligned} A_0 = \iint_D dx\,dy = 4\int_0^{\pi/2} d\theta \int_0^{r(\theta)} \varrho\, d\varrho = \\ = 2\exp\left[-\frac{2\lambda_0}{\varrho_0 \beta X}\right] \int_0^{\pi/2} \exp\left[-\frac{2\gamma}{\beta}\sin^2\theta\right] d\theta\,. \end{aligned} \tag{3.7}$$

The definite integral on the right-hand side can be evaluated exactly by using properties of Bessel functions (cf. Gradshteyn and Ryzhik [3, 8.431])

$$\begin{aligned} \int_0^{\pi/2} \exp\left[-\frac{2\gamma}{\beta}\sin^2\theta\right] d\theta = \frac{1}{2}\exp\left[-\frac{\gamma}{\beta}\right] \int_0^{\pi/2} \exp\left[\frac{\gamma}{\beta}\cos 2\theta\right] d(2\theta) = \\ = \frac{\pi}{2}\exp\left[-\frac{\gamma}{\beta}\right] I_0\left(\frac{\gamma}{\beta}\right), \end{aligned} \tag{3.8}$$

where I_0 is the Bessel function of imaginary argument. Thus from (3.7) and (3.8),

$$A_0 = \pi I_0\left(\frac{\gamma}{\beta}\right) \exp\left[-\frac{\gamma}{\beta}\right] \exp\left[-\frac{2\lambda_0}{\varrho_0 \beta X}\right],$$

and (3.6) becomes

$$r = \sqrt{\frac{A_0}{\pi I_0(\gamma/\beta)}} \exp\left[\frac{\gamma}{2\beta}\sin 2\theta\right] \quad \text{with } 0 \leqslant \theta \leqslant 2\pi\,. \tag{3.9}$$

The radius of the circle of equal area is $r_0 = \sqrt{A/\pi}$.

4. A comparison between the eccentricities.

If we define as the « eccentricity » of the figure of maximum displacement the ratio

$$\varepsilon = \frac{r_{\max} - r_{\min}}{r_{\max}}$$

in both the cases considered above, it is immediately seen that $\varepsilon = 1/(1 + \alpha)$ in three-dimensional elasticity and $\varepsilon = 1 - \exp[-\gamma/2\beta]$ in plane elasticity. Since $\alpha > 1$ and $\gamma/\beta < 1$, the eccentricity is less than $\frac{1}{2}$ in the first case and greater than $1 - \exp[-\frac{1}{2}] \simeq 0{,}39347$ in the second. A property of this kind has been noticed in other isoperimetric problems (cf. Pólya and Szegö [4]).

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