E. BAROZZI
U. MASSARI

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Regularity of Minimal Boundaries with Obstacles.

E. BAROZZI - U. MASSARI (*)

In this paper, we shall prove some regularity results for minimal boundaries with obstacles.

The problem can be formulated in the following way:

Let Ω be an open set of $\mathbb{R}^n$ and M a measurable subset of Ω. We are interested in studying the regularity of a set $E \subset \Omega$, containing M and minimizing surface area in the class of all subsets F of Ω with $M \subset F$ and $F \triangle E \subset\subset \Omega$.

M. Miranda proved by geometrical methods that if the obstacle M is of class C^1, then the minimal boundary ∂E is C^1 in a neighbourhood of ∂M (see [1]).

In the special non parametric case in which $\Omega = A \times \mathbb{R}$ (A open set of $\mathbb{R}^{n-1}$) and $M = \{(y, t);\ y \in A,\ t < \psi(y)\}$ ($\psi\colon A \to \mathbb{R}$ a given function), the minimal set E is the subgraph of a function $u\colon A \to \mathbb{R}$ minimizing the area integral in the class of all functions $v\colon A \to \mathbb{R}$, with $v \geqslant \psi$ and support$(v - u) \subset\subset A$ and the problem is reduced to study the regularity of u.

In this special case, M. Giaquinta and L. Pepe proved by methods of functional analysis that if $\psi \in H^{2,p}(A)$ with $p > n-1$, then the function $u \in H^{2,p}_{\mathrm{loc}}(A)$, hence $u \in C^{1,\alpha}$ (see [2]).

Moreover, if $\psi \in C^2(A)$, then $u \in C^{1,1}(A)$ (see Brezis-Kinderlehrer [3]).

In this paper we study the parametric case by using the same idea introduced by Giaquinta-Pepe for the non parametric version of the problem. In particular we assume that the « mean curvature of the

(*) Indirizzo degli AA.: Dipartimento di Matematica, Libera Università di Trento, 38050 Povo (Trento).

obstacle is bounded from above by a function $g \in L^p(\Omega)$ with $p > n$ » (see definition 1) and we prove that a minimal set with respect to an obstacle minimizes, without any conditions, a functional of the type:

$$\mathcal{F}(F) = \int_\Omega |D\varphi_F| + \int_\Omega H(x)\,\varphi_F(x)\,dx\,. \tag{1}$$

We can then apply to $\mathcal{F}$ the regularity results of [4] and [5]. We obtain in this way that the reduced boundary $\partial^* E$ of a minimum E is a $C^{1,\alpha}$-manifold (of dimension $n-1$) and the k-dimensional Hausdorff measure of the singular set $\partial E - \partial^* E$ is zero for every $k \in \mathbb{R}$ with $k > n-8$.

The paper is divided in two sections. In this first, we state and discuss our assumption on the mean curvature of the obstacle and prove the regularity result. In the second one we present some applications of our result to the non parametric case.

We wish to thank M. Miranda for his stimulating remarks.

1. Let Ω be an open set of $\mathbb{R}^n$; for a function $f \in L^1_{\text{loc}}(\Omega)$ we denote

$$\int_\Omega |Df| = \sup\left\{\int_\Omega f \operatorname{div} g\,dx;\ g \in [C_0^1(\Omega)]^n,\ |g| \leqslant 1\right\}. \tag{2}$$

We say that a measurable set F has finite perimeter in Ω if its characteristic function φ_F satisfies $\int_\Omega |D\varphi_F| < +\infty$.

If M is a measurable subset of Ω, we say that a set E has minimal perimeter in Ω with respect to the obstacle M if:

$$\begin{cases} \text{i)} & M \subset E; \\ \text{ii)} & \int_{\Omega'} |D\varphi_E| < +\infty\,, \qquad \int_{\Omega'} |D\varphi_E| \leqslant \int_{\Omega'} |D\varphi_F| \end{cases} \tag{3}$$

for every $\Omega' \subset\subset \Omega$ and $F \subset \Omega$ with $M \subset F$ and

$$F \,\Delta\, E = (F - E) \cup (E - F) \subset\subset \Omega'.$$

From the lower semicontinuity of the perimeter with respect to the $L^1_{loc}(\Omega)$-convergence and the compactness theorem ([6], theorems 2.2 and 2.4), it is not hard to prove the existence of sets minimizing the perimeter with respect to a fixed obstacle.

DEFINITION 1. We say that the mean curvature of M is less or equal than $g \in L^1_{loc}(\Omega)$ if, for every $\Omega' \subset\subset \Omega$ and F with $F \subset M$ and $F \Delta M \subset\subset \Omega'$, we have:

$$\int_{\Omega'} |D\varphi_M| \leqslant \int_{\Omega'} |D\varphi_F| + \int_{M-F} g(x)\, dx \,. \tag{4}$$

To illustrate the meaning of condition (4), it is useful to consider the non parametric situation, i.e. when $\Omega = A \times \mathbb{R}$ and $M = \{(y, t);\ y \in A,\ t < \psi(y)\}$. If we suppose $\psi \in C^2(A)$, the mean curvature of ∂M at $(y, \psi(y))$ is given by:

$$-\frac{1}{n-1} \operatorname{div}(T\psi)(y) \tag{5}$$

where:

$$T\psi(y) = \frac{D\psi(y)}{\sqrt{1 + |D\psi(y)|^2}} \,. \tag{6}$$

(Here the mean curvature is considered with respect to the inner normal vector, so that convex sets have non negative mean curvature).

Denoting

$$C^+(y) = \max\{0, -\operatorname{div}(T\psi)(y)\} \,, \tag{7}$$

we obtain, for every $\varphi \in C^1_0(A)$, $\varphi \leqslant 0$:

$$\int_A \varphi C^+(y)\, dy \leqslant -\int_A \varphi \operatorname{div}(T\psi)\, dy = \int_A T\psi \cdot D\varphi\, dy$$

and then:

$$\frac{d}{dt}\Big[\int_A \sqrt{1 + |D(\psi + t\varphi)|^2}\, dy - \int_A C^+(y)(\psi + t\varphi)\, dy\Big]_{t=0} \geqslant 0 \,.$$

As the function in brackets is convex, we have:

$$\int_A \sqrt{1+|D\psi|^2}\,dy - \int_A C^+(y)\psi\,dy \leqslant$$

$$\leqslant \int_A \sqrt{1+|D(\psi+\varphi)|^2}\,dy - \int_A (\psi+\varphi)\,C^+(y)\,dy$$

which is condition (4) for sets $F = \{(y,t);\ y \in A,\ t < \psi + \varphi\}$ with $g(y,t) = C^+(y)$.

Of course the same considerations hold if ψ is twice weakly differentiable.

We get now the following result:

LEMMA. Let E be a set minimizing the perimeter in Ω with respect to the obstacle M. If M satisfies condition (4) then E minimize the functional (1) with $H(x) = -\varphi_M(x)\,g(x)$, i.e. for every $\Omega' \subset\subset \Omega$ and F with $F \Delta E \subset\subset \Omega'$ there holds:

$$\int_{\Omega'} |D\varphi_E| + \int_{\Omega'} \varphi_E(x)\,H(x)\,dx \leqslant \int_{\Omega'} |D\varphi_F| + \int_{\Omega'} \varphi_F(x)\,H(x)\,dx\,. \tag{8}$$

PROOF. Let F be a measurable set such that $F \Delta E \subset\subset \Omega'$; from inequality I of [7], pag. 362, we have:

$$\int_{\Omega'} |D\varphi_E| \leqslant \int_{\Omega'} |D\varphi_{F\cup M}| \leqslant \int_{\Omega'} |D\varphi_F| + \int_{\Omega'} |D\varphi_M| - \int_{\Omega'} |D\varphi_{F\cap M}|\,. \tag{9}$$

On the other hand, assumption (4) gives:

$$\int_{\Omega'} |D\varphi_M| - \int_{\Omega'} |D\varphi_{F\cap M}| \leqslant \int_{\Omega'\cap(M-F)} A(x)\,dx = \tag{10}$$

$$= \int_{\Omega'} \varphi_M(x)\,A(x)\,dx - \int_{\Omega'} \varphi_F(x)\,\varphi_M(x)\,A(x)\,dx\,.$$

From (9) and (10), we obtain (8).

We can now apply the regularity results of [5] to get the following theorem:

THEOREM. Let E be a set minimizing perimeter in Ω with respect to the obstacle M and suppose that M satisfies condition (4) with:

$$g(x) \in L^p_{loc}(\Omega), \qquad p > n; \tag{11}$$

then $\partial^* E \cap \Omega \in C^{1,\alpha}$ and $H_k((\partial E - \partial^* E) \cap \Omega) = 0$ for every $k > n - 8$ [1].

REMARK. Let M be a bounded set satisfying the internal sphere condition with radius r, i.e. for every $x \in M$ there exists a sphere B_r of radius r such that $x \in \bar{B}_r \subset M$, then the condition (4) holds with $g(x) = nr^{-1}$ (see [9]).

If the boundary of M contain an outward angle or cusp, condition (4) is not satisfied for any g.

2. We return now to the non parametric case. In this case the minimal boundary ∂E is the graph of a function u minimizing the area integral in the class

$$\left\{v \in L^1(A);\ \int_\Omega |Dv| < +\infty,\ v \geqslant \psi,\ \text{support}\,(v - u) \subset\subset A\right\}.$$

With the help of the preceding theorem, we derive some kind of regularity for the minimizing function u.

First of all we observe that, if ψ is twice weakly differentiable and $C^+(y) \in L^p_{loc}(A)$, $p > n$, then we obtain $C^{1,\alpha}$-regularity for $\partial^* E \cap \Omega$. Nevertheless we note that, in this case, it is sufficient to assume that:

$$C^+(y) \in L^p_{loc}(A), \qquad p > n - 1 \tag{12}$$

to obtain the regularity of $\partial^* E \cap \Omega$; infact, for $\bar{B}_\varrho \subset \Omega$, the minimum

[1] Here $\partial^* E$ denotes the reduced boundary of the set E, i.e. the set of points $x \in \partial E$ such that there exists the limit:

$$\lim_{\varrho \to 0} \frac{\int_{B_\varrho(x)} D\varphi_E}{\int_{B_\varrho(x)} |D\varphi_E|} = \nu(x)$$

and $|\nu(x)| = 1$ (see [8]), and H_k denotes the k-dimensional Hausdorff measure.

property of E (or of u) allows us to write

$$\int_{\bar{B}_\varrho} |D\varphi_E| + \int_{B_\varrho} \varphi_E H(y, t)\, dy\, dt \leqslant \int_{\bar{B}_\varrho} |D\varphi_F| + \int_{B_\varrho} \varphi_F H(y, t)\, dy\, dt \tag{13}$$

for every measurable set F such that $F = E$ in $\Omega - \bar{B}_\varrho$, where we have denoted $H(y, t) = -\varphi_M(y, t)\, C^+(y)$.

Therefore:

$$\int_{\bar{B}_\varrho} |D\varphi_E| - \int_{\bar{B}_\varrho} |D\varphi_F| \leqslant \int_{B_\varrho} |H(y, r)|\, |\varphi_E - \varphi_F|\, dy\, dt \leqslant$$
$$\leqslant 2\varrho \int_{B'_\varrho} C^+(y)\, dy \leqslant \| C^+\|_{L^p(B'_\varrho)} 2\omega_{n-1}^{1-1/p}\, \varrho^{n-(n-1)/p}$$

(B'_ϱ is the projection of B_ϱ on $t = 0$). It follows:

$$\psi(E, \varrho) = \int_{\bar{B}_\varrho} |D\varphi_E| - \inf\left\{ \int_{\bar{B}_\varrho} |D\varphi_F|;\ F = E \text{ in } \Omega - \bar{B}_\varrho \right\} \leqslant \tag{14}$$
$$\leqslant 2 \| C^+\|_{L^p(B'_\varrho)}\, \omega_{n-1}^{1-1/p}\, \varrho^{n-1+\varepsilon}$$

with $\varepsilon = (p - n + 1)/p$.

The the regularity theorem is a consequence of inequality (14). (See e.g. [5]).

REMARKS:

1) In the case $E = \{(y, t);\ y \in A,\ t < u(y)\}$ and $C^+ \in L^p_{\mathrm{loc}}(A)$ with $p > n - 1$, from well-known regularity theorems it follows that $\partial^* E - \partial M$ is a $(n-1)$-dimensional analytic manifold, whose mean curvature is zero and $\partial^* E - \partial M = \partial E - \partial M$. Moreover the projection A_0 of $\partial E - \partial M$ on A is an open set and $u|_{A_0}$ is an analytic function (see [7]).

2) As contact points are concerned, i.e. $x \in \partial E \cap \partial M \cap (\Omega = A \times \mathbb{R})$, they can be regular even if u is not. By example, consider for $n = 3$ $(y \in \mathbb{R}^2)$:

$$\psi(y) = \begin{cases} \lg \dfrac{2 + \sqrt{3}}{|y| + \sqrt{|y|^2 - 1}} & 1 < |y| < 2\,; \\ \lg (2 + 3) + \sqrt{1 - |y|^2} & |y| \leqslant 1\,. \end{cases}$$

Then the subgraph M of ψ satisfies assumption (4) with:

$$C^+(y) = \begin{cases} 0 & 1 < |y| < 2 \,; \\ 2 & |y| \leqslant 1 \,. \end{cases}$$

It is easy to see that $E = M$ has minimal perimeter with respect to the obstacle M and $\partial E \in C^{1,\alpha}$ (see [1] and [10]), but $u = \psi \notin C^1$.

3) We finally note that, if $(y, t) \in \partial E \cap \partial M$ and ∂M is a C^1-manifold in a neighbourhood of (y, t), then ∂E is also regular at (y, t) and the normal vectors to ∂E and ∂M at (y, t) coincide. (See [1]). It follows in particular that, if $\psi \in H^{2,p}(A)$ with $p > n - 1$, then $u \in C^{1,\alpha}$.

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