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On the Connection between the Real and the Complex Interpolation Method for Several Banach Spaces.

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SUMMARY - The objective of this paper is to exhibit some connections between the real and the complex interpolation method for 2^n Banach spaces. A version of the Lions-Peetre interpolation method for 2^n Banach spaces and some properties of the complex method involving multiple Poisson integrals are presented. Applications to spaces with a dominant mixed derivatives are given.

Introduction.

The study of the intepolation spaces of several Banach spaces by real methods has been made by Yoshikawa [15], Sparr [14] and Fernandez [4], and by complex method by Lions [7], Favini [3] and Fernandez [5].

The aim of this paper is to exhibit some connections between the real and the complex interpolation methods for several Banach spaces and to give applications to the spaces with a dominant mixed derivative.

First we give a version of a real interpolation method among 2^n

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Banach spaces along the lines of Lions-Peetre [8]. These spaces thus constructed are similar to some studied by one of the authors in [4] and [6]. We recall the definition of the complex method for 2^n Banach spaces and then give some new properties of these spaces along the lines of Calderón [2] and Peetre [13]. Using the Hausdorff-Young theorem for L^P spaces with mixed norms, as given by Benedek-Panzone in [1], and borrowing some ideas from Peetre [13] we give a connection between the real and the complex interpolation space among 2^n Banach spaces. As a by—product we show that for Hilbert spaces the two interpolation spaces coincide. Our development is carried out in the context of the L^P spaces with mixed norms of Benedek-Panzone [1].

As an application of the theory we give some relationships between the Lipschitz spaces of Nikol'skii [11] and of potential spaces of Lizorkin-Nikol'skii [10].

1. Generalities on interpolation for 2^n Banach spaces.

Let us denote by $\square$ the set of $k = (k_1, \ldots, k_n) \in \mathbb{R}^n$ such that $k_j = 0$ or 1. We have $\square = \{0, 1\}$ when $n = 1$, and $\square = \{(0, 0), (1, 0), (0,1), (1, 1)\}$ when $n = 2$. The families of objects we shall consider will take indices in $\square$.

We shall consider families of 2^n Banach spaces $(E_k | k \in \square)$ embedded in one and a same linear Hausdorff space V. Such a family will be called an admissible family (of Banach spaces (in V)).

If $\mathbb{E} = (E_k | k \in \square)$ is an admissible family of Banach spaces, the linear hull $\sum \mathbb{E}$ and the intersection $\bigcap \mathbb{E}$ can be introduced in the usual way. They are Banach spaces under the norms

$$\|x\|_{\Sigma \mathbb{E}} = \inf\left\{\sum_k \|x_k\|_{E_k} | x = \sum_k x_k;\ x_k \in E_k,\ k \in \square\right\}$$

and

$$\|x\|_{\cap \mathbb{E}} = \max\{\|x\|_{E_k} | k \in \square\}\,.$$

The spaces $\sum \mathbb{E}$ and $\bigcap \mathbb{E}$ are continuously embedded in V.

A Banach space E which satisfy

$$\bigcap \mathbb{E} \subset E \subset \sum \mathbb{E}$$

will be called an *intermediate space* (with respect to $\mathbb{E}$).

(Hereafter $\subset$ will denote continuous embeddings).

A pair of Banach spaces (E, F), intermediate respect to the admissible families $\mathbb{E} = (E_k | k \in \square)$ (in V) and $\mathbb{F} = (F_k | k \in \square)$ (in W) respectively, has the *interpolation property* if for every linear mapping from $\sum \mathbb{E}$ into $\sum \mathbb{F}$ such that

$$T|E_k\colon E_k \to F_k \qquad (k \in \square)$$

it follows that

$$T|E\colon E \to F$$

(we agree that $\to$ will denote *bounded* linear mappings).

REMARK. Observe that $T\colon E_k \to F_k$, $k \in \square$, implies $T\colon \sum \mathbb{E} \to \sum \mathbb{F}$.

2. The real interpolation spaces $(\mathbb{E}; \Theta; P)$.

Let $\mathbb{E} = (E_k | k \in \square)$ be an admissible family of Banach spaces. For $t = (t_1, \ldots, t_n) > 0$ and $k = (k_1, \ldots, k_n) \in \square$ we set $t^k = t_1^{k_1} \ldots t_n^{k_n}$. For $y \in \bigcap \mathbb{E}$ we define

$$J(t; y) = \max_k t^k \|y\|_{E_k} .$$

Observe that $J(1, \ldots, 1; y) = \|y\|_{\cap \mathbb{E}}$ and for each t fixed, $J(t; y)$ is a *functional norm in* $\bigcap \mathbb{E}$.

Now, let us denote by L_*^P the L^P space, on $\mathbb{R}_+^n = \mathbb{R}_+ \times \ldots \times \mathbb{R}_+$, with respect to the Haar measure $d^*t = dt_1/t_1 \ldots dt_n/t_n$. If F is a Banach space, $L_*^P(F)$ is the L_*^P space of the strongly measurable functions $g\colon \mathbb{R}_+^n \to F$ such that $\|g(t)\|_F \in L_*^P$. For notations and results on L^P spaces, with mixed norms see [1].

With the above notation we have the following proposition.

PROPOSITION 2.1. *Assume that* $0 < \Theta = (\theta_1, \ldots, \theta_n) < 1$ *and* $1 \leqslant P = (p_1, \ldots, p_n) \leqslant \infty$. *If* $u\colon \mathbb{R}_+^n \to \bigcap \mathbb{E}$ *is a function such that* $\|u(t)\|_{E_k} \in L_*^1$, $k \in \square$, *then the following conditions are equivalent*

2.1(1) $$t^{-\Theta} J(t; u(t)) \in L_*^P$$

and

2.1(2) $$t^{k-\Theta} u(t) \in L_*^P(E_k)\,, \qquad k \in \square\,.$$

PROOF. Indeed, the following inequalities hold:

$$t^{k-\Theta}\|u(t)\|_{E_k} \leqslant \max_k t^{k-\Theta}\|u(t)\|_{E_k} \leqslant t^{-\Theta}J(t;\, u(t)) ,$$

and

$$t^{-\Theta}J(t;\, u(t)) \leqslant \max_k t^{k-\Theta}\|u(t)\|_{E_k} .$$

Now, we introduce the spaces $(\mathbf{E};\Theta;P)$.

DEFINITION 2.2. *We define* $(\mathbf{E};\Theta;P)$ *to be the space of all elements* $x \in \sum \mathbf{E}$ *for which there exists a function* $u\colon \mathbf{R}^n_+ \to \bigcap \mathbf{E}$, *with* $\|u(t)\|_{\Sigma\mathbf{E}} \in L^1_*$, *which satisfy* 2.1(1) *or* 2.1(2) *and such that*

2.2(1) $$x = \int_{\mathbf{R}^n_+} u(t)\, d^*t \qquad (\textit{in } \textstyle\sum \mathbf{E}) .$$

PROPOSITION 2.3. *The space* $(\mathbf{E};\Theta;P)$ *is an intermediate Banach space under any one of the following equivalent norms*

2.3(1) $$\|x\|_{\Theta;P} = \inf \|t^{-\Theta}J(t;\, u(t))\|_{L^P_*} ,$$

2.3(2) $$\|x\|_{\Theta;P} = \inf \max \|t^{k-\Theta}u(t)\|_{L^P_*(E_k)} ,$$

where the infimum is taken on all u *which satisfy* 2.2(1).

It will be convenient to work also with an interpolation space slightly more general than the spaces $(\mathbf{E};\Theta;P)$ just introduced.

DEFINITION 2.4. *Let* $\mathbf{E} = (E_k | k \in \square)$ *be an admissible family of Banach spaces. Given* $1 \leqslant P_0 = (p_0^1, \dots, p_0^n)$, $P_1 = (p_1^1, \dots, p_1^n) \leqslant \infty$, *let us set* $\mathbf{P} = (P_k = (p^1_{k_1}, \dots, p^n_{k_n}) | k = (k_1, \dots, k_n) \in \square)$. *Now, if* $0 < \Theta = (\theta_1, \dots, \theta_n) < 1$, *we define* $(\mathbf{E};\Theta;\mathbf{P})$ *to be the space of all* $x \in \sum \mathbf{E}$ *for which there is a function* $u\colon \mathbf{R}^n_+ \to \bigcap \mathbf{E}$ *with* $u \in L^1_*(E_k)$, $k \in \square$, *such that* 2.2(1) *holds and the following conditions are satisfied*

2.4(1) $$t^{k-\Theta}u(t) \in L^{P_k}_*(E_k) , \qquad k \in \square .$$

We equip $(\mathbf{E};\Theta;\mathbf{P})$ *with the norm*

2.4(2) $$\|x\|_{\Theta;\mathbf{P}} = \inf_u \max_{k\in\square} \|t^{k-\Theta}u(t)\|_{L^{P_k}_*(E_k)} .$$

We see at once that the following proposition holds.

PROPOSITION 2.5. *If* $\mathbf{P} = (P_k = P | k \in \square)$ *it follows that*

2.5(1) $$(\mathbf{E}; \Theta; P) = (\mathbf{E}; \Theta; \mathbf{P}) .$$

The spaces $(\mathbf{E}; \Theta; P)$ and $(\mathbf{E}; \Theta; \mathbf{P})$ have the so called interpolation property. For the proof and further properties of these spaces see Fernandez [4] and [6].

3. The complex interpolation spaces $[\mathbf{E}; \Theta]$.

We shall recall briefly the notion of the complex method of interpolation for 2^n Banach spaces. For the proofs see Fernandez [5].

Let $\mathbf{E} = (E_k | k \in \square)$ be an admissible family of Banach spaces.

3.1. The spaces of all $\sum \mathbf{E}$-valued functions $f(z)$ defined, continuous and bounded on the n-strip S^n (product of n unit strips)

$$S = \{z = s + it | 0 \leqslant s \leqslant 1, \ t \in \mathbf{R}\}$$

which are holomorphic on the interior of S^n, with respect to the norm of $\sum \mathbf{E}$, and such that $f(k + it) \in E_k$ and are E_k-continuous and bounded for all $k \in \square$ will be denoted by $H(\mathbf{E})$.

The space $H(\mathbf{E})$ endowed with the norm

3.1(1) $$\|f\|_{H(\mathbf{E})} = \max \{\|f(k + it)\|_{E_k} | k \in \square\}$$

becomes a Banach space.

3.2. For $0 < \Theta = (\theta_1, \ldots, \theta_n) < 1$, we set

3.2(1) $$[\mathbf{E}; \Theta] = \{x \in \sum \mathbf{E} | \exists f \in H(\mathbf{E}), \ f(\Theta) = x\} .$$

This spaces is a intermediate Banach space under the norm

3.2(2) $$\|x\|_{[\mathbf{E}; \Theta]} = \inf \{\|f\|_{H(\mathbf{E})} | f(\Theta) = x\} .$$

Also, the spaces $[\mathbf{E}; \Theta]$ have the interpolation property.

EXEMPLE 1. Let $P_0 = (P_0^1, \ldots, P_0^n)$ and $P_1 = (P_1^1, \ldots, P_1^n)$ be given with $1 \leqslant P_0$, $P_1 \leqslant \infty$. Consider $(P_k)_{k \in \square}$ the sequence of admissible powers associated with P_0 and P_1 ($P_k = (P_{k_1}^1, \ldots, P_{kj}^j, \ldots, P_{k_n}^n)$ where $k_j = 0$ or 1), and set $L^{P_k} = L^{P_k}(X, \mu)$, $k \in \square$. Then

$$[(L^{P_k})_{k \in \square}; \Theta] = L^P$$

where $1/P = (1-\Theta)/P_0 + \Theta/P_1$.

EXEMPLE 2. For $S = (s_1, \ldots, s_n) \in \mathbb{R}^n$, let $H^{S,P}(\mathbb{R}^n)$ be the space of $u \in S'(\mathbb{R}^n)$ such that

$$\|u\|_{H^{S,P}} = \|\mathcal{F}^* \prod_j (1 + |x_j|^2)^{s_j/2} \mathcal{F} u\|_{L^P} < \infty .$$

If $(S_k | k \in \square)$ is a family of admissible parameters associated with $S_0 = (s_0^1, \ldots, s_0^n)$ and $S_1 = (s_1^1, \ldots, s_1^n)$, that is $S_k = (s_{k_1}^1, \ldots, s_{kj}^j, \ldots, s_{kn}^n)$ where $k_j = 0$ or 1, and $(P_k | k \in \square)$ is a family of admissible powers associated with P_0 and P_1, where $1 \leqslant P_0$, $P_0 \leqslant \infty$. We have

$$[(H^{S_k, P_k}(\mathbb{R}^n)_{k \in \square}; \Theta)] = H^{S,P}(\mathbb{R}^n)$$

where $S = (1-\Theta) S_0 + \Theta S_1$ and $1/P = (1-\Theta)/P_0 + \Theta P_1$. For the proof and details see [5].

4. A characterization of $[E; \Theta]$ involving the Poisson kernel.

4.1. The Poisson kernels for the unit strip S will be denoted by $P_0(s, y)$ and $P_1(s, y)$. They can be obtained from the Poisson kernel for the half-plane by mapping conformally the half plane onto the strip. Explicitly these kernels are

4.1(1) $$P_0(s, y) = \frac{1}{2} \frac{\operatorname{sen} \pi s}{\cos h\pi y - \cos \pi s},$$

4.1(2) $$P_1(s, y) = \frac{1}{2} \frac{\operatorname{sen} \pi s}{\cos h\pi y + \cos \pi s}.$$

4.2. For $k = (k_1, \ldots, k_n) \in \square$, let us set the k-Poisson kernel for the poly-strip S^n:

4.2(1) $$P_k(S, y) = \prod_{j=1}^{n} P_{k_j}(s_j, y_j)$$

here $S = (s_1, \ldots, s_n)$ with $0 \leqslant s_j \leqslant 1$ and $y_1 = (y_1, \ldots, y_n) \in \mathbf{R}^n$.

PROPOSITION 4.3. *For all $f \in H(\mathbb{E})$ and $S = (s_1, \ldots, s_n)$ with $0 < S < 1$ we have*

4.3(1) $$\log \|f(S)\|_{[\mathbf{E}; S]} \leqslant \sum_{k \in \square} \int_{\mathbf{R}^n} \log \|f(k + it)\|_{E_k} P_k(S, t)\, dt\,.$$

PROOF. Let g_k be a bounded infinitely differentiable function such that

$$g_k(t) \geqslant \log \|f(k + it)\|_{E_k}\,.$$

Let F be an analytic function such that

$$\operatorname{Re}\{F(z)\} = \sum_{k \in \square} \int_{\mathbf{R}^n} g_k(t) P_k(z, t)\, dt\,.$$

Such a function exists and $\operatorname{Re}\{F(k + it)\} = g_k(t)$. Furthermore, the differentiability of g_k implies that $F(z)$ is continuous in $0 \leqslant S \leqslant 1$. Consequently

$$\exp\{-F(z)\} f(z) \in H(\mathbb{E})$$

and since

$$\|\exp\{-F(k + it)\} f(k + it)\|_{E_k} \leqslant \exp\{-g_k(t)\} \|f(k + it)\|_{E_k} \leqslant 1\,,$$

it follows that

$$\|\exp\{-F(Z)\} f(Z)\|_{H(\mathbf{E})} \leqslant 1\,.$$

Consequently

$$\|\exp\{-F(S)\} f(S)\|_{[\mathbf{E}; S]} \leqslant 1$$

and

$$\|f(S)\|_{[\mathbf{E}; S]} \leqslant \exp F(S)\,.$$

Hence

$$\log \|f(S)\|_{[\mathbf{E}; S]} \leqslant \operatorname{Re} F(S) \leqslant \sum_{k \in \square} \int_{\mathbf{R}^n} g_k(t) P_k(S, t)\, dt\,.$$

Takink now a decreasing sequence of functions $g_{k,\nu}$ converging to $\log\|f(k+it)\|_{E_k}$, $k\in\square$, respectively, and passing to the limit we obtain the result.

COROLLARY 4.4. *For* $f\in H(\mathrm{E})$ *and* $0<S=(s_1,..,s_n)<1$, *we have*

$$\|f(S)\|_{[\mathbf{E};S]}\leqslant\prod_{k\in\square}\left\{\frac{1}{s(k)}\int_{\mathbf{R}^n}\|f(k+it)\|_{E_k}P_k(S,t)\,dt\right\}^{s(k)} \tag{4.4(1)}$$

where $s(k)=\prod_j\{1-k_j+(-1)^{k_j+1}s_j\}=\prod_j s(k_j)$, *and*

$$\|f(S)\|_{[\mathbf{E};S]}\leqslant\sum_{k\in\square}\int_{\mathbf{R}^n}\|f(k+it)\|_{E_k}P_k(S,t)\,dt\,. \tag{4.4(2)}$$

PROOF. We observe that

$$\int_{\mathbf{R}^n}P_k(S,t)\,dt=s(k)\qquad(k\in\square)\,.$$

From this and from Jensen's inequality it follows that

$$s(k)\exp\left\{\frac{1}{s(k)}\int_{\mathbf{R}^n}\log\|f(k+it)\|_{E_k}P_k(S,t)\,dt\right\}\leqslant\int_{\mathbf{R}^n}\|f(k+it)\|_{E_k}P_k(S,t)\,dt\,,$$

for all $k\in\square$. Now, from 4.3(1) and these inequalities, we obtain

$$\begin{aligned}\|f(S)\|_{[\mathbf{E};S]}&\leqslant\exp\left\{\sum_{k\in\square}\int_{\mathbf{R}^n}\log\|f(k+it)\|_{E_k}P_k(S,t)\,dt\right\}=\\&=\prod_{k\in\square}\exp\left\{s(k)\frac{1}{s(k)}\int_{\mathbf{R}^n}\log\|f(k+it)\|_{E_k}P_k(S,t)\,dt\right\}\leqslant\\&\leqslant\prod_{k\in\square}\left(\exp\left\{\frac{1}{s(k)}\int_{\mathbf{R}^n}\log\|f(k+it)\|_{E_k}P_k(S,t)\,dt\right\}\right)^{s(k)}\leqslant\\&\leqslant\prod_{k\in\square}\left\{\frac{1}{s(k)}\int_{\mathbf{R}^n}\|f(k+it)\|_{E_k}P_k(S,t)\,dt\right\}^{s(k)},\end{aligned}$$

this gives 4.4(1).

It remains to prove 4.4(2). The following inequality holds

$$\exp\left\{\sum_{k\in\square} a_k\right\} \leqslant \sum_{k\in\square} s(k)\exp\{a_k/s(k)\}\,.$$

This follows by induction from the well known case $n = 1$:

$$e^{a_0+a_1} \leqslant (1-s)\,e^{a_0/(1-s)} + s e^{a_1/s}\,.$$

Let us set

$$a_k = \int_{\mathbf{R}^n} \log\|f(k+it)\|_{E_k} P_k(S, t)\,dt\,,$$

in the above inequality. Again, by 4.3(1), and the above inequality we get 4.4(2):

$$\|f(S)\|_{[\mathbf{E};S]} \leqslant \exp\left\{\sum_{k\in\square} \int_{\mathbf{R}^n} \log\|f(k+it)\|_{E_k} P_k(S, t)\,dt\right\} \leqslant$$

$$\leqslant \sum_{k\in\square} s(k)\exp\left\{\frac{1}{s(k)}\int_{\mathbf{R}^n} \log\|f(k+it)\|_{E_k} P_k(S, t)\,dt\right\} \leqslant$$

$$\leqslant \sum_{k\in\square} \int_{\mathbf{R}^n} \|f(k+it)\|_{E_k} P_k(S, t)\,dt\,.$$

PROPOSITION 4.5. *Let $a \in [\mathbf{E}; \Theta]$ and $f \in H(E)$ be such that $f(\Theta) = a$. Suppose that*

4.5(1) $$f(k+it) \in L^{Q_k}(E_k) \qquad (k \in \square)$$

where $(Q_k|k \in \square)$ is a family of admissible parameters associated to Q_0 and Q_1 and such that $1 < Q_k \leqslant \infty$. Then

4.5(2) $$\|a\|_\Theta \leqslant C \max_k \|f(k+it)\|_{L^{Q_k}(E_k)}\,.$$

PROOF. Due to corollary 4.4 we have

$$\|a\|_{[\mathbf{E};\Theta]} = \|f(\Theta)\|_{[\mathbf{E};\Theta]} \leqslant \sum_{k\in\square} \int_{\mathbf{R}^n} \|f(k+it)\|_{E_k} P_k(\Theta; t)\,dt \leqslant$$

$$\leqslant \sum_{k\in\square} \|f(k+it)\|_{L^{Q_k}(E_k)} \|P_k(\Theta; t\|_{L^{Q_k'}} \leqslant$$

$$\leqslant \left(\sum_{k\in\square} \|P_k(\Theta; t)\|_{L^{Q_k'}(E_k)}\right) \max_{k\in\square} \|f(k+it)\|_{L^{Q_k}(E_k)}\,.$$

PROPOSITION 4.6 *Let f be a continuous and bounded $\sum \mathbb{E}$-valued function on the polystrip S^n which is analytic on the interior of S^n and such that*

5.6(1) $$f(k+it) \in L^{Q_k}(E_k) \qquad (k \in \square)$$

where $(Q_k | k \in \square)$ are admissible parameters, with $1 \leqslant Q_k \leqslant \infty$, and associated to $Q_0 = (q_0^1, \ldots, q_0^n)$ and $Q_1 = (q_1^1, \ldots, q_1^n)$. Then, if $f(\Theta) = a$ it follows that $a \in [\mathbb{E}; \Theta]$.

PROOF. The assertion will be done if we show that there is a Cauchy sequence (a_j) in $[\mathbb{E}, \Theta]$ such that $a_j \to a$ in $\sum \mathbb{E}$, as $j \to \infty$.

Let (φ_j) be a sequence of non-negative continuous functions on $\mathbb{R}^n$ such that

(i) $\displaystyle\int_{\mathbb{R}^n} \varphi_j(t)\, dt = 1;$

(ii) $\varphi_j(t) = 0$, if $|t| \geqslant 1/j$.

Now, let f be given as in the hypothesis, and let us set

$$f_j(z) = \int_{\mathbb{R}^n} f(k+it)\, \varphi_j(t)\, dt$$

and

$$a_j = f_j(\Theta) = \int_{\mathbb{R}^n} f(\Theta + it)\, \varphi_j(t)\, dt\,.$$

We shall show that $f_j \in H(\mathbb{E})$ and $a_j \in [\mathbb{E}; \Theta]$. First, we observe that f_j is $\sum \mathbb{E}$-holomorphic in S^n. Also, it is bounded:

$$\|f_j(z)\|_{\Sigma \mathbb{E}} \leqslant \int_{\mathbb{R}^n} \|f(z+it)\|_{\Sigma \mathbb{E}} \varphi_j(t)\, dt \leqslant \sup_{\omega \in S^n} \|f(\omega)\|_{\Sigma \mathbb{E}}\,.$$

From Minkowski's inequality, we have

$$f_j(k+it) \in L^{Q_k}(E_k)\,, \qquad k \in \square\,.$$

Since $\varphi_j \in L^{Q_k'}$, $k \in \square$, the Hölder inequality implies that $f_j(k+it)$ is

E_k-bounded:

$$\|f_j(k+it)\|_{E_k} \leqslant \int_{\mathbf{R}^n} \|f(k+it)\|_{E_k} \varphi_j(t)\,dt$$
$$\leqslant \|f(k+it)\|_{L^{Q_k}(E_k)} \|\varphi_j\|_{L^{Q'_k}}\,.$$

This inequality also implies that $f_j(k+it)$ is E_k-continuous. Thus, it follows that $f_j \in H(\mathbf{E})$ and consequently $a_j = f_j(\Theta) \in [\mathbf{E}; \Theta]$.

Now, the inequality

$$\|a_j - a\|_{\Sigma \mathbf{E}} = \left\| \int_{\mathbf{R}^n} f(\Theta + if)\,\varphi_j(t)\,dt - \int_{\mathbf{R}^n} a\varphi_j(t)\,dt \right\|_{\Sigma \mathbf{E}}$$
$$\leqslant \int_{|t|\leqslant 1/j} \|f(\Theta + it) - f(\Theta)\|_{\Sigma \mathbf{E}} \varphi_j(t)\,dt$$

implies that $a_j \to a$ in $\sum \mathbf{E}$, as $j \to \infty$.

It remains to show that (a_j) is a Cauchy sequence in $[\mathbf{E}; \Theta]$. We shall use the inequality 4.5(2):

$$\|a_n - a_m\|_{[\mathbf{E};\Theta]} \leqslant C \max_{k \in \square} \|(f_n - f_m)(k+it)\|_{L^{Q_k}(E_k)} \leqslant$$
$$\leqslant C \max_{k\in\square} \left\{ \left\| \int_{\mathbf{R}^n} \|f(k+i(t+x)) - f(k+it)\|_{E_k} \varphi_n(x)\,dx \right\|_{L^{Q_k}} + \right.$$
$$\left. + \left\| \int_{\mathbf{R}^n} \|f(k+i(t+x)) - f(k+it)\|_{E_k} \varphi_m(x)\,dx \right\|_{L^{Q_k}} \right\} \leqslant$$
$$\leqslant C \max_{k\in\square} \left\{ \int_{\mathbf{R}^n} \|f(k+i(t+x)) - f(k+it)\|_{L^{Q_k}(E_k)} \varphi_n(x)\,dx + \right.$$
$$\left. + \int_{\mathbf{R}^n} \|f(k+i(t+x)) - f(k+it)\|_{L^{Q_k}(E_k)} \varphi_m(x)\,dx \right\} \leqslant$$
$$\leqslant C \max_{k\in\square} \left\{ \int_{|x|\leqslant 1/n} \|f(k+i(t+x)) - f(k+it)\|_{L^{Q_k}(E_k)} \varphi_n(x)\,dx + \right.$$
$$\left. + \int_{|x|\leqslant 1/m} \|f(k+i(t+x)) - f(k+it)\|_{L^{Q_k}(E_k)} \varphi_m(x)\,dx \right\}.$$

Now, given $\varepsilon > 0$, there is an integer N such that

$$\|f(k+i(x+t)) - f(k+it)\|_{L^{Q_k}(E)} < \varepsilon \qquad (k \in \square)$$

if $|x| \leqslant \delta = 1/N$. Hence, for $n, m \geqslant N$ it follows that

$$\|a_n - a_m\|_{[\mathrm{E};\Theta]} \leqslant C \max\Big\{\varepsilon \int_{|t|\leqslant 1/n} \varphi_n(t)\,dt + \varepsilon \int_{|t|\leqslant 1/m} \varphi_m(t)\,dt\Big\} \leqslant 2C\varepsilon\,.$$

The proof is complete.

5. The Hausdorff-Young theorem, for L^P spaces with mixed norm, and spaces of type P.

We recall the Hausdorff-Young theorem in the form given by Benedek-Panzone [1].

THEOREM 5.1. *Let Ff be the Fourier transform of $f \in S'(\mathbf{R}^n)$, and $P = (p_1, \ldots, p_n)$ such that $1 \leqslant p_n \leqslant p_{n-1} \leqslant \ldots \leqslant p_1 \leqslant 2$. Then*

5.1(1) $$\|\mathcal{F}f\|_{L^{P'}} \leqslant C(P)\|f\|_{L^P}\,,$$

where $1/P + 1/P' = 1$ and $C(P) = 1$. If $1 \leqslant P \leqslant 2$ and the components of P are not monotonically non-increasing then 5.1(1) *does not hold for any $C(P)$.*

Now, following Peetre [13] and the above theorem we set.

DEFINITION 5.2. *Let E be a given Banach space and $P = (p_1, \ldots, p_n)$ an n-tuple with $1 \leqslant p_n \leqslant \ldots \leqslant p_1 \leqslant 2$. If for some constant $C(P)$ it holds that*

$$\|\mathcal{F}f\|_{L^{P'}(E)} \leqslant C(P)\|f\|_{L^P(E)} \qquad (1/P + 1/P' = 1)$$

for all $f \in L^P(E)$, the space E will be called of type P.

When $p_1 = \ldots = p_n = p$ theorem 5.1, reduces to the usual Hausdorff-Young theorem, and the definition 5.2, coincides with Peetre's definition 2.1 in [13].

EXEMPLES 5.3.

5.3(1) Banach spaces are of type $1 = (1, \ldots, 1)$;

5.3(2) Hilbert spaces are of type $2 = (2, \ldots, 2)$;

5.3(3) (J. Peetre) the space $L^P(\mathbf{R}^m)$ is of type P if $1 \leqslant P \leqslant 2$;

5.3(4) If E_k is of type P_k $(k \in \square)$, where $(P_k | k \in \square)$ is a family of powers associated with P_0 and P_1, then $((E_k | k \in \square); \Theta; P)$ is of type P, where $1/P = (1-\Theta)/P_1 + \Theta/P_2$.

Indeed, by hypothesis we have

$$F: L^{P_k}(E_k) \to L^{P'_k}(E_k)\,, \qquad (1/P_k + 1/P'_k = 1)\,.$$

Thus

$$F: ((L^{P_k}(E_k))_k; \Theta; P) \to ((L^{P'_k}(E_k); \Theta; P))\,.$$

But

$$((L^{P_k}(E_k))_k; \Theta; P) = (L^P((E_k)_k; \Theta; P))$$

and

$$((L^{P'_k}(E_k))_k; \Theta; P) \subset L^{P'}((E_k)_k; \Theta; P))\,.$$

5.3(5) If E is reflexive then E and the dual spaces E' are either of type P.

6. A connection between the real and the complex method.

As in the case $n = 1$, we know that

$$(\mathbf{E}; \Theta; 1) \subset [\mathbf{E}; \Theta] \subset (\mathbf{E}; \Theta; \infty)$$

where $1 = (1, \dots, 1)$ and $\infty = (\infty, \dots, \infty)$. We shall give a generalization of this result.

THEOREM 6.1. *Let* $\mathbf{E} = (E_k | k \in \square)$ *be an admissible family of Banach spaces and* $(\mathbf{P} = (P_k | k \in \square))$ *a family of admissible powers associated with* P_0 *and* P_1. *Now, if* E_k *is of type* P_k, $k \in \square$, *then*

6.1(1) $$(\mathbf{E}; \Theta; \mathbf{P}) \subset [\mathbf{E}; \Theta] \subset (\mathbf{E}; \Theta; \mathbf{P}')\,.$$

PROOF. We follow the ideas of [13] (see also [5]). Let $a \in (\mathbf{E}; \Theta; \mathbf{P})$ and $u = u(t) \in L^1_*(\sum \mathbf{E})$ such that

$$a = \int_{\mathbf{R}^n_+} u(t)\, d_* t$$

with

$$t^{k-\Theta} u \in L_*^{P_k}(E_k)\,, \qquad k \in \Box\,.$$

Let us set

$$U(z) = Mu(z) = \int_{\mathbf{R}_+^n} t^{z-\Theta} u(t)\, d_* t$$

$$= \int_0^\infty \ldots \int_0^\infty t_1^{z_1-\theta_1} \ldots t_n^{z_n-\theta_n} u(t_1, \ldots, t_n)\, d_* t_1 \ldots d_* t_n\,,$$

the n-dimensional Mellin transformation of u.

We see that $U(z)$ is a $\sum \mathbf{E}$-valued holomorphic function with

$$a = U(\Theta) = U(\theta_1, \ldots, \theta_n)\,.$$

But, by the change of variables $t_j = \exp(-s_j)$, we get

$$U(x+iy) = \int_{\mathbf{R}_+^n} t^{iy} t^{x-\Theta} u(t)\, d_* t$$

$$= \int_{\mathbf{R}^n} e^{-iy\cdot s} e^{(x-\Theta)\cdot s} v(s)\, ds$$

where $v(s) = u(\exp(-s))$. But

$$e^{(k-\Theta)\cdot s} v(s) \in L^{P_k}(E_k)$$

and since E_k is of type P_k is follows that

$$U(k+iy) \in L^{P_k'}(E_k)\,.$$

By proposition 4.6 it follows that $a \in [\mathbf{E}; \Theta]$.

CONVERSE. Let $a \in [\mathbf{E}; \Theta]$. Then, there is $u \in H(\mathbf{E})$ such that $a = u(\Theta) = u(\theta_1, \ldots, \theta_n)$. Moreover $u(k+iy)$ is E_k-bounded and continuous, for any $k \in \Box$. But, here we can replace $u(z)$ by $u(z) \exp(z - \Theta)^2$. Thus we can suppose that

$$u(k+iy) \in L^{P_k}(E_k)\,.$$

If we set

$$U(x) = (2\pi)^{-n}\int_{\mathbf{R}^n} x^{(s-\Theta)+it} u(s+it)\,dt$$

(the inverse of the n-dimensional Mellin transformation) we shall have

$$a = \int_{\mathbf{R}_+^n} U(t)\,d_*t$$

and

$$t^{k-\Theta}\,U(t) \in L_*^{P_k'}(E_k) \qquad (k \in \Box)\,.$$

We see the 2.2(1) and 2.4(1) are satisfied and thus $a \in (\mathbf{E}; \Theta; \mathbf{P}')$ as desired.

When the elements of an admissible family are Hilbert spaces we have the following result.

THEOREM 6.2. *Let $H = (H_k | k \in \Box)$ be an admissible family of Hilbert spaces. Then*

$$(H; \Theta; 2) = [H; \Theta]\,.$$

PROOF. Since Hilbert spaces are of type $2 = (2, \ldots, 2)$ we have

$$(H; \Theta; 2) \subset [H; \Theta] \subset (H; \Theta; 2)\,.$$

7. Applications.

If $M = (m_1, \ldots, m_n) \in \mathbf{N}^n$ and $1 < P = (p_1, \ldots, p_n) < \infty$, let us consider the Sobolev Nikols'kii spaces (see [11]):

$$W^{M,P} = W^{M,P}(\mathbf{R}^n) = \{u \in S'(\mathbf{R}^n) | D^\alpha u \in L^P(\mathbf{R}^n),\ \alpha \leqslant M\}\,,$$

and for $S = (s_1, \ldots, s_n)$ the potential space of Lizorkin-Nikols'kii (see [10]):

$$H^{S,P} = H^{S,P}(\mathbf{R}^n) = \left\{u \in S'(\mathbf{R}^n) | \mathcal{F}^* \prod_{j=1}^n (1 + |x_j|^2)^{s_j/2}\, \mathcal{F} u \in L^P(\mathbf{R}^n)\right\}$$

We see that $W^{0,P} = H^{0,P} = L^P$.

Now, given $M = (m_1, \ldots, m_n) \in \mathbb{N}^n$, let $(M_k)_{k\in\square}$ be the family of admissible parameters associated with $M = (m_1, \ldots, m_n)$ and $0 = (0, \ldots, 0)$ (that is $M_k = (m_{k_1}, \ldots, m_{k_n})$, with $m_{k_j} = 0$ or m_j). Let us set

$$B_P^{S,Q} = ((W^{M_k,P})_{k\in\square}; \Theta; Q) ,$$

where $1 < P = (p_1, \ldots, p_n) < \infty$, $1 \leqslant Q = (q, \ldots, q) \leqslant \infty$, $S = (s_1, \ldots, s_n)$ and $\Theta = (\theta_1, \ldots, \theta_n)$ with $\theta_j = s_j/m_j$, $0 < s_j < m_j$, $j = 1, \ldots, n$.

On the other hand we know that the spaces $W^{M,P}$ and $H^{M,P}$ are isomorphic via the Mihlin-Lizorkin theorem ([9] and [10]). Thus, we have

$$H^{S,P} = [(W^{M_k,P})_{k\in\square}; \Theta]$$

where S, M_k and P are given as above.

The Mihlin-Lizorkin theorem implies also that $W^{M,P}$ is isomorphic to L^P. Thus, if $1 \leqslant p_n \leqslant \ldots \leqslant p_1 \leqslant 2$ and $P = (p_1, \ldots, p_n)$, the spaces $W^{M,P}$ is of type P.

Now, by theorem 6.1 we have

7.0(1) $\qquad B_P^{S,P} \subset H^{S,P}; \qquad (1 < p_n \leqslant \ldots \leqslant p_1 \leqslant 2;\ P = (p_1, \ldots, p^n))$

and

7.0(2) $\qquad H^{S,Q} \subset B_Q^{S,Q}; \qquad (2 \leqslant q_1 \leqslant \ldots \leqslant q_n < \infty,\ Q = (q_1, \ldots, q_n))$.

Moreover, theorem 6.2 implies that

$$H^{S,2} = B_2^{S,2}.$$

The embeddings 7.0(1) and 7.0(2) hold for also $P = 1$ and $P' = \infty$, but not by theorem 6.1.

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