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V. V. Rumjantsev

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On Stability Problem of a Top.

V. V. RUMJANTSEV (*)

This paper deals with the stability of a heavy gyrostate [1] on a horizontal plane. The gyrostate is considered as a rigid body with a rotor rotating freely (without friction) about an axis invariably connected with the body leaning on a plane by a convex surface, i.e. the top in a broad sence of this word. For mechanician the top is a symple and principal object of study [2] attracting investigators' attention.

1. Let $\xi\eta\zeta$ be the fixed coordinate system with the origin in some point of a horizontal plane and vertically up directed axis ζ with unit vector $\boldsymbol{\gamma}$; $Ox_1x_2x_3$ is the coordinate system rigidly connected with the body with the origin in centre of mass of gyrostate and axis x_3 coincided with one of its principal central axes of inertia. Two another axes x_1 and x_2 are parallel to the directions of the main curvatures of the body surface for the point P of intersection of the negative semi-axis x_3 with this surface tangent plane to which in the point P is supposed perpendicular to the axis x_3.

It is possible to write the equations of motion of the top, for example, in the form of the laws of a momentum and of angular momentum and also of a constancy of the vector $\boldsymbol{\gamma}$.

These equations contain the reaction of a plane $\boldsymbol{R}$ for expression of which it is necessary to make a supposition about the character of an interaction of the top with the plane. In the case of an ideal smooth plane

$$\boldsymbol{R} = R\boldsymbol{\gamma}\,. \tag{1.1}$$

(*) Indirizzo dell'A.: Vyc. Centr., ul. Vavilova 40, Moscow 117333, U.S.S.R.

In the case of an ideal rough plane the velocity of the contact point of the top with the plane

$$\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{r} = 0 . \tag{1.2}$$

Here $\boldsymbol{v}$ and $\boldsymbol{\omega}$ are the vectors of the velocity of the centre of mass and of momentary body angular velocity, $\boldsymbol{r}$ is the radius-vector of the contact point.

In the case of a plane with sliding friction

$$\boldsymbol{R} \cdot (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{r}) \leqslant 0 \tag{1.3}$$

independently from the hypothesis on character of sliding friction.

For all these cases the mechanical energy of the top does not increase

$$M\boldsymbol{v}^2 + \boldsymbol{\omega} \cdot \boldsymbol{\theta} \cdot \boldsymbol{\omega} + 2Mg\zeta_0 \leqslant \text{const} \tag{1.4}$$

being constant in the first and second cases. Here $\boldsymbol{\theta}$ is the central tensor of inertia of the transformed body [3], M is the mass of the top, g is the gravitational acceleration, $\zeta_0 = -\boldsymbol{r} \cdot \boldsymbol{\gamma}$ is the vertical coordinate of the centre of mass. The relation between the vectors $\boldsymbol{r}$ and $\boldsymbol{\gamma}$ defines by the equation of the body surface, so [4]

$$x_i = -l_i \gamma_i + \dots \ (i = 1, 2) , \qquad x_3 = -l + \tfrac{1}{2}(l_1 \gamma_1^2 + l_2 \gamma_2^2) + \dots . \tag{1.5}$$

Here l_i are the main radii of curvature of the body surface in the point P, l is the distance between the points O and P, ω_i are the projections of the vector $\boldsymbol{\omega}$ on axes x_i $(i = 1, 2, 3)$, dots denote terms of more high order. It follows from the (1.5) that

$$\zeta_0 = l + \tfrac{1}{2}[(l_1 - l)\gamma_1^2 + (l_2 - l)\gamma_2^2] + \dots . \tag{1.6}$$

Accordingly Lyapunov's stability theorem from the (1.4) and (1.6) we get the sufficient conditions of stability

$$l_i > l \qquad (i = 1, 2) \tag{1.7}$$

on the equilibrium of a heavy gyrostate

$$\boldsymbol{v} = \boldsymbol{\omega} = 0\,, \quad \gamma_1 = \gamma_2 = 0\,, \quad \gamma_3 = 1 \tag{1.8}$$

on a horizontal plane with respect to the values $\boldsymbol{v}$, $\boldsymbol{\omega}$, $\boldsymbol{\gamma}$.

The inequalities (1.7) are also the necessary stability conditions of equilibrium of the top with not rotating rotor [4].

In the case (1.1) the equations of motion have the integral of the angular momentum

$$G_\zeta \equiv (\boldsymbol{\theta}\cdot\boldsymbol{\omega} + \boldsymbol{k})\cdot\boldsymbol{\gamma} = \text{const} \tag{1.9}$$

and also the integral of constancy of the projection of momentary angular velocity

$$\omega_3 = \text{const} \tag{1.10}$$

if the gyrostate has dynamical and geometrical symmetry about the axis x_3 and gyrostatic moment $\boldsymbol{k} = \mathbf{const}$ is parallel to this axis $(k_1 = k_2 = 0,\ k_3 = k)$.

If the surface of axisymmetrical gyrostate in the neighbourhood of the point P is spherical with the centre on the axis x_3 then the equations of motion in the general case (1.3) of sliding friction have the integral [5]

$$(\boldsymbol{\theta}\cdot\boldsymbol{\omega} + \boldsymbol{k})\cdot\boldsymbol{r} = \text{const}\,. \tag{1.11}$$

In all these cases the equations of motion have the solution

$$\boldsymbol{v} = 0\,, \quad \omega_1 = \omega_2 = \gamma_1 = \gamma_2 = 0\,, \quad \omega_3 = \omega\,, \quad \gamma_3 = 1 \tag{1.12}$$

describing the permanent rotation of the gyrostate about vertical axis x_3 with arbitrary angular velocity ω, if the gyrostatic moment $\boldsymbol{k}$ is parallel to this axis $(k_1 = k_2 = 0)$. If the $\boldsymbol{k}$ is not parallel to the axis x_3 then the solution (1.12) is possible only for $\omega = 0$.

On the motion (1.12) the top leans on a plane by the point $P(x_1 = x_2 = 0,\ x_3 = -l)$ and the reaction $\boldsymbol{R} = g\boldsymbol{\gamma}$.

Consider the stability of the solution (1.12) using the integral relation (1.4) and first integrals (1.9)-(1.11) for construction the Lyapunov's function.

2. We start from simplest case (1.1) of an ideal smouth plane. In this case only vertical force $(R-g)\gamma$ acts on the gyrostate, so the projection of the velocity of the mass centre on a horizontal plane remaines constant. Without restriction of generality we shall consider this projection be equal to zero, i.e. the mass centre of gyrostate moves only on immobile vertical.

At first consider the case when the axis of rotor is parallel to the axis x_1 ($k_1 = k$, $k_2 = k_3 = 0$) and the solution (1.12) is possible only for $\omega = 0$. Using the integrals of energy (1.4) and of angular momentum (1.9) we constract Lyapunov's function

$$
\begin{aligned}
(2.1) \qquad V = M\left(\frac{d\zeta_0}{dt}\right)^2 + A_1\omega_1^2 + A_2\omega_2^2 - 2F\omega_1\omega_2 + A_3\omega_3^2 + \\
+ 2Mg\zeta_0 + \lambda[(A_1\omega_1 - F\omega_2 + k)\gamma_1 + (A_2\omega_2 - F\omega_1)\gamma_2 + \\
+ A_3\omega_3(1-\gamma_1^2-\gamma_2^2)^{\frac{1}{2}}]^2 .
\end{aligned}
$$

Here A_i are moments of inertia of the gyrostate about the axes x_i ($i = 2, 3$), A_1-moment of inertia of the body about axis x_1, F-centrifugal moment of inertia for axes x_1 and x_2, $\lambda = \text{const}$. Obviously, $dV/dt = 0$.

The conditions for the positive definiteness of the function (2.1) are reduced to the inequalities

$$
(2.2) \qquad l_2 > l, \quad k^2 + MgA_3(l_1 - l) > 0
$$

which accordingly to Lyapunov's theorem are sufficient stability conditions of the equilibrium (1.8) of gyrostate with gyrostatic moment parallel to the axis x_1 with respect to variables ω_i, γ_i ($i = 1, 2, 3$) and $d\zeta_0/dt$. It is possible to show [4] that the equilibrium is unstable for opposite sign in one of the inequalities (2.2).

The conditions (2.2) contain as particular cases the conditions of stability [5] of Gervat's gyroscop with flat rectilinear support ($l_2 = \infty$, $l_1 = 0$) and with flat circular support of radius ϱ ($l_2 = \varrho$, $l_1 = 0$). It follows from the conditions (2.2) that unstable for $l_1 < l$ position of the equilibrium of the top is stabilized by rotation of rotor with gyroscopic moment k satisfying to the second condition of the inequalities (2.2).

Consider the stability of the gyrostate with the dynamical and geometrical symmetry about the axis x_3 when $A_1 = A_2$, $l_1 = l_2$, $k_1 = k_2 = 0$.

We shall define the orientation of the gyrostate about its centre of mass by Euler's angles θ, ψ, φ and the position of a rotor with the axis parallel to the axis x_3 about the body by an angle α. It is easily to see [4] that coordinates ψ, φ and α are cyclic, and the first integrals of Lagrange's equations

$$(2.3)\qquad \begin{cases} A_1\dot{\psi}\sin^2\theta + [A_3(\dot{\varphi}+\dot{\psi}\cos\theta)+k]\cos\theta = G_\zeta \\ A_3(\dot{\varphi}+\dot{\psi}\cos\theta)+k = H\,, \qquad \dot{\varphi}+\dot{\psi}\cos\theta+\dot{\alpha} = \Omega \end{cases}$$

correspond to them.

Eliminating the variables $\dot{\psi}$ and $\dot{\varphi}$ from integral energy (1.4), represent it in the form

$$(2.4)\qquad [A_1 + Mf'^2(\theta)]\dot{\theta}^2 + W(\theta) = \text{const}$$

Here

$$(2.5)\qquad W(\theta) = \frac{(G_\zeta - H\cos\theta)^2}{2A_1\sin^2\theta} + Mgf(\theta)$$

is the transformed potential energy; $\zeta = f(\theta)$ is the function defining by the shape of the surface of rotation about the axis x_3 bounding the body, $f' = df/d\theta$, A_1 is the central equatorial moment of inertia of gyrostate, A_3 is the axial moment of inertia of the body, G_ζ, H, Ω are constants of the integration.

The equation

$$W'(\theta) = \frac{1}{A_1\sin^3\theta}(G_\zeta - H\cos\theta)(H - G_\zeta\cos\theta) + Mgf'(\theta) = 0$$

defines for $\theta \neq 0, \pi$ the two-parametrical family of solutions

$$\theta = \theta_0(G_\zeta, H)$$

describing the regular precessions of a heavy gyrostate on a plane. These motions are stable if

$$W''(\theta_0) = A_1^{-1}\sin^{-4}\theta_0[(G_\zeta - H\cos\theta_0)^2 - 2\cos\theta_0(G_\zeta - H\cos\theta_0)\cdot \\ \cdot(H - G_\zeta\cos\theta_0) + (H - G_\zeta\cos\theta_0)^2] + Mgf''(\theta_0) > 0$$

and are unstable if the inequality (2.6) has the opposite sign [6].

In the case $\theta = 0$ from (2.3) we have $G_\xi = H$ and the function (2.5) has the next form

$$W(\theta) = \frac{H^2}{2A_1} \frac{1 - \cos\theta}{1 + \cos\theta} + Mgf(\theta) .$$

The equation $W'(\theta) = 0$ holds for any value H and $\theta = 0$ if $f'(0) = 0$; this solution is similar to the solution (1.12). It is stable if

$$W''(0) = \frac{H^2}{4A_1} + Mgf''(0) > 0 \tag{2.7}$$

and unstable if the inequality (2.7) has the opposite sign [4].

In the case $W''(0) = 0$ the solution of stability problem of motion (1.12) depends on members of more high order in the expansion of the $W(\theta)$ in Macloren's serie. So if $f'''(0) \neq 0$ then the motion is unstable; if

$$f'''(0) = 0 , \quad f''''(0) - 2f''(0) > 0 \tag{2.8}$$

then the motion (1.12) is stable. Therefore

$$H^2 + 4A_1 Mg(l_1 - l) \geqslant 0 \qquad \big(l_1 = f(0) + f''(0)\big) \tag{2.9}$$

is necessary and sufficient stability condition of motion (1.12) of axis-symmetrical gyrostate on a plane if the (2.8) holds. For example, in the case of a gyrostate leaning on a plane by a needle ($l_1 = 0$) when $f(\theta) = l\cos\theta$ the conditions (2.8) hold and the inequality (2.9) takes the form

$$H^2 \geqslant 4A_1 Mgl . \tag{2.10}$$

The condition (2.10) has the form of necessary and sufficient stability condition of rotation of a heavy gyrostate with a fixed point about vertical in which however A_1 denotes the moment of inertia for a fixed point whereas in the (2.10) this value denotes the moment of inertia for the centre of mass. The last is less than the first by Ml^2, so stability of a gyrostate on a plane is required (with another equale conditions) smaller value of cinetic moment H than stability of a gyrostate with a fixed point. This difference is due to the fact that the mass centre of gyrostate on a plane rather then the point of support plays the role of a «fixed» point for the steady motion. In the case of equilibrium of gyrostate ($\omega = 0$) on a plane the condition (2.9) takes the form of the inequality

$$k^2 + 4A_1Mg(l_1 - l) \geqslant 0 \ . \tag{2.11}$$

Comparing the (2.11) with the (2.2) we see that for stabilization of unstable equilibrium of gyrostate in the case $k_1 = k_2 = 0$, $k_3 = k$ the value k must be twice as much than in the case $k_1 = k$, $k_2 = k_3 = 0$, provided the value A_1 in (2.11) is equal to the value A_3 in (2.2).

Note that formulae (2.3)-(2.11) remain correct in the case of variable angular velocity of rotor [4].

3. Consider the rotation (1.12) of nonsymmetric gyrostate with constant gyrostatic moment $\boldsymbol{k}$ ($k_1 = k_2 = 0$, $k_3 = k$) on an ideal rough plane (1.2). Putting $\omega_3 = \omega + \xi$, $\gamma_3 = 1 - \eta$ it is possible to wrote the equations of perturbed motion in the form [4].

$$\begin{aligned} &A\ddot{\gamma}_2 + F\ddot{\gamma}_1 + [(A + B - A_3 - Mll_1)\omega - k]\dot{\gamma}_1 + \\ &+ F\omega^2\gamma_1 + [(Mll_2 + A_3 - B)\omega + \omega k - Mg(l - l_2)]\gamma_2 = X_1 \\ &B\ddot{\gamma}_1 + F\ddot{\gamma}_2 - [(A + B - A_3 - Mll_2)\omega - k]\dot{\gamma}_2 + F\omega^2\gamma_2 + \\ &+ [(Mll_1 + A_3 - B)\omega + \omega k - Mg(l - l_1)]\gamma_1 = X_2 \ , \\ &\dot{\xi} = X_3 \ , \quad \dot{\eta} = X_4 \ . \end{aligned} \tag{3.1}$$

Here $A = A_1 + Ml^2$, $B = A_2 + Ml^2$ and nonlinear members X_s ($s = 1, \dots, 4$) vanish for $\gamma_i = \dot{\gamma}_i = 0$ ($i = 1, 2$).

The characteristic equation for linearized equations (3.1)

$$\lambda^2(a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4) = 0 \tag{3.2}$$

has two zero-roots and four roots with negative real part if

$$a_1 > 0 \ , \quad a_4 > 0 \ , \quad a_1a_2a_3 - a_1^2a_4 - a_0a_3^2 > 0 \ . \tag{3.3}$$

Here

$$
\begin{aligned}
&a_0 = AB - F^2\,,\\
&a_1 = MF\omega l(l_1 - l_2)\,,\\
&a_3 = a_1\omega^2\,,\\
&a_2 = [A(A_3 - A) + B(A_3 - B) + Ml(Al_1 + Bl_2) - 2F^2]\omega^2 + \\
&\quad + (A+B)\omega k - Mg[A(l-l_1) + B(l-l_2)] + \\
&\quad + [(A+B-A_3)\omega - k]^2 - M\omega l(l_1+l_2)\cdot \\
&\quad \cdot[(A+B-A_3)\omega - k] + M^2\omega^2 l^2 l_1 l_2 \\
&a_4 = [(A_3 - A + Mll_1)\omega^2 + \omega k - Mg(l-l_1)]\cdot \\
&\quad \cdot[(A_3 - B + Mll_2)\omega^2 + \omega k - Mg(l-l_2)] - F^2\omega^4\,.
\end{aligned}
\tag{3.4}
$$

Accordingly Lyapunov-Malkin's theorem [7] the motion (1.12) is stable with respect to the variables ω_i, γ_i $(i = 1, 2, 3)$ and asymptotic stable with respect to the variables ω_s $(s = 1, 2)$, γ_i provided the conditions (3.3). The motion (1.12) will be unstable accordingly Lyapunov's theorem on instability by first approximation when extremely one of the unequalities (3.3) has the opposite sign.

It has place partial asymptotic stability under the conditions (3.3) though the system is conservative one and there are no external dissipative forces. Partial asymptotic stability is caused by nonholonomic constraint (1.2); the gyrostate is subjected (in first approximation) by potential, dissipative-accelerating and gyroscopic forces (see (3.1)); the stability depends on spin direction: the first from the inequalities (3.3) (see (3.4)) holds only if

$$
\operatorname{sign}\omega = \operatorname{sign}[F(l_1 - l_2)]
\tag{3.5}
$$

two another inequalities (3.3) restrict the values ω and k in the stable motion.

It is interesting to note, that the motion (1.12) with very small velocity $\omega = \varepsilon$ and $k = 0$ will unstable even in the case of static stability, when the conditions (1.7) hold: the third from the conditions (3.3) will have the opposite sign. Instability is explained by the action of accelerating forces; gyroscopic stabilization has no place when gyroscopic forces are weak.

If extremely one from the conditions

$$F = 0\,, \quad l_1 = l_2 \tag{3.6}$$

holds then $a_1 = a_3 = 0$ and the conditions (3.3) have no place; partical asymptotic stability is not possible in this case. The necessary simple stability conditions are reduced to the inequalities

$$a_2 > 0\,, \quad a_4 > 0\,, \quad a_2^2 - 4a_0 a_4 > 0\,.$$

4. Consider at last the stability of the rotation (1.12) of axisymmetrical gyrostate leaning on a plane with friction (1.3) by spherical support of radius ϱ with the centre on the axis x_3. The coordinate of the sphere centre is equal to $\varrho - l$ and coordinates of sphere contact point with the plane and the high of centre of mass are

$$x_i = -\varrho\gamma_i \quad (i = 1, 2)\,, \qquad x_3 = \varrho(1 - \gamma_3) - l\,, \qquad \zeta_0 = \varrho - (\varrho - l)\gamma_3\,.$$

Under $\omega_3 = \omega + \xi$, $\gamma_3 = 1 - \eta$ the energetic correlation (1.4) and the integrals (1.11) and $\gamma^2 = 1$ take the form

$$V_1 = M\boldsymbol{v}^2 + A_1(\omega_1^2 + \omega_2^2) + A_3(\xi^2 + 2\omega\xi) + 2Mg(\varrho - l)\eta \leqslant \text{const}$$

$$V_2 = A_1(\omega_1\gamma_1 + \omega_2\gamma_2) - A_3\xi\eta - H\eta + A_3\frac{l}{\varrho}\xi = \text{const}$$

$$V_3 = \gamma_1^2 + \gamma_2^2 + \eta^2 - 2\eta = 0\,, \quad H = A_3\omega + k$$

Consider the function

$$\begin{aligned} V = {} & V_1 + 2\lambda V_2 + \mu V_3 + \tfrac{1}{4}(A_3 - A_1)\lambda^2 V_3^2 = \\ = {} & M\boldsymbol{v}^2 + A_1(\omega_1^2 + \omega_2^2) + A_3\xi^2 + 2\lambda[A_1(\omega_1\gamma_1 + \omega_2\gamma_2) - A_3\xi\eta] + \\ & + \mu(\gamma_1^2 + \gamma_2^2) + [\mu + (A_3 - A_1)\lambda^2]\eta^2 + \ldots \end{aligned} \tag{4.1}$$

where

$$\lambda = -\frac{l}{\varrho}\omega\,, \quad \mu = Mg(\varrho - l) - H\lambda\,.$$

It is easily to see that the function (4.1) will be positive definite when [5]

$$(4.2) \qquad \left(H - A_1\omega\frac{\varrho}{l}\right)\frac{\varrho}{l}\omega + Mg(\varrho - l) > 0$$

and $dV/dt \leqslant 0$. Therefore the (4.2) is the sufficient stability condition of the motion of an axisymmetrical gyrostate with spherical support for every law of friction of form (1.3) with respect to the variables $\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\gamma}$. Similarly [2] it is possible to show that this condition is necessary for stability in the case of viscous friction. So in the last case the inequality (4.2) is the necessary and sufficient stability condition with respect to all variables and asymptotic stability condition with respect to the variables v_i, γ_i, ω_s $(s = 1, 2)$ because all conditions Lyapunov-Malkin's theorem hold.

In the case $k = 0$, $H = A_3\omega$ the inequality (4.2) becomes the stability condition of the rotation alone rigid body. Comparing it with (4.2) we conclude that it is possible to make the motion (1.12) of the gyrostate for given value $\omega \neq 0$ to be stable or unstable by wishing under proper choice of the value and the sign of the gyrostatic moment k independently from stability or instability of alone rigid body.

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