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1. Introduction.

Let E and F denote Banach spaces with scalars in $\mathbb{K}$ ($\mathbb{C}$ or $\mathbb{R}$) and $\mathcal{L}(E, F)$ the space of bounded linear operators mapping E into F. We denote by $\mathcal{F}(E, F)$ and $\mathcal{K}(E, F)$ the subspace of all finite rank operators and the closed subspace of all compact operators, respectively. T. Kato in his treatment of perturbation theory ([5]) introduced the closed subspace of the *strictly singular* operators that we will denote by $\mathcal{S}(E, F)$. We recall that $A\colon E \to F$ is a strictly singular operator if given any infinite dimensional subspace M of E, A restricted to M is not an isomorphism, *i.e.* a linear homeomorphism. When $E = F$, we denote by $\mathcal{F}(E)$, $\mathcal{K}(E)$, $\mathcal{S}(E)$, the ideals $\mathcal{F}(E, E)$, $\mathcal{K}(E, E)$, $\mathcal{S}(E, E)$, respectively. We recall that $A \in \mathcal{L}(E) = \mathcal{L}(E, E)$ is said to be a *Fredholm operator* if the quantities $\alpha(A) =$ dimension of the null space $N(A)$, $\beta(A) =$ codimension of the range $A(E)$, are both finite. Each class $\mathcal{J}$ of operators which is an ideal and verifies

I) $\mathcal{J} \supseteq \mathcal{F}(E)$,

II) $I - A$ is a Fredholm operator for each $A \in \mathcal{J}$,

is called a Φ-ideal. It is well known that $\mathcal{F}(E)$, $\mathcal{K}(E)$ (see [4]) and $\mathcal{S}(E)$ (see [5]) are examples of *Φ-ideal.* The Φ-ideals play a fun-

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damental role in the theory of *Riesz operators*. The class $\mathfrak{R}(E)$ of Riesz operators is defined as follows

$$\mathfrak{R}(E) = \{A \in \mathfrak{L}(E): \lambda I - A \text{ is a Fredholm operator for each } \lambda \neq 0\} .$$

The class $\mathfrak{R}(E)$ generally is not an ideal and the Riesz-Schauder theory holds for the spectrum of such operators. We will say that $A \in \mathfrak{L}(E, F)$ is *relatively regular* if there exists $B \in \mathfrak{L}(F, E)$ such that $ABA = A$. The operator B is called a *generalized inverse* of A. It is easy to verify that a generalized inverse need not be uniquely determined. In fact if $ABA = A$ the operator $C = BAB$ satisfies the equality $ACA = A$. The concept of relative regularity in the infinite dimensional case has been introduced by F. V. Atkinson ([1]); it plays an essential role in the algebraic theory of Fredholm operators in satured algebras developed in the monograph ([4]) of H. Heuser.

It is well known that

THEOREM I. *Let A: $E \to F$ a compact operator. $A(E)$ is closed if and only if A is a finite rank operator.*

This theorem is not trivial; it is a consequence of Schauder's Theorem which says that $A \in \mathfrak{L}(E, F)$ is compact if and only if the dual operator $A': F' \to E'$ is also compact. As we will see, for strictly singular operators $A: E \to F$, the equivalence

$$(*) \qquad A(E) \text{ closed} \Leftrightarrow A \text{ is a finite rank operator}$$

generally does not hold. Our purpose in this note is to determine conditions such that an operator belonging to $\mathcal{S}(E, F)$, or to any Φ-ideal, or to the class of the Riesz operators, becomes a finite rank operator. In § 2 we will give a sufficient condition for E such that the equivalence $(*)$ is true. In the case $E = F$, we will show that in a Φ-ideal the subset of the relatively regular operators coincides with the ideal $\mathcal{F}(E)$ (§ 3). Moreover it is shown that in a infinite dimensional *complex* Banach space, a relatively regular Riesz operator having a generalized inverse which commutes with it is again a finite rank operator. I should like to express my gratitude to H. Heuser for several valuable discussions of the topics covered in this paper.

2. Strictly singular operators and superprojective spaces.

For some spaces E, studied by R. J. Whitley, the analogue of Theorem 1 is still valid when we replace the world « compact » by « strictly singular ». But for an arbitrary Banach space E that is not true, as we show by means of the following example ([2]). Let $E = l_1$ and F any infinite dimensional separable reflexive Banach space. Since F is separable there exists by a Theorem of Banach-Mazur ([2], p. 63, Corollary II.4.5) a bounded operator $A: l_1 \to F$ such that $A(l_1) = F$. Since in l_1 the weak convergence is the same as the norm convergence, A is strictly singular ([6], Theorem 1.2) and $A(l_1)$ is a closed infinite dimensional space. With a method due to Phillips unplublished, but referred in [6] applied to the example just considered, it is possible to construct a strictly singular endomorphism which has closed range but is not a finite rank operator. Let $G = l_1 \times F = \{(x, y): x \in l_1, y \in F\}$; G is a Banach space with norm $\|(x, y)\| = \max(\|x\|, \|y\|)$. The endomorphism $B: G \to G$ defined by $B(x, y) = (0, Ax)$ is strictly singular ([6]) and $B(G) = \{0\} \times F$ is a closed infinite dimensional subspace of G.

Let us recall the concept of subprojective space and superprojective space introduced by R. J. Whitley [6]. A normed linear space E is *subprojective* if, given any closed infinite dimensional subspace M of E, there exists a closed infinite dimensional subspace N contained in M and a continuous projection of E onto N.

E is *superprojective* if, given any closed subspace M with infinite codimension, there exists a closed subspace N containing M, where N has infinite codimension and there is a bounded projection of E onto N.

The spaces l^p, $1 < p < \infty$, are subprojective and superprojective. The spaces l_1 and c_0 are subprojective but not superprojective. The spaces $L_p(S, \Sigma, \mu)$ in the special case where S is $[0, 1]$, Σ is the Lebesgue measurable subsets of $[0, 1]$ and μ is the Lebesgue measure, are subprojective when $2 \leqslant p < \infty$, are superprojective when $1 < p \leqslant 2$ ([6]). Each Hilbert space is, of course, a superprojective and a subprojective space.

If E is a reflexive superprojective space we have an analogue of Theorem 1.

THEOREM 2. *Let E be a reflexive and superprojective Banach space, F a Banach space, $A: E \to F$ a strictly singular operator. $A(E)$ is closed if and only if A is finite rank operator.*

PROOF. Let $A(E)$ be closed and let $A_0: E \to A(E)$ be defined by $A_0 x = Ax$ for each $x \in E$. A_0 is a bounded surjective operator, hence $A_0': A(E)' \to E'$ has a bounded inverse ([4], Proposition 97.1), *i.e.* A_0' is a linear homeomorphism of $A(E)'$ onto some subspace of E'. Since A_0 is strictly singular, its conjugate A_0' must also be strictly singular ([6], Corollary 4.7 and Corollary 2.3), and so it follows that $\dim A(E)'$ is finite. Hence also $\dim A(E)$ is finite ■

COROLLARY 1. *Let E be a Hilbert space, F a Banach space, $A: E \to F$ a strictly singular operator. $A(E)$ is closed if and only if A is finite rank operator.*

PROOF. An Hilbert space is reflexive and superprojective ■

COROLLARY 2. *Let E be a reflexive and subprojective Banach space, $A \in \mathcal{L}(E)$, $A(E)$ closed. Then $A' \in \mathcal{S}(E')$ if and only if $A' \in \mathcal{F}(E')$.*

PROOF. $A(E)$ being closed, it follows that $A'(E')$ is closed. Since E is subprojective and reflexive its dual space E' must be superprojective ([6], Corollary 4.7). ■

COROLLARY 3. *Let E be a reflexive, subprojective and superprojective, Banach space $A \in \mathcal{L}(E)$, $A(E)$ closed. The following conditions are equivalent*:

I) $A \in \mathcal{S}(E)$,

II) $A' \in \mathcal{S}(E')$,

III) $A \in \mathcal{F}(E)$,

IV) $A' \in \mathcal{F}(E')$.

PROOF. I) $\Rightarrow$ II) follows by Corollary 2.3 and Corollary 4.7 of [6]. II) $\Rightarrow$ I) follows by Theorem 2.2 of [6]. I) $\Leftrightarrow$ III) is Theorem 2. II) $\Leftrightarrow$ IV) is Corollary 2. ■

S. Goldberg and E. Thorp have shown that every bounded linear operator from l_p to l_q, $1 < p$, $q < \infty$, $p \neq q$, is strictly singular ([3], Theorem *a*) and note). The spaces l_p, $p \neq 1$, being reflexive and superprojective, it follows by Theorem 2 that the finite rank operators from l_p to l_q, $1 < p$, $q < \infty$, $p \neq q$, are exactly those which have closed range.

3. Relative regularity and Riesz operators.

We first need the following lemma whose proof may be found in [4] (see p. 125, problem 1 and Theorem 32.1).

LEMMA. *Let E and F be Banach spaces. $A \in \mathfrak{L}(E, F)$ is relatively regular if and only if $A(E)$ is closed and there exists a bounded projection of E onto $N(A)$ and a bounded projection of F onto $A(E)$.*

PROPOSITION 1. *Let $A\colon E \to F$ be a strictly singular operator. If*

I) *$A(E)$ is closed*

II) *there exists a bounded projection of E onto $N(A)$*
then A is a finite rank operator.

PROOF. By hypothesis there exists a topological complement of $N(A)$, *i.e.* $E = N(A) \oplus U$ with U closed. If we define $A_0 u = Au$ for each $u \in U$, it is obvious that A_0 maps the Banach space U onto the Banach space $A(E)$, moreover A_0 is injective. From the open mapping Theorem it follows that A_0 is a linear homeomorphism. Since A is strictly singular we must have $\dim U < \infty$ and hence also $\dim A(E) < \infty$. ■

If $A \in \mathfrak{L}(E, F)$ is relatively regular, the hyphoteses I) and II) of Proposition 1 are verified by the Lemma, so the strictly singular operators which are also relatively regular have finite rank. When $E = F$ we may generalize the last proposition to each Φ-ideal.

PROPOSITION 2. *Let A belong to a Φ-ideal $\mathfrak{J}$.*
A is relatively regular if and only if A is a finite rank operator.

PROOF. Let A be relatively regular. Consequently there exists a $B \in \mathfrak{L}(E)$ such that $ABA = A$. The operator $P = AB$ is trivially a projection, moreover $A \in \mathfrak{J}$ implies $P \in \mathfrak{J}$. From the definition of Φ-ideal, $I - P$ is a Fredholm operator, *i.e.* $\dim N(I - P) = \dim P(E) < \infty$. It follows that $A = PA \in \mathcal{F}(E)$. Viceversa if A is a finite rank operator there exists a bounded projection of E onto $A(E)$ ([4], Proposition 24.2), hence A is relatively regular ([4], p. 131, Problem 3). ■

Let $A \in \mathfrak{L}(E)$ such that $A^n \in \mathcal{F}(E)$ for some nonnegative integer n. A^n being a finite rank operator, there exists a non negative integer $m \geqslant n$

such that A^m is a relatively regular operator (see [4], p. 132, Problem 5). Conversely if A^m is relatively regular for some nonnegative integer m, and A belongs to a Φ-ideal $\mathfrak{J}$, since $A^m \in \mathfrak{J}$, by Proposition 2 we have

PROPOSITION 3. *Let $A \in \mathfrak{J}$, $\mathfrak{J}$ a Φ-ideal. $A^n \in \mathcal{F}(E)$ for some nonnegative n if and only if A^m is relatively regular for some $m \geqslant n$.*

Because of Proposition 2 it is natural to ask under which conditions a relatively regular Riesz operator is also a finite rank operator. The following theorem, which may have an independent interest, will permit us to give a sufficient condition in the case of a complex Banach space. We first recall that $A \in \mathfrak{L}(E)$ is a *Semifredholm* operator if $A(E)$ is closed and at least one of the quantities $\alpha(A)$, $\beta(A)$ is finite. The *ascent* of an operator A is the smallest nonnegative integer p, when it exists, such that $N(A^p) = N(A^{p+1})$. The *descent* of A is the smallest nonnegative integer q, when it exists, such that $A^q(E) = A^{q+1}(E)$. If $N(A^n)$ is contained properly in $N(A^{n+1})$ for each integer n, we define $p = \infty$: Similarly if $A^n(E)$ contains properly $A^{n+1}(E)$ for each nonnegative integer n, we define $q = \infty$. If p, q are both finite they coincide ([4], Proposition 38.3) and we will say that « A has finite chains ». A systematic study relating the four quantities $\alpha(A)$, $\beta(A)$, p, q, is found in [4].

THEOREM 3. *Let E be a* complex *infinite dimensional Banach space and A a Riesz operator. The descent q of A is finite and $A^q(E)$ is closed if and only if A has finite chains and A^q is a finite rank operator.*

PROOF. Let $M = A^q(E)$. M is a closed invariant subspace under A, hence the restriction A_q of A on M is a Riesz operator ([4], Proposition 52.8). The operator $A_q\colon M \to M$ is surjective and bounded, hence the conjugate $A_q'\colon M' \to M'$ has a bounded inverse, in particular $\alpha(A_q') = 0$. Moreover A_q' is a Riesz operator since it is the conjugate of a Riesz operator ([4], Proposition 52.7). $A_q(M) = M$ being closed, $A_q'(M')$ is also closed ([4], Proposition 97), hence A_q' is a Semifredholm operator. Let us suppose $\dim M' = \infty$. Then for some complex λ, $\lambda I' - A_q'$ is not a Fredholm operator ([4], Proposition 51.9). But since A_q' is a Riesz operator we must have $\beta(A_q') = \infty$. Therefore the index of $A_q' = \alpha(A_q') - \beta(A_q')$ must be infinite and a stability Theorem due to Kato (see [2], Corollary V.1.7.) implies that the index of $\lambda I' - A_q'$ must be infinite in some annulus $0 < |\lambda| < \varrho$, contradicting the fact that A_q' is a Riesz operator. Hence $\dim M' = \dim A^q(E) < \infty$. But A^q is a finite rank operator if and only if 0 is a pole of the resolvent

$R = (\lambda I - A)^{-1}$ of A ([4], p. 230, Problem 2) and this happens if and only if A has finite chains ([4], Proposition 50.2). ∎

REMARK. It is easy to verify that a projection P which is also a Riesz operator is a finite rank operator, in fact $\alpha(I - P) = \dim P(E) < \infty$. The last theorem, for $q = 2$, shows that this property is, more generally, true for each Riesz operator which has the following properties: $A(E)$ closed, $A^2(E) = A(E)$.

COROLLARY 3. *Let E be a* complex *infinite dimensional Banach space and A a relatively regular Riesz operator. If a generalized inverse B of A commutes with A then A is a finite rank operator.*

PROOF. By hypothesis $A(E)$ is closed, since the operator AB is a projection of E onto $A(E)$ it follows

$$A^2(E) = A(A(E)) = A(AB(E)) = ABA(E) = A(E) . \quad \blacksquare$$

BIBLIOGRAPHY

[1] F. V. ATKINSON, *On relativity regular operators*, Acta Sci. Math. Szeged, **15** (1953), pp. 38-56.

[2] S. GOLDBERG, *Unbounded linear operators with applications*, McGraw-Hill, New York, 1966.

[3] S. GOLDBERG - E. THORP, *On some open questions concerning strictly singular operators*, Proc. Amer. Math. Soc., **14** (1963), pp. 334-336.

[4] H. HEUSER, *Funktionalanalysis*, Stuttgart, 1975.

[5] T. KATO, *Perturbation theory for nullity, deficiency, and other quantities of linear operators*, J. Analyse Math., **6** (1958), pp. 261-322.

[6] R. J. WHITLEY, *Strictly singular operators and their conjugates*, Trans. Am. Math. Soc., **113** (1964), pp. 252-261.

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