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**Fiber products and Morita duality for commutative rings**

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# Fiber Products and Morita Duality for Commutative Rings.

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The lack of examples of commutative rings with a Morita duality has been noticed by many authors (e.g. P. Vamos [8], B. Ballet [1], etc.). Apart from complete Noetherian local rings and maximal valuation rings, only a few other «sporadic» examples ([8]) seem to exist in the literature. In this paper we show that fiber product is a useful tool for the construction of commutative rings with a Morita duality starting from known examples; with it we are able to construct a number of examples of such rings.

In particular the examples we obtain in this way allow us to give a complete characterization of all the trees with a finite number of maximal chains which are order-isomorphic to spectra of commutative rings with a Morita duality. I. Kaplansky has observed that the spectra $X$ of commutative rings possess the following two properties (as partially ordered sets):

(K1) Every chain in $X$ has a least upper bound and a greatest lower bound;

(K2) If $x, y \in X$ and $x < y$ then there exist elements $x_1, y_1 \in X$ with $x \leqslant x_1 < y_1 \leqslant y$ such that there is no element of $X$ properly between $x_1$ and $y_1$.

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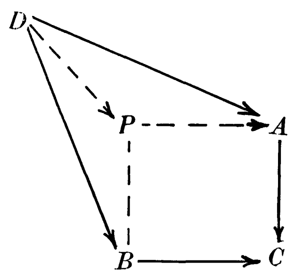
We prove that every tree $X$ with a finite number of maximal chains and with properties (K1) and (K2) is order-isomorphic to the spectrum of a commutative ring with a Morita duality.

Therefore all the trees with a finite number of maximal chains which are order-isomorphic to spectra of commutative rings are also order-isomorphic to spectra of commutative rings with a Morita duality.

## 1. Notation and elementary results.

All rings in this note are commutative with identity, and ring morphisms respect the identities.

Let $A, B, C$ be commutative rings, and let $\alpha: A \to C$, $\beta: B \to C$ be ring morphisms. Then there exist a commutative ring $P$ and two ring morphisms $\alpha': P \to B$ and $\beta': P \to A$ such that $\beta\alpha' = \alpha\beta'$, with the following property: given any ring $D$ and any two ring morphisms $\varphi: D \to A$ and $\psi: D \to B$ such that $\alpha\varphi = \beta\psi$, there exists a unique morphism $\omega: D \to P$ such that $\varphi = \beta'\omega$ and $\psi = \alpha'\omega$.




Disregarding $\alpha, \beta, \alpha', \beta'$, $P$ is called the *fiber product* of $A$ and $B$ over $C$, denoted by $A \times_C B$. It is unique up to isomorphism in the obvious way. The easiest way to visualize $A \times_C B$ is as a subring of $A \times B$ (direct product of the rings $A$ and $B$), by defining $A \times_C B = \{(a, b) \in A \times B | \alpha(a) = \beta(b)\}$ (the operations are of course by components) and by taking the projections on the first and the second factor as $\beta'$ and $\alpha'$ respectively.

It is easy to generalize the above construction to any finite number of rings $A_1, \ldots, A_n$ and morphisms $\alpha_1, \ldots, \alpha_n$, with $\alpha_i: A_i \to C$.

It is also easy to prove the existence of the following canonical isomorphisms:

1) $A \times_C B \cong B \times_C A$;

2) $(A_1 \times_C A_2) \times_C A_3 \cong A_1 \times_C (A_2 \times_C A_3)$;

3) $C \times_C A \cong A$.

The isomorphism in 1) is the exchange; in 3) the morphism $C \to C$ is the identity; 2) is improved in the following lemma.

1.1. LEMMA. *Let $A_1, A_2, A_3, C_1, C_2$ be commutative rings, and $\alpha_{11}: A_1 \to C_1$, $\alpha_{21}: A_2 \to C_1$, $\alpha_{22}: A_2 \to C_2$, $\alpha_{32}: A_3 \to C_2$ be ring morphisms. Then $\alpha_{22}$ canonically induces a morphism $\tilde{\alpha}_{22}: A_1 \times_{C_1} A_2 \to C_2$ and $\alpha_{21}$ canonically induces a morphism $\tilde{\alpha}_{21}: A_2 \times_{C_2} A_3 \to C_1$. Constructing the fiber product with respect to these morphisms, there is a canonical isomorphism $(A_1 \times_{C_1} A_2) \times_{C_2} A_3 \cong A_1 \times_{C_1} (A_2 \times_{C_2} A_3)$.*

The proof is standard. Another lemma which plays a fundamental role in the sequel is the following:

1.2. LEMMA. *Let $A, B, C$ be commutative rings, and $\alpha: A \to C$, $\beta: B \to C$ be ring morphisms. Let $C' = \alpha(A) \cap \beta(B)$, $A' = \alpha^{-1}(C')$, $B' = \beta^{-1}(C')$. Then $A', B', C'$ are subrings of $A, B, C$ respectively, and there is a canonical isomorphism $A \times_C B \cong A' \times_{C'} B'$.*

This proof is standard too. Note that in Lemma 1.2 the morphisms $A' \to C'$, $B' \to C'$ used in the construction of the fiber product $A' \times_{C'} B'$ are the restrictions of $\alpha$ and $\beta$. These restrictions are surjective ring morphisms. Since we shall essentially employ the fiber product to construct new types of rings with a Morita duality, Lemma 1.2 says that we may limit ourselves to the case in which the two morphisms $\alpha: A \to C$ and $\beta: B \to C$ are surjective. This hypothesis considerably simplifies our approach. Hence *from now on whenever we construct the fiber product $A \times_C B$, we shall suppose that the two morphisms $\alpha: A \to C$ and $\beta: B \to C$ are surjective.* We conclude this section with three elementary lemmas which will be useful in the sequel.

1.3. LEMMA. *Let $A, B, C$ be commutative rings, $\alpha: A \to C$, $\beta: B \to C$ surjective ring morphisms. Then the ring morphisms $\alpha': A \times_C B \to B$ and $\beta': A \times_C B \to A$ are surjective.*

The proof is trivial (think of $A \times_C B$ as a subring of $A \times B$ and of $\alpha'$ and $\beta'$ as the restrictions of the canonical projections).

Recall that a ring is *local* if it has a unique maximal ideal.

1.4. LEMMA. *Let $A$, $B$, $C$ be commutative rings, $\alpha: A \to C$, $\beta: B \to C$ surjective ring morphisms. Then $A \times_C B$ is local if and only if $A$ and $B$ are local.*

PROOF. The necessity follows from Lemma 1.3. The proof of the sufficiency is standard as soon as $A \times_C B$ is viewed as a subring of $A \times B$; in this case the maximal ideal of $A \times_C B$ is $(A \times_C B) \cap (\mathfrak{M}_A \times \mathfrak{M}_B)$, where $\mathfrak{M}_A$ $(\mathfrak{M}_B)$ is the maximal ideal of $A$ $(B)$.

1.5. LEMMA. *Let $A$, $B$, $C$ be commutative rings, $\alpha: A \to C$, $\beta: B \to C$ surjective ring morphisms. Then $A \times_C B$ is a Noetherian ring if and only if $A$ and $B$ are Noetherian rings.*

PROOF. The necessity follows from Lemma 1.3. Sufficiency: Consider $A$ and $B$ as $A \times_C B$-modules via $\beta'$ and $\alpha'$; then the $A \times_C B$-submodules of $A$ and $B$ are exactly their ideals. Hence $A$ and $B$ are Noetherian $A \times_C B$-modules, and therefore $A \times B$ is a Noetherian $A \times_C B$-module. But $A \times_C B$ is a submodule of $A \times B$.

## 2. Spectrum of a fiber product.

In Section 5 we need a description of the spectrum of a fiber product. The fiber product $A \times_C B$ of two rings $A$ and $B$ over a third ring $C$ may be viewed as a pasting of the two rings $A$ and $B$ along $C$. It follows that its spectrum $\operatorname{Spec}(A \times_C B)$ is obtained by pasting together $\operatorname{Spec}(A)$ and $\operatorname{Spec}(B)$ along two closed sets homeomorphic to $\operatorname{Spec}(C)$. This is better specified in Proposition 2.1.

Let $A$, $B$, $C$ be commutative rings and $\alpha: A \to C$, $\beta: B \to C$ be surjective ring morphisms. Consider the topological space $X = \operatorname{Spec}(A) \cup \operatorname{Spec}(B)$, disjoint union of $\operatorname{Spec}(A)$ and $\operatorname{Spec}(B)$ with the topology in which the open sets are exactly the union of an open set of $\operatorname{Spec}(A)$ and an open set of $\operatorname{Spec}(B)$. If $P \in \operatorname{Spec}(A)$ and $Q \in \operatorname{Spec}(B)$, set $P \sim Q$ if $P \supseteq \ker \alpha$, $Q \supseteq \ker \beta$ and $\alpha(P) = \beta(Q)$. Consider the equivalence relation in $X$ generated by $\sim$. Call this equivalence relation $\sim$ too. Then it is possible to consider the quotient topological space $(\operatorname{Spec}(A) \cup \operatorname{Spec}(B))/\sim$ of the topological space $X = \operatorname{Spec}(A) \cup \operatorname{Spec}(B)$ modulo the equivalence relation $\sim$.

2.1. PROPOSITION. $\operatorname{Spec}(A \times_C B)$ *is canonically homeomorphic to* $(\operatorname{Spec}(A) \cup \operatorname{Spec}(B))/\sim$.

PROOF. Apply the functor Spec to the commutative diagram

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\beta'} & A \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \alpha} \\ B & \xrightarrow[\beta]{} & C \end{array}$$

of commutative rings and surjective ring morphisms, and get the commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(A \times_C B) & \xleftarrow{\operatorname{Spec}(\beta')} & \operatorname{Spec}(A) \\ {\scriptstyle \operatorname{Spec}(\alpha')}\uparrow & & \uparrow{\scriptstyle \operatorname{Spec}(\alpha)} \\ \operatorname{Spec}(B) & \xleftarrow[\operatorname{Spec}(\beta)]{} & \operatorname{Spec}(C) \end{array}$$

of topological spaces and continuous maps. Spec $(\alpha)$, Spec $(\beta)$, Spec $(\alpha')$ and Spec $(\beta')$ are homeomorphisms of Spec $(C)$, Spec $(C)$, Spec $(B)$, Spec $(A)$ onto $V(\ker \alpha)$, $V(\ker \beta)$, $V(\ker (\alpha'))$, $V(\ker \beta')$ respectively (here if $I$ is an ideal of a ring $R$, $V(I)$ is the closed set of Spec $(R)$ consisting of all prime ideals of $R$ containing $I$).

Thus if $X$ is the disjoint union of Spec $(A)$ and Spec $(B)$ and $\varphi: X \to \operatorname{Spec}(A \times_C B)$ is the map which restricted to Spec $(A)$ coincides with Spec $(\beta')$ and restricted to Spec $(B)$ coincides with Spec $(\alpha')$, then $\varphi$ is a continuous map. Furthermore $\ker \alpha' \cap \ker \beta' = 0$, so every prime ideal of $A \times_C B$ contains either $\ker \alpha'$ or $\ker \beta'$. This means that $\varphi$ is surjective. Using the fact that the restrictions of $\varphi$ to Spec $(A)$ and Spec $(B)$ are injective, it is also easy to check that the kernel of $\varphi$ is the equivalence relation $\sim$. In order to prove the theorem it only remains to show that the continuous map $\varphi$ is a quotient map, i.e. that the topology of Spec $(A \times_C B)$ is the finest for which $\varphi$ is continuous, i.e. that if $Y$ is a subset of Spec $(A \times_C B)$ and $\varphi^{-1}(Y)$ is closed in $X$ then $Y$ is closed in Spec $(A \times_C B)$. But if $Y$ is a subset of Spec $(A \times_C B)$ and $\varphi^{-1}(Y) = \operatorname{Spec}(\beta')^{-1}(Y) \cup \operatorname{Spec}(\alpha')^{-1}(Y)$ is closed in $X$, then $\operatorname{Spec}(\beta')^{-1}(Y)$ is closed in Spec $(A)$. But Spec $(\beta')$ is a homeomorphism of Spec $(A)$ onto $V(\ker \beta')$. It follows that $Y \cap V(\ker \beta')$ is closed in $V(\ker \beta')$, hence it is closed in Spec $(A \times_C B)$. Similarly $Y \cap V(\ker \alpha')$ is closed in Spec $(A \times_C B)$. Thus $Y = (Y \cap V(\ker \beta')) \cup (Y \cap V(\ker \alpha'))$ is closed in Spec $(A \times_C B)$.

Intuitively Proposition 2.1 says that Spec $(A)$ and Spec $(B)$ each contains a closed set homeomorphic to Spec $(C)$ and that if we paste Spec $(A)$ and Spec $(B)$ together by identifying these two closed sets, we get Spec $(A \times_C B)$ (under the hypothesis that $\alpha\colon A \to C$ and $\beta\colon B \to C$ are surjective, of course).

## 3. Injective envelope of a simple module in a local fiber product.

In Section 2 we studied the spectrum of the fiber product of two rings. This paper is devoted to the study of the commutative rings with a Morita duality. Now a ring with a Morita duality is linearly compact (in the discrete topology) [6, Theorem 1] and every linearly compact commutative ring is the direct product of a finite number of local rings [10, Proposition 14]. Hence without loss of generality we only have to study the local case.

Now by Lemma 1.4, if $\alpha\colon A \to C$ and $\beta\colon B \to C$ are surjective, $A \times_C B$ is local if and only if both $A$ and $B$ are local; in this case $C$ is local too; moreover it is immediate to see that in this case the unique (up to isomorphism) simple modules over $A$, $B$, $C$ and $A \times_C B$ are isomorphic (as $A \times_C B$-modules). Hence we need to study the injective envelope of the unique (up to isomorphism) simple $A \times_C B$-module, i.e. the minimal injective cogenerator in the category of all $A \times_C B$-modules. The situation, described in the following theorem, is the best we could hope for: since $A \times_C B$ is the pull-back of $A$ and $B$ over $C$, the minimal injective cogenerator in the category of all $A \times_C B$-modules is the push-out of the minimal injective cogenerators in the categories of all $A$- and $B$-modules over the minimal injective cogenerator in the category of all $C$-modules.

In the sequel if $R$ is a ring, $I$ an ideal of $R$ and $M$ an $R$-module, then $\mathrm{Ann}_M I$ denotes the set of all $x \in M$ such that $Ix = 0$ and $E_R(M)$ denotes the injective envelope of $M$.

3.1. Theorem. *Let $A$, $B$, $C$ be local rings, and $\alpha\colon A \to C$, $\beta\colon B \to C$ be surjective ring morphisms. Let $A \times_C B$ be the fiber product of $A$ and $B$ over $C$, and let $S$ be the unique (up to isomorphism) simple $C$-module.*

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\beta'} & A \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \alpha} \\ B & \xrightarrow[\beta]{} & C \end{array}$$

*Then*

1) *$S$ is the unique (up to isomorphism) simple $A$-, $B$- and $A \times_C B$-module.*

2) *$\alpha$ induces a monomorphism of $A$-modules (and therefore of $A \times_C B$-modules) $\alpha^*\colon E_C(S) \to E_A(S)$, whose image is $\operatorname{Ann}_{E_A(S)} \ker \alpha$; similarly for $\beta$.*

3) *If $P$ is the push-out of the following diagram of $A \times_C B$-modules*

$$\begin{array}{ccc} P & \dashleftarrow & E_A(S) \\ \uparrow & & \uparrow \alpha^* \\ E_B(S) & \xleftarrow[\beta^*]{} & E_C(S) \end{array}$$

*then $P$ is the injective envelope of the simple $A \times_C B$-module $S$.*

PROOF:

1) is obvious.

2) easily follows from [7, Proposition 2.27]; note that $\alpha^*(\alpha(a)e) = a\alpha^*(e)$ for $a \in A$, $e \in E_C(S)$.

3). Let us view $A \times_C B$ as a subring of $A \times B$. Then if $P$ is the push-out of the above diagram, we have $P = (E_A(S) \oplus E_B(S))/M$, where $M = \{(\alpha^*(e), -\beta^*(e)) | e \in E_C(S)\}$; the multiplication of an element $(a, b) \in A \times_C B$ by an element $\overline{(x, y)} \in P$, where $x \in E_A(S)$, $y \in E_B(S)$ and the bar denotes reduction modulo $M$, is defined by $(a, b)\overline{(x, y)} = \overline{(ax, by)}$. Note that the map $\varepsilon_A\colon E_A(S) \to P$ (dotted in the above diagram) is simply the canonical embedding of $E_A(S)$ into $E_A(S) \oplus E_B(S)$ followed by the reduction modulo $M$. It is clearly injective. Similarly for $\varepsilon_B\colon E_B(S) \to P$. We want to prove that $P = E_{A \times_C B}(S)$. Let us show that $P$ is an injective $A \times_C B$-module. The proof is divided into four steps.

STEP 1: *For each ideal $I$ in $A$ every $A \times_C B$-morphism $\varphi\colon I \to P$ extends to an $A \times_C B$-morphism $A \to P$. Similarly for $B$.*

In fact if $I$ is an ideal of $A$ and $\varphi\colon I \to P$ is an $A \times_C B$-morphism, then $\varphi(I) \subseteq \operatorname{Ann}_P(\ker \beta')$ because $I \ker \beta' = 0$. Now

$$\begin{aligned}\mathrm{Ann}_P(\ker\beta') &= \{\overline{(x, y)} \in P \mid x \in E_A(S),\ y \in E_B(S),\ (0 \times \ker\beta)(x, y) \subseteq M\}\\ &= \{\overline{(x, y)} \in P \mid x \in E_A(S),\ y \in E_B(S),\ \ker\beta\cdot y = 0\}\\ &= \{\overline{(x, y)} \in P \mid x \in E_A(S),\ y \in E_B(S),\ y \in \beta^*(E_C(S))\}\,.\end{aligned}$$

But if $x \in E_A(S)$ and $y \in \beta^*(E_C(S))$ then $(x, y) \equiv (x + \alpha^*\beta^{*-1}(y), 0)$ mod $M$, and thus $\mathrm{Ann}_P(\ker\beta') = \varepsilon_A(E_A(S))$. Hence $\varphi(I) \subseteq \varepsilon_A(E_A(S))$, and thus there exists an $A \times_C B$-morphism $\psi\colon I \to E_A(S)$ such that $\varphi = \varepsilon_A\psi$. But $\psi$ is also an $A$-morphism and $E_A(S)$ is $A$-injective. Therefore there exists an $A$-morphism $\tilde{\psi}\colon A \to E_A(S)$ extending $\psi$. Thus $\varepsilon_A\tilde{\psi}\colon A \to P$ is an $A \times_C B$-morphism extending $\varphi$. Similarly for $B$.

STEP 2: $\mathrm{Ext}^1_{A\times_C B}(A, P) = 0$.

Let $\varphi\colon \ker\beta' \to P$ be an $A \times_C B$-morphism. Since $\ker\beta' = 0 \oplus \ker\beta$, by the first step $\varphi$ extends to an $A \times_C B$-morphism $0 \oplus B \to P$; this in turn trivially extends to an $A \times_C B$-morphism $A \times B \to P$, whose restriction to $A \times_C B$ is an $A \times_C B$-morphism extending $\varphi$. Therefore every $A \times_C B$-morphism $\ker\beta' \to P$ extends to an $A \times_C B$-morphism $A \times_C B \to P$.

Consider the exact sequence $0 \to \ker\beta' \to A \times_C B \to A \to 0$. By applying the functor $\mathrm{Hom}_{A\times_C B}(-, P)$, we get an exact sequence

$$\mathrm{Hom}\,(A \times_C B, P) \to \mathrm{Hom}\,(\ker\beta', P) \to \mathrm{Ext}^1(A, P) \to \mathrm{Ext}^1\,(A \times_C B, P).$$

In this sequence the last module in zero and we have just proved that the first morphism is surjective. Hence $\mathrm{Ext}^1_{A\times_C B}(A, P) = 0$.

STEP 3: *If $I$ is an ideal of $A$, $\mathrm{Ext}^1_{A\times_C B}(A/I, P) = 0$. Similarly for $B$.*

Consider the exact sequence $0 \to I \to A \to A/I \to 0$. By applying the functor $\mathrm{Hom}_{A\times_C B}(-, P)$ we get the exact sequence $\mathrm{Hom}\,(A, P) \to$ $\to \mathrm{Hom}\,(I, P) \to \mathrm{Ext}^1\,(A/I, P) \to \mathrm{Ext}^1\,(A, P)$. By the first step the first morphism is surjective and by the second step the last module is zero. Thus $\mathrm{Ext}^1_{A\times_C B}(A/I, P) = 0$. Similarly for $B$.

STEP 4: *$P$ is an injective $A \times_C B$-module.*

We must show that if $I$ is an ideal of $A \times_C B$, every morphism $I \to P$ extends to a morphism $A \times_C B \to P$. We shall even show that every morphism $I \to P$ extends to a morphism $A \times B \to P$. To do that it is enough to prove that $\mathrm{Ext}^1_{A\times_C B}(A\oplus B/I, P) = 0$. From the exact sequence $0 \to (I+B)/I \to (A\oplus B)/I \to (A\oplus B)/(I+B) \to 0$ we get the exact sequence

$$(*)\qquad \mathrm{Ext}^1_{A\times_C B}\big((A\oplus B)/(I+B), P\big) \to \mathrm{Ext}^1_{A\times_C B}\big((A\oplus B)/I, P\big) \to \to \mathrm{Ext}^1_{A\times_C B}\big((I+B)/I, P\big)\,.$$

But $(A\oplus B)/(I+B) \cong A/A\cap(I+B)$ and $(I+B)/I \cong B/B\cap I$, where $A\cap(I+B)$ and $B\cap I$ are $A\times_C B$-submodules of $A$ and $B$ respectively, i.e. ideals in $A$ and $B$. By the third step, the first and the last module in $(*)$ are zero. Hence $P$ is an injective $A\times_C B$-module.

Let us show that $P$ is an indecomposable $A\times_C B$-module. Let $\varphi$ be a non-zero $A\times_C B$-endomorphism of $P$ such that $\varphi^2 = \varphi$. It is enough to prove that $\varphi$ is the identity morphism. Now $\varphi\big(\mathrm{Ann}_P(\ker\beta')\big) \subseteq \subseteq \mathrm{Ann}_P(\ker\beta')$. In the proof of Step 1 we have seen that $\mathrm{Ann}_P(\ker\beta') = = \varepsilon_A\big(E_A(S)\big)$. Hence $\varphi\varepsilon_A E_A(S) \subseteq \varepsilon_A E_A(S)$. Therefore $\varphi$ induces an $A\times_C B$-endomorphism of $\varepsilon_A\big(E_A(S)\big)$, i.e. an $A$-endomorphism of $E_A(S)$, which coincides with its own square. Since $E_A(S)$ is an indecomposable $A$-module, it follows that either $\varphi\varepsilon_A = 0$ or $\varphi\varepsilon_A = \varepsilon_A$. Similarly either $\varphi\varepsilon_B = 0$ or $\varphi\varepsilon_B = \varepsilon_B$. Since $\varphi \neq 0$ and $\varepsilon_A E_A(S) + \varepsilon_B E_B(S) = P$, we must have either $\varphi\varepsilon_A = \varepsilon_A$ or $\varphi\varepsilon_B = \varepsilon_B$. If for instance $\varphi\varepsilon_A = \varepsilon_A$, then $\varphi\varepsilon_A(S) = \varepsilon_A(S)$. But $\varepsilon_A(S) = \varepsilon_B(S)$. Hence $\varphi\varepsilon_B(S) = \varepsilon_B(S) \neq 0$, from which $\varphi\varepsilon_B = \varepsilon_B$. Hence we must have both $\varphi\varepsilon_A = \varepsilon_A$ and $\varphi\varepsilon_B = \varepsilon_B$. But then if $\overline{(x, y)} \in P$, $\varphi\overline{(x, y)} = \varphi\big(\varepsilon_A(x) + \varepsilon_B(y)\big) = \varphi\varepsilon_A(x) + \varphi\varepsilon_B(y) = = \varepsilon_A(x) + \varepsilon_B(y) = \overline{(x, y)}$. Hence $\varphi$ is the identity of $P$ and $P$ is indecomposable.

Therefore $P$ is an indecomposable injective module and contains the simple $A\times_C B$-module $\varepsilon_A(S) \cong S$. Hence $P = E_{A\times_C B}(S)$.

## 4. Fiber products and Morita duality.

We essentially know three operations of commutative rings which preserve the property « to have a Morita duality »:

1) Direct product: If $A$ and $B$ are rings with a Morita duality, then $A\times B$ has a Morita duality. (Note that every commutative ring with a Morita duality is a finite direct product of local rings with a Morita duality.)

2) Homomorphic images: If $A$ is a ring with a Morita duality and $I$ is an ideal of $A$, then $A/I$ has a Morita duality.

3) Taking linearly compact extensions and conversely: Let $A \subseteq B$ be commutative rings, and let $B$ be a linearly compact $A$-module. Then $A$ has a Morita duality if and only if $B$ has a Morita duality. This is a theorem due to Peter Vamos [8, Theorem 2.14].

In this section we prove that such a property is also preserved by fiber product; assuming, without loss of generality, that the fiber product is local, we shall prove the following statement:

4) Let $A, B, C$ be local rings and let $\alpha: A \to C$, $\beta: B \to C$ be surjective ring morphisms. Then $A \times_C B$ has a Morita duality if and only if $A$ and $B$ have a Morita duality.

We must notice that we also know other operations between commutative rings which preserve the property « to have a Morita duality », but all such operations easily follow from the operations 1), 2) and 3). For instance:

*a*) Taking finitely generated integral extensions (this follows from 3) and is the case of Example 2.4 in [8]);

*b*) Taking split extensions, by joining a linearly compact module as a nilpotent ideal (see [5, Theorem 10]; this is the case of Example 2.4 in [8], too);

*c*) Group rings: if $R$ is a commutative ring with a Morita duality and $G$ is a finite abelian group, then the group ring $R[G]$ has a Morita duality (it follows from *a*)); etc.

The interest in proving 4) lies in the fact that with the operation 4) it is possible to construct a number of examples of commutative rings with a Morita duality. The (local) commutative rings with a Morita duality known up to now are essentially the following:

i) complete Noetherian local rings;

ii) maximal valuation rings;

iii) rings with a prime ideal $P \neq 0$ such that 1) $P$ is comparable to every ideal of $R$, that is $I \subseteq P$ or $P \subseteq I$ for all ideals $I$ of $R$; 2) the canonical morphism $R \to R_P$ is injective; 3) $R_P$ has a Morita duality; 4) $R/P$ has a linearly compact field of fractions. This class of rings was discovered by P. Vamos [8]. Müller ([8, page 285]) has noticed that such rings $R$ are the fiber product of the rings $R/P$ and $R_P$ (but here the morphisms are not both surjective).

All other known examples of commutative rings with a Morita duality are obtained from these rings with operations 1), 2) and 3). This is the case, for instance, for the domains with linearly compact field of fractions ([8]).

The following result has an elementary proof, but since it plays an essential role in the paper, we shall give two different proofs of it.

4.1. THEOREM. *Let $A, B, C$ be local rings, and let $\alpha: A \to C$, $\beta: B \to C$ be surjective ring morphisms. Then $A \times_C B$ has a Morita duality if and only if $A$ and $B$ have a Morita duality.*

FIRST PROOF. The necessity is obvious, since $A$ and $B$ are homomorphic images of $A \times_C B$. The sufficiency will be proved by using Müller's Theorem 1 [6] and our Theorem 3.1. A ring $R$ has a Morita duality if and only if $R$ and the minimal injective cogenerator are linearly compact. Thus if $A$ and $B$ have a Morita duality, $A$ and $B$ are linearly compact $A \times_C B$-modules, and therefore $A \times_C B \subseteq A \oplus B$ is linearly compact. On the other hand the minimal injective cogenerators are linearly compact $A$- and $B$-modules. By Theorem 3.1 the $A \times_C B$-minimal injective cogenerator is linearly compact; hence $A \times_C B$ has a Morita duality.

SECOND PROOF. Let us prove the sufficiency by using Müller's Theorem 1 [6] and Vamos' Theorems 2.14 [8]. If $A$ and $B$ have a Morita duality, then $A$ and $B$ are linearly compact rings. Hence they are linearly compact $A \times_C B$-modules, from which $A \times B$ is a linearly compact $A \times_C B$-module. Since $A \times B$ has a Morita duality, $A \times_C B$ has a Morita duality too.

The last part of this paper is devoted to the study of the local rings with a Morita duality which are obtained as fiber products. Note that if $A, B, C$ are rings and $\alpha: A \to C$, $\beta: B \to C$ are surjective ring morphisms, then $A \times_C B$ is a complete Noetherian local ring if and only if $A$ and $B$ are complete Noetherian local rings (this follows, for instance, from 1.4, 1.5 and 4.1); hence by taking fiber products (via surjective ring morphisms) of Noetherian rings with a Morita duality we still obtain rings of the same type. On the contrary we get new examples of commutative rings with a Morita duality by taking fiber products (via surjective ring morphisms) either of a

Noetherian ring and a valuation ring or of two valuation rings, and by iterating such constructions a finite number of times. This is what we shall do in the next section.

## 5. Particular cases.

Let $A, B, C$ be commutative rings and let $\alpha: A \to C$, $\beta: B \to C$ be surjective ring morphisms. Assume $A$ is a complete Noetherian local ring and $B$ is a maximal valuation ring. Then $C$ must be a Noetherian maximal valuation ring. Hence $C$ has to be of one of the following three types: 1) a field; 2) a local artinian principal ideal ring which is not a field; 3) a complete DVR.

In this case $\operatorname{Spec}(A \times_C B)$ is $\operatorname{Spec}(A)$ to which the chain of all prime non-maximal ideals of $B$ (in cases 1) and 2)) or the chain of all prime ideals of coheight $>1$ of $B$ (in case 3)) has been pasted under the maximal ideal $\mathfrak{M}_A$ of $A$ (in cases 1) and 2)) or under the prime ideal $\alpha^{-1}(0)$ of coheight 1 in $A$ (in the case 3)).

The Noetherianity of $A$ and the fact that $B$ is a valuation ring (i.e. every ideal in $B$ is the union of a chain of principal ideals) « mix » in $A \times_C B$ in the following sense: every ideal of $A \times_C B$ is the union of an ascending chain of finitely generated ideals. If we fix an ideal $I$ of $A \times_C B$, this can easily be seen by considering the exact sequence $0 \to I \cap \ker \beta' \to I \xrightarrow{\beta'} \beta'(I) \to 0$ and by taking a set of generators of $I$ consisting of a finite number of elements of $I$ whose $\beta'$-images generate $\beta'(I)$ and of a set of generators of $I \cap \ker \beta'$.

5.1. Example. Let $C = k$ be a field, $X_1, \ldots, X_n$ indeterminates, $A = k[[X_1, \ldots, X_n]]$ the power-series ring, $G$ a totally ordered abelian group, $B = k[[G]]$ the long power-series ring relative to $k$ and $G$. Then $A \times_C B$, fiber product of a complete Noetherian local ring and a maximal valuation domain, is a commutative local ring with a Morita duality.

5.2. Example. Let $k$, $A$ and $G$ be as in the above example, let $G \oplus \mathbb{Z}$ have the lexicographic order, $B = k[[G \oplus \mathbb{Z}]]$ be the long power-series ring, $C = A/(X_2, \ldots, X_n) \cong k[[\mathbb{Z}]]$.

5.3. Example. Let $k, A, G, B$ be as in Example 5.2, and let $n$ be an integer $> 1$. Take $C = A/(X_1^n, X_2, \ldots, X_n) \cong B/I$, where $I$ is the ideal of $B$ consisting of all elements of $B$ with valuation $\geqslant (0, n)$.

5.4. EXAMPLE. Let $p$ be a prime number, $A$ the ring of $p$-adic integers, $G$ a totally ordered abelian group, $C = \mathbb{Z}/p\mathbb{Z}$, $B = C[\![G]\!]$ the long power-series ring.

Examples 5.2, 5.3, 5.4, like Example 5.1, are local rings with a Morita duality, which are fiber products of a Noetherian ring and a valuation domain.

Now, on the contrary, let $A, B, C$ be maximal valuation rings, and let $\alpha: A \to C$, $\beta: B \to C$ be surjective ring morphisms. By Proposition 2.1 the spectrum of $A \times_C B$ looks like a reverse $Y$. The fact that $A$ and $B$ are valuation rings (i.e. every finitely generated ideal is principal) is inherited by $A \times_C B$ in the following way: every finitely generated ideal of $A \times_C B$ can be generated by two elements.

5.5. EXAMPLE. Let $k$ be a field, $G_1, G_2, H$ be totally ordered abelian groups, $G_1 \oplus H$, $G_2 \oplus H$ have the lexicographic order. Set $A = k[\![G_1 \oplus H]\!]$, $B = k[\![G_2 \oplus H]\!]$, $C = k[\![G]\!]$, long power-series rings. Then $A \times_C B$ is local, has a Morita duality, is the fiber product of two valuation rings and every finitely generated ideal can be generated by two elements.

Such a construction can be easily iterated to any finite number of maximal valuation rings $A_1, \ldots, A_n$, $C_1, \ldots, C_{n-1}$ and surjective morphisms $\alpha_i: A_i \to C_i$, $\beta_i: A_{i+1} \to C_i$, $i = 1, \ldots, n-1$. In this case the spectrum of $A_1 \times_{C_1} A_2 \times_{C_2} \cdots \times_{C_{n-1}} A_n$ is a tree with at most $n$ maximal chains and every finitely generated ideal can be generated by $n$ elements.

Let us invert such a result with a construction which is curiously dual to a construction due to S. Wiegand [9]. Recall that a *chain* of a partially ordered set $X$ is a totally ordered subset of $X$. A *tree* is a partially ordered set $X$ with maximum such that $\{y \in X | y \geqslant x\}$ is a chain of $X$ for all $x \in X$.

S. Wiegand [9] proved that if $X$ is a finite tree, then there exists a Bezout domain $R$ such that Spec $(R)$ is order anti-isomorphic to $X$ and $R_P$ is a maximal valuation domain for all $P \in$ Spec $(R)$. We shall prove that if $X$ is a finite tree, then there exists a local ring $R$ with a Morita duality such that Spec $(R)$ is order isomorphic to $X$ and $R/P$ is a maximal valuation domain for all $P \in$ Spec $(R)$. In fact we shall prove much more, allowing $X$ to be infinite but with a finite number $n$ of maximal chains, and we shall obtain that the $R$ constructed will have the property that every finitely generated ideal of $R$ can be generated by $n$ elements.

Of course not every tree with a finite number of maximal chains is order-isomorphic to the spectrum of a commutative ring. In fact let $X$ be a partially ordered set and suppose that $X$ is order-isomorphic to Spec $(R)$ for some commutative ring $R$. Then, as noted by I. Kaplansky [3, page 6, Theorems 9 and 11] (see also [4]), the following two properties hold:

(K1) Every chain in $X$ has a supremum (sup) and an infimum (inf).

(K2) If $x, y \in X$ and $x < y$ then there exist elements $x_1, y_1 \in X$ such that $x \leqslant x_1 < y_1 \leqslant y$ and there does not exist an element of $X$ properly between $x_1$ and $y_1$.

Recall the definition of lexicographic product [2, Chap. III, § 15, Exercise 3]. Let $\{G_\lambda\}_{\lambda \in \Lambda}$ be a family of totally ordered abelian groups, and assume that $\Lambda$ is totally ordered under a relation $<$. Let $G = \prod_{\lambda \in \Lambda} G_\lambda$ be the direct product of the groups $G_\lambda$. We consider the elements of $G$ as functions $f\colon \Lambda \to \bigcup_{\lambda \in \Lambda} G_\lambda$ such that $f(\lambda) \in G_\lambda$ for each $\lambda$ in $\Lambda$. For $f \in G$, we define the *support* of $f$, denoted $S(f)$, to be $\{\lambda \in \Lambda | f(\lambda) \neq 0\}$. Let $L = \{f \in G | S(f)$ is a well-ordered subset of $\Lambda\}$. Then $L$ is a subgroup of $G$. We define a relation $\leqslant$ on $L$ as follows. If $f, g \in L$, $f \neq g$, and if $\lambda$ is the first element of $S(g - f)$, then $f \leqslant g$ if and only if $f(\lambda) \leqslant g(\lambda)$. Then the relation $\leqslant$ is a total order compatible with the group operation on $L$. The group $L$, under the relation $\leqslant$, is called the *lexicographic product* of the groups $\{G_\lambda\}_{\lambda \in \Lambda}$.

Also recall that a subgroup $H$ of a totally ordered group $G$ is *convex* if $y \in H$ whenever $x \in H$, $y \in G$ and $0 \leqslant y \leqslant x$.

First we need a lemma.

5.6. Lemma. *Let* $(X, \leqslant)$ *be a totally ordered set with properties* (K1) *and* (K2). *Let* $Y = \{y \in X | y$ *has an immediate predecessor in* $X\}$. *Let* $\preccurlyeq$ *be the order relation in* $Y$ *defined by* $y_1 \preccurlyeq y_2$ *if* $y_1, y_2 \in Y$ *and* $y_1 \geqslant y_2$ *in* $X$. *Let* $L$ *be the lexicographic product of a family of totally ordered groups isomorphic to* $\mathbb{Z}$ *indexed by* $(Y, \preccurlyeq)$. *Then the set of all convex subgroups of* $L$ *ordered by* $\subseteq$ *is order isomorphic to* $(X, \leqslant)$.

Proof. Let $\mathcal{C}(L)$ be the set of all convex subgroups of $L$.

Define a map $\varphi\colon \mathcal{C}(L) \to X$: if $C \in \mathcal{C}(L)$, $C \neq 0$, $\varphi(C) = \sup_X\Big(\bigcup_{f \in C} S(f)\Big)$, $\varphi(0) = \inf_X(X)$. Note that $\varphi$ is well-defined because $X$ is totally ordered and has property (K1). Furthermore if $C_1, C_2 \in \mathcal{C}(L)$ and $C_1 \subseteq C_2$, then $\varphi(C_1) \leqslant \varphi(C_2)$.

Define a map $\psi\colon X \to \mathcal{C}(L)$: if $x \in X$, $\psi(x) = \{f \in L | \sup_X(S(f)) \leqslant x\} \cup \cup \{0\}$. It is easy to check that $\psi(x)$ is a convex subgroup of $L$ and that if $x_1 \leqslant x_2$, then $\psi(x_1) \subseteq \psi(x_s)$.

Let us show that $\varphi$ and $\psi$ are inverses of each other. If $x \in X$, $\varphi\psi(x) = x$, i.e. $\sup_X\big(\cup \{S(f) | f \in L, \sup_X(S(f)) \leqslant x\}\big) = x$, because $x = \sup_X(\{y \in Y | y \leqslant x\})$ by (K2). Let us show that if $C \in \mathcal{C}(L)$, then $\psi\varphi(C) = C$. We have to show that for all $f \in L$, $f \in C$ if and only if $\sup_X(S(f)) \leqslant \sup_X\Big(\bigcup_{g \in C} S(g)\Big)$. The «only if» is trivial. Conversely if $\sup_X(S(f)) \leqslant \sup_X\Big(\bigcup_{g \in C} S(g)\Big)$, then there exists $h \in C$ such that $\sup_X(S(f)) \leqslant \sup_X(S(h))$; otherwise, since $\sup_X(S(f)) = \inf_Y(S(f)) \in Y$, $\sup_X(S(f))$ would have an immediate predecessor $y$ in $X$; it would follow that $y \geqslant \sup_X(S(g))$ for all $g \in C$, i.e. $\sup_X(S(f)) > y \geqslant \sup_X\Big(\bigcup_{g \in C} S(g)\Big)$, contradiction. Hence $h$ exists. But then if $n = |f(\sup_X S(f))| + 1 \in \mathbb{Z}$, $n|h| \in C$ and $0 \leqslant |f| \leqslant n|h|$. Since $C$ is convex, $|f| \in C$ and thus $f \in C$.

We are ready for our last theorem.

5.7. THEOREM. *Let $X$ be a tree with a finite number $n$ of maximal chains. The following are equivalent:*

(i) *$X$ has properties* (K1) *and* (K2).

(ii) *There exists a commutative ring $R$ such that* Spec $(R) \cong X$ *(as an ordered set).*

(iii) *There exists a commutative ring $R$ such that:*

1) Spec $(R) \cong X$ *(as an ordered set)*;
2) *$R$ is local*;
3) *$R$ has a Morita duality*;
4) *$R/P$ is a maximal valuation domain for all $P \in$* Spec $(R)$;
5) *Every finitely generated ideal of $R$ can be generated by $n$ elements.*

PROOF. (iii) $\Rightarrow$ (ii) is trivial and (ii) $\Rightarrow$ (i) is Kaplansky's remark. We only have to prove that (i) $\Rightarrow$ (iii). Let $X$ be a tree with a finite number $n$ of maximal chains and with properties (K1) and (K2). Let $C_1, \dots, C_n$ be the maximal chains of $X$. Then $C_1, \dots, C_n$, as ordered subsets of $X$, satisfy (K1) and (K2), too. Let $L_1, \dots, L_n$ be the ordered groups corresponding to $C_1, \dots, C_n$ via Lemma 5.6 ($L_i$ is a suitable lexicographic product of copies of $\mathbb{Z}$, whose set of convex subgroups is order-isomorphic to $C_i$).

Then $C_i \cap C_{i+1}$ is a chain in $X$, hence it has an inf in $X$. Such an inf belongs to both $C_i$ and $C_{i+1}$ ($i = 1, \ldots, n-1$). Let $x_i \in C_i \cap C_{i+1}$ be the inf of $C_i \cap C_{i+1}$. Now let $L_i'$ be the convex subgroup of $L_i$ consisting of all elements $f \in L_i$ such that $x \geqslant x_i$ for all $x \in S(f)$. Then $L_i'$ is canonically isomorphic to a convex subgroup of $L_{i+1}$ ($i = 1, \ldots, n-1$). Now let $k$ be any field and let $A_1 = k[\![L_1]\!], \ldots, A_n = k[\![L_n]\!]$, $C_1 = k[\![L_1']\!], \ldots, C_{n-1} = k[\![L_{n-1}']\!]$ be the long power-series rings over $L_1, \ldots, L_n, L_1', \ldots, L_{n-1}'$ respectively. Let $\alpha_i \colon A_i \to C_i, \beta_i \colon A_{i+1} \to C_i$, $i = 1, \ldots, n-1$, be the surjective canonical ring morphisms induced by the embeddings $L_i' \to L_i$, $L_i' \to L_{i+1}$ respectively. Then clearly $R = A_1 \times_{C_1} A_2 \times_{C_2} \ldots \times_{C_{n-1}} A_n$ has the properties required in (iii).

REMARK. Of course with the fiber product it is possible to construct examples of rings with a Morita duality by starting from rings which are neither Noetherian nor valuation, too.

5.8. EXAMPLE. Let $A$ be Vamos' Example (3.3) in [8]. If $C = k$ is a field and $B = k[\![X_1, \ldots, X_n]\!]$, where $X_1, \ldots, X_n$ are indeterminates over $k$, then $A \times_C B$ is a local ring with a Morita duality.

5.9. EXAMPLE. Let $p$ be a prime number, $C$ be the field of $p$ elements, $A$ the ring of $p$-adic integers, $B$ the ring of [8, Example (3.3)].

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