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1. Introduction.

Let $E$ and $F$ denote Banach spaces with scalars in $K$ (C or R) and $\mathcal{L}(E, F)$ the space of bounded linear operators mapping $E$ into $F$. We denote by $\mathcal{F}(E, F)$ and $\mathcal{K}(E, F)$ the subspace of all finite rank operators and the closed subspace of all compact operators, respectively. T. Kato in his treatment of perturbation theory ([5]) introduced the closed subspace of the strictly singular operators that we will denote by $S(E, F)$. We recall that $A : E \to F$ is a strictly singular operator if given any infinite dimensional subspace $M$ of $E$, $A$ restricted to $M$ is not an isomorphism, i.e. a linear homeomorphism. When $E = F$, we denote by $\mathcal{F}(E)$, $\mathcal{K}(E)$, $S(E)$, the ideals $\mathcal{F}(E, E)$, $\mathcal{K}(E, E)$, $S(E, E)$, respectively. We recall that $A \in \mathcal{L}(E) = \mathcal{L}(E, E)$ is said to be a Fredholm operator if the quantities $\alpha(A) = \dim \ker A$, $\beta(A) = \codim \text{rng } A$, are both finite. Each class $J$ of operators which is an ideal and verifies

I) $J \supseteq \mathcal{F}(E),$

II) $I - A$ is a Fredholm operator for each $A \in J,$

is called a $\Phi$-ideal. It is well known that $\mathcal{F}(E)$, $\mathcal{K}(E)$ (see [4]) and $S(E)$ (see [5]) are examples of $\Phi$-ideal. The $\Phi$-ideals play a fun-

damental role in the theory of Riesz operators. The class $\mathcal{R}(E)$ of Riesz operators is defined as follows

$$\mathcal{R}(E) = \{ A \in \mathcal{L}(E) : \lambda I - A \text{ is a Fredholm operator for each } \lambda \neq 0 \}.$$ 

The class $\mathcal{R}(E)$ generally is not an ideal and the Riesz-Schauder theory holds for the spectrum of such operators. We will say that $A \in \mathcal{L}(E, F)$ is relatively regular if there exists $B \in \mathcal{L}(F, E)$ such that $ABA = A$. The operator $B$ is called a generalized inverse of $A$. It is easy to verify that a generalized inverse need not be uniquely determined. In fact if $ABA = A$ the operator $C = BAB$ satisfies the equality $ACA = A$. The concept of relative regularity in the infinite dimensional case has been introduced by F. V. Atkinson ([1]); it plays an essential role in the algebraic theory of Fredholm operators in saturated algebras developed in the monograph ([4]) of H. Heuser.

It is well known that

**Theorem I.** Let $A : E \to F$ a compact operator. $A(E)$ is closed if and only if $A$ is a finite rank operator.

This theorem is not trivial; it is a consequence of Schauder’s Theorem which says that $A \in \mathcal{L}(E, F)$ is compact if and only if the dual operator $A' : F' \to E'$ is also compact. As we will see, for strictly singular operators $A : E \to F$, the equivalence

$$(*) \quad A(E) \text{ closed } \iff A \text{ is a finite rank operator}$$

generally does not hold. Our purpose in this note is to determine conditions such that an operator belonging to $\mathcal{S}(E, F)$, or to any $\Phi$-ideal, or to the class of the Riesz operators, becomes a finite rank operator. In § 2 we will give a sufficient condition for $E$ such that the equivalence $(*)$ is true. In the case $E = F$, we will show that in a $\Phi$-ideal the subset of the relatively regular operators coincides with the ideal $\mathcal{R}(E)$ (§ 3). Moreover it is shown that in a infinite dimensional complex Banach space, a relatively regular Riesz operator having a generalized inverse which commutes with it is again a finite rank operator. I should like to express my gratitude to H. Heuser for several valuable discussions of the topics covered in this paper.
2. Strictly singular operators and superprojective spaces.

For some spaces $E$, studied by R. J. Whitley, the analogue of Theorem 1 is still valid when we replace the world "compact" by "strictly singular". But for an arbitrary Banach space $E$ that is not true, as we show by means of the following example ([2]). Let $E = l_1$ and $F$ any infinite dimensional separable reflexive Banach space. Since $F$ is separable there exists by a Theorem of Banach-Mazur ([2], p. 63, Corollary II.4.5) a bounded operator $A : l_1 \to F$ such that $A(l_1) = F$. Since in $l_1$ the weak convergence is the same as the norm convergence, $A$ is strictly singular ([6], Theorem 1.2) and $A(l_1)$ is a closed infinite dimensional space. With a method due to Phillips unpublished, but referred in [6] applied to the example just considered, it is possible to construct a strictly singular endomorphism which has closed range but is not a finite rank operator. Let $G = l_1 \times F = \{(x, y) : x \in l_1, y \in F\}; G$ is a Banach space with norm $\|(x, y)\| = \max (\|x\|, \|y\|)$. The endomorphism $B : G \to G$ defined by $B(x, y) = (0, Ax)$ is strictly singular ([6]) and $B(G) = \{0\} \times F$ is a closed infinite dimensional subspace of $G$.

Let us recall the concept of subprojective space and superprojective space introduced by R. J. Whitley [6]. A normed linear space $E$ is subprojective if, given any closed infinite dimensional subspace $M$ of $E$, there exists a closed infinite dimensional subspace $N$ contained in $M$ and a continuous projection of $E$ onto $N$.

$E$ is superprojective if, given any closed subspace $M$ with infinite codimension, there exists a closed subspace $N$ containing $M$, where $N$ has infinite codimension and there is a bounded projection of $E$ onto $N$.

The spaces $l_p$, $1 < p < \infty$, are subprojective and superprojective. The spaces $l_1$ and $c_0$ are subprojective but not superprojective. The spaces $L_p(S, \Sigma, \mu)$ in the special case where $S$ is $[0, 1]$, $\Sigma$ is the Lebesgue measurable subsets of $[0, 1]$ and $\mu$ is the Lebesgue measure, are subprojective when $2 < p < \infty$, are superprojective when $1 < p < 2$ ([6]). Each Hilbert space is, of course, a superprojective and a subprojective space.

If $E$ is a reflexive superprojective space we have an analogue of Theorem 1.

**Theorem 2.** Let $E$ be a reflexive and superprojective Banach space, $F$ a Banach space, $A : E \to F$ a strictly singular operator. $A(E)$ is closed if and only if $A$ is finite rank operator.
PROOF. Let $A(E)$ be closed and let $A_0 : E \to A(E)$ be defined by $A_0 x = Ax$ for each $x \in E$. $A_o$ is a bounded surjective operator, hence $A'_0 : A(E)' \to E'$ has a bounded inverse ([4], Proposition 97.1), i.e. $A'_0$ is a linear homeomorphism of $A(E)'$ onto some subspace of $E'$. Since $A_0$ is strictly singular, its conjugate $A'_0$ must also be strictly singular ([6], Corollary 4.7 and Corollary 2.3), and so it follows that $\dim A(E)'$ is finite. Hence also $\dim A(E)$ is finite.

**COROLLARY 1.** Let $E$ be a Hilbert space, $F$ a Banach space, $A : E \to F$ a strictly singular operator. $A(E)$ is closed if and only if $A$ is finite rank operator.

**PROOF.** An Hilbert space is reflexive and superprojective.

**COROLLARY 2.** Let $E$ be a reflexive and subprojective Banach space, $A \in \mathcal{L}(E)$, $A(E)$ closed. Then $A' \in \mathcal{S}(E')$ if and only if $A' \in \mathcal{F}(E')$.

**PROOF.** $A(E)$ being closed, it follows that $A'(E')$ is closed. Since $E$ is subprojective and reflexive its dual space $E'$ must be superprojective ([6], Corollary 4.7).

**COROLLARY 3.** Let $E$ be a reflexive, subprojective and superprojective, Banach space $A \in \mathcal{L}(E)$, $A(E)$ closed. The following conditions are equivalent:

1) $A \in \mathcal{S}(E)$,
2) $A' \in \mathcal{S}(E')$,
3) $A \in \mathcal{F}(E)$,
4) $A' \in \mathcal{F}(E')$.

**PROOF.** I) $\Rightarrow$ II) follows by Corollary 2.3 and Corollary 4.7 of [6]. II) $\Rightarrow$ I) follows by Theorem 2.2 of [6]. I) $\Leftrightarrow$ III) is Theorem 2. II) $\Leftrightarrow$ IV) is Corollary 2.

S. Goldberg and E. Thorp have shown that every bounded linear operator from $l_p$ to $l_q$, $1 < p, q < \infty$, $p \neq q$, is strictly singular ([3], Theorem a) and note). The spaces $l_p$, $p \neq 1$, being reflexive and superprojective, it follows by Theorem 2 that the finite rank operators from $l_p$ to $l_q$, $1 < p, q < \infty$, $p \neq q$, are exactly those which have closed range.
3. Relative regularity and Riesz operators.

We first need the following lemma whose proof may be found in [4] (see p. 125, problem 1 and Theorem 32.1).

**Lemma.** Let $E$ and $F$ be Banach spaces. $A \in \mathcal{L}(E, F)$ is relatively regular if and only if $A(E)$ is closed and there exists a bounded projection of $E$ onto $N(A)$ and a bounded projection of $F$ onto $A(E)$.

**Proposition 1.** Let $A : E \to F$ be a strictly singular operator. If

I) $A(E)$ is closed

II) there exists a bounded projection of $E$ onto $N(A)$

then $A$ is a finite rank operator.

**Proof.** By hypothesis there exists a topological complement of $N(A)$, i.e. $E = N(A) \oplus U$ with $U$ closed. If we define $A_0 u = Au$ for each $u \in U$, it is obvious that $A_0$ maps the Banach space $U$ onto the Banach space $A(E)$, moreover $A_0$ is injective. From the open mapping theorem it follows that $A_0$ is a linear homeomorphism. Since $A$ is strictly singular we must have $\dim U < \infty$ and hence also $\dim A(E) < \infty$. 

If $A \in \mathcal{L}(E, F)$ is relatively regular, the hypotheses I) and II) of Proposition 1 are verified by the Lemma, so the strictly singular operators which are also relatively regular have finite rank. When $E = F$ we may generalize the last proposition to each $\Phi$-ideal.

**Proposition 2.** Let $A$ belong to a $\Phi$-ideal $\mathcal{J}$.

$A$ is relatively regular if and only if $A$ is a finite rank operator.

**Proof.** Let $A$ be relatively regular. Consequently there exists a $B \in \mathcal{L}(E)$ such that $ABA = A$. The operator $P = AB$ is trivially a projection, moreover $A \in \mathcal{J}$ implies $P \in \mathcal{J}$. From the definition of $\Phi$-ideal, $I - P$ is a Fredholm operator, i.e. $\dim N(I - P) = \dim P(E) < \infty$. It follows that $A = PA \in \mathcal{F}(E)$. Viceversa if $A$ is a finite rank operator there exists a bounded projection of $E$ onto $A(E)$ ([4], Proposition 24.2), hence $A$ is relatively regular ([4], p. 131, Problem 3). 

Let $A \in \mathcal{L}(E)$ such that $A^n \in \mathcal{F}(E)$ for some nonnegative integer $n$. $A^n$ being a finite rank operator, there exists a non negative integer $m \geq n$
such that $A^m$ is a relatively regular operator (see [4], p. 132, Problem 5). Conversely if $A^m$ is relatively regular for some nonnegative integer $m$, and $A$ belongs to a $\Phi$-ideal $J$, since $A^m \in J$, by Proposition 2 we have

**Proposition 3.** Let $A \in J$, $J$ a $\Phi$-ideal. $A^n \in \mathcal{F}(E)$ for some nonnegative $n$ if and only if $A^m$ is relatively regular for some $m \geq n$.

Because of Proposition 2 it is natural to ask under which conditions a relatively regular Riesz operator is also a finite rank operator. The following theorem, which may have an independent interest, will permit us to give a sufficient condition in the case of a complex Banach space. We first recall that $A \in \mathcal{L}(E)$ is a Semifredholm operator if $A(E)$ is closed and at least one of the quantities $\alpha(A)$, $\beta(A)$ is finite. The ascent of an operator $A$ is the smallest nonnegative integer $p$, when it exists, such that $N(A^p) = N(A^{p+1})$. The descent of $A$ is the smallest nonnegative integer $q$, when it exists, such that $A^q(E) = A^{q+1}(E)$. If $N(A^n)$ is contained properly in $N(A^{n+1})$ for each integer $n$, we define $p = \infty$: Similarly if $A^n(E)$ contains properly $A^{n+1}(E)$ for each nonnegative integer $n$, we define $q = \infty$. If $p$, $q$ are both finite they coincide ([4], Proposition 38.3) and we will say that « $A$ has finite chains ». A systematic study relating the four quantities $\alpha(A)$, $\beta(A)$, $p$, $q$, is found in [4].

**Theorem 3.** Let $E$ be a complex infinite dimensional Banach space and $A$ a Riesz operator. The descent $q$ of $A$ is finite and $A^q(E)$ is closed if and only if $A$ has finite chains and $A^q$ is a finite rank operator.

**Proof.** Let $M = A^q(E)$. $M$ is a closed invariant subspace under $A$, hence the restriction $A_q$ of $A$ on $M$ is a Riesz operator ([4], Proposition 52.8). The operator $A_q : M \rightarrow M$ is surjective and bounded, hence the conjugate $A'_q : M' \rightarrow M'$ has a bounded inverse, in particular $\alpha(A'_q) = 0$. Moreover $A'_q$ is a Riesz operator since it is the conjugate of a Riesz operator ([4], Proposition 52.7). $A_q(M) = M$ being closed, $A'_q(M')$ is also closed ([4], Proposition 97), hence $A'_q$ is a Semifredholm operator. Let us suppose $\dim M' = \infty$. Then for some complex $\lambda$, $\lambda I - A'_q$ is not a Fredholm operator ([4], Proposition 51.9). But since $A'_q$ is a Riesz operator we must have $\beta(A'_q) = \infty$. Therefore the index of $A'_q = \alpha(A'_q) - \beta(A'_q)$ must be infinite and a stability Theorem due to Kato (see [2], Corollary V.1.7.) implies that the index of $\lambda I - A'_q$ must be infinite in some annulus $0 < |\lambda| < q$, contradicting the fact that $A'_q$ is a Riesz operator. Hence $\dim M' = \dim A^q(E) < \infty$. But $A^q$ is a finite rank operator if and only if 0 is a pole of the resolvent
$R = (\lambda I - A)^{-1}$ of $A$ ([4], p. 230, Problem 2) and this happens if and only if $A$ has finite chains ([4], Proposition 50.2).

REMARK. It is easy to verify that a projection $P$ which is also a Riesz operator is a finite rank operator, in fact $\alpha(I - P) = = \dim P(E) < \infty$. The last theorem, for $q = 2$, shows that this property is, more generally, true for each Riesz operator which has the following properties: $A(E)$ closed, $A^2(E) = A(E)$.

COROLLARY 3. Let $E$ be a complex infinite dimensional Banach space and $A$ a relatively regular Riesz operator. If a generalized inverse $B$ of $A$ commutes with $A$ then $A$ is a finite rank operator.

PROOF. By hypothesis $A(E)$ is closed, since the operator $AB$ is a projection of $E$ onto $A(E)$ it follows

$$A^2(E) = A(A(E)) = A(AB(E)) = ABA(E) = A(E).$$

BIBLIOGRAPHY


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