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Contributions to foundations of probability calculus on the basis of the modal logical calculus MC^{ν} or MC^{ν}_{*}

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Contributions to Foundations of Probability Calculus on the Basis of the Modal Logical Calculus MC^v or MC_*^v.

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PART 2

On a Known Existence Rule for the Probability Calculus.

8. Introduction (**)

By means of the probability calculus, and precisely its formulation PC_* [PC] based on MC_*^v [MC^v]—cf. Part I, N. 5—we can calculate the values of certain relative probabilities, such as $\mathfrak{T}_{\alpha,\beta\vee\gamma}$, under certain conditions, e.g. $\alpha\beta \supset^{\cap} \sim\gamma$, when we know the values of certain other probabilities, e.g. $\mathfrak{T}_{\alpha,\beta}$ and $\mathfrak{T}_{\alpha,\gamma}$. Thus we can also deduce the existence of some probabilities. However the need of asserting the existence of various other probabilities was felt. Therefore the so called existence rule was formulated. For instance the philosopher Reichenbach states it as follows:

RULE 8.1. « If the numerical value p of a probability implication $(A \ni_p B)$, provided the probability implication exists, is determined

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(**) Bressan's contribution to this work belongs to the sphere of activity of the C.N.R. (Consiglio Nazionale delle Ricerche) in the academic years 1978-79 and 1979-80.

The summary of this paper is included in the one of Part I, i.e. [2].

by given probability implications according to the rules of the calculus, then this probability implication ($A \ni_p B$) exists »—cf. [4] p. 53 (1).

This rule is substantially used also by some probabilists, e.g. by Dore—cf. [3] p. 66. Such rules are said to be metalogical by all users of them—cf. the preceding footnote—and even meta-probabilistic by some of them (2). Let us add that these formulations also have an intuitive and not very precise character.

By the reasons above, in [1] the existence rule, a logic-probabilistic rule on a pair with inference rules such as modus ponens, is replaced with an (existence) axiom scheme, as a proposal to be tested.

In the present paper the afore-mentioned axiom scheme is considered—cf. A9.1—and it is hinted at its reduction to an admittedly more complex single axiom, A9.1′ which is meaningful also in $\boldsymbol{ML}^{\nu}$. Then the test of A9.1 proposed in [1], is performed with a positive result. A precise existence rule, Rule 10.1, strictly weaker than A9.1, is derived from this axiom scheme in PC^* [PC]; and the main existence assertions ordinarily reached on the basis of the intuitive Rule 8.1, —cf. [2]—, are rigorously and explicitly derived by the new rule. The probability calculus PC^* [PC] supplemented with A9.1 [A9.1′] will be called the probability theory PT^* [PT].

9. On a rigorous axiom scheme in MC_*^{ν} that can replace Rule 8.1, and a single axiom equivalent to it.

In order to formulate the axiom scheme A9.1 below, to be substituted for Rule 8.1, we consider the following conditions (a) to (d), which refer to $\boldsymbol{MC}_*^{\nu}$.

(1) In [4], p. 53, Reichenbach says about Rule 8.1: « The rule of existence is not an axiom of the calculus; it is a rule formulated in the metalanguage, analogous to the rule of inference or the rule of substitution.

It must be given an interpretation even in the formal treatment of the calculus. There must exist a formula that can be demonstrated in the calculus and that expresses the probability under consideration as a mathematical function of the given probabilities, with the qualification that the function be unique and free from singularities for the numerical values used.

This is what is meant by the expression "determined according to the rules of the calculus" ».

(2) In [3], p. 53, about the existence rule it is said that its metalogical and metaprobabilitistic character is evident, and that its basis must be found in the intuitive content of the concept of probability.

(*a*) α_1 *to* α_n $(n \geqslant 0)$ *are distinct propositional variables,*

(*b*) p_1 *to* p_m *and* u_1 *to* u_μ $(m \geqslant 0,\ \mu > 0)$ *are distinct variables of type* $t_{\mathbf{R}}$, *the type of real numbers; and*

$$(9.1) \qquad \gamma_\alpha \equiv_D \bigwedge_{i=1}^{m} \alpha_{r_i} \ni_{p_i} \alpha_{s_i}\,, \qquad \delta_{\alpha u} \equiv_D \bigwedge_{j=1}^{\mu} \alpha_{\sigma_j} \ni_{u_j} \alpha_{\tau_j}\,,$$

where $r_i, s_i, \sigma_j, \tau_j \in \{1, \ldots, n\}$ *for* $i = 1, \ldots, m$ *and* $j = 1, \ldots, \mu$;

(*c*) $\mathcal{E}_\alpha$ *is any assertion equivalent to* $\Box\mathcal{E}_\alpha$ *and built by means of only* α_1 *to* α_n, $\sim$, $\wedge$, $\Box$, *and parentheses (so that* p_i, u_i, *and* $v_i\forall$ *fail to occur in* $\mathcal{E}_\alpha$);

(*d*) $\mathfrak{D}_p$ *is an assertion equivalent to* $\Box\mathfrak{D}_p$, *that belongs to the theory* RT *of real numbers and contains no variables different from* p_1 *to* p_m.

Now we can formulate the existence axiom stated in [1].

A9.1 (*existence*)

$$(\Box)\, \mathfrak{D}_p \gamma_\alpha \mathcal{E}_\alpha (\exists^{(1)} u_1, \ldots, u_\mu)(\exists \alpha_1, \ldots, \alpha_n) \gamma_\alpha \mathcal{E}_\alpha \delta_{\alpha u} \supset \bigwedge_{j=1}^{\mu} (\exists u_j)(\alpha_{\sigma_j} \ni_{u_j} \alpha_{\tau_j})\,.$$

Intuitively, by (*a*) to (*d*), A9.1 means the following:

Let us consider (i) a mathematical condition $\mathfrak{D}_p$ on the real numbers p_1 to p_m—e.g. $p_1 + p_2 < 1$—, (ii) the conditions γ_α and $\delta_{\alpha u}$ on the propositional variables α_1 to α_n and the real-valued variables p_1 to p_m and u_1 to u_μ, defined by (9.1), (iii) a non-contingent (modally constant) condition $\mathcal{E}_\alpha$ on only α_1 to α_n, that can be expressed in the (modal) propositional calculus, and (iv) the assumptions

A) *conditions* $\mathfrak{D}_{p'}$, $\gamma_{\alpha'}$ *and* $\mathcal{E}_{\alpha'}$ *hold,*

B) *condition* α'_{σ_j} *can occur* $(j = 1, \ldots, \mu)$,

and

C) *if conditions* $\gamma_{\alpha''}$, $\mathcal{E}_{\alpha''}$, $\delta_{\alpha'\mu'}$, *and* $\delta_{\alpha''u''}$ *also hold, then* $u'_j = u''_j$ $(j = 1, \ldots, \mu)$.

Assumptions A) *to C*) *necessarily imply the existence of real numbers* u_1 *to* u_μ *such that* α_{σ_j} *implies* α_{τ_j} *with the probability* u_j $(j = 1, \ldots, \mu)$.

In MC^{ν}_{*} A5.r can be regarded as a single axiom $(r = 1, \ldots, 8)$ whereas A9.1 is necessarily an axiom scheme. In order to reduce it briefly to a single axiom (meaningful also in MC^{ν}), in connection with (a) we use a variable A of type $(t_{\mathbf{N}})$ (capable of expressing properties of natural numbers). In connection with (b) we introduce in MC^{ν}_{*} the class

$$C_{\mu,n} =_D \{1, \ldots, n\}^{\{1,\ldots,\mu\}} \tag{9.2}$$

formed by the mappings f of type $(t_{\mathbf{N}}:t_{\mathbf{N}})$, such that $f(x) \leqslant \mu$ for $x \leqslant_{\mathcal{N}} n$, and $f(x) = a^*$ otherwise. We can define

$$U_\mu =_D [0, 1]^{\{1,\ldots,\mu\}} \qquad \text{where } [0, 1] =_D \{x \in \mathbf{R}\colon 0 \leqslant x \leqslant 1\}, \tag{9.3}$$

in a similar way. Then (9.1) can be replaced by

$$\begin{cases} \gamma_{A,p,m,r,s} \equiv_D (\forall i \leq_{\mathbf{N}} m)\, A(r_i) \ni_{p_i} A(s_i)\,, \\ \delta_{A,u,\mu,\sigma,\tau} \equiv_D (\forall j \leq_{\mathbf{N}} \mu)\, A(\sigma_j) \ni_{u_j} A(\tau_j)\,. \end{cases} \tag{9.1'}$$

In connection with (c), first, we consider the additional variables $\mathcal{M}$, m, and q, of the respective types $(t_{\mathbf{N}})$, $t_{\mathbf{N}}$, and $t_{\mathbf{N}}$; second, we introduce a substitutum for $\mathcal{E}_\alpha$ by setting

$$\begin{aligned} \mathcal{E}_{A,\mathcal{M},m,q} \equiv_D {} & (i) \cdot [i \leq_{\mathbf{N}} m \supset .\mathcal{M}(i) \equiv^{\cap} A(i)] \wedge \\ & \wedge \{m < i \leq q \supset (\exists r, s < i)[\mathcal{M}(i) \equiv^{\cap} \sim \mathcal{M}(r). \vee .\mathcal{M}(i) \equiv \mathcal{M}(r) \wedge \mathcal{M}(s). \vee \\ & \vee .\mathcal{M}(i) \equiv^{\cap} \Box \mathcal{M}(r). \vee .\mathcal{M}(i) \equiv^{\cap} (x)\ x = x]\}\,. \end{aligned} \tag{9.4}$$

In connection with (d) we consider $\mathbf{R}$ as a constant of MC^{ν} (hence of MC^{ν}_{*}) that expresses the natural absolute concept of real numbers; furthermore we replace $\mathfrak{D}_p$ with $\mathfrak{D}_{p,\mathfrak{R},m}$ where p, $\mathfrak{R}$, and m are variables of the respective types $(t_{\mathbf{N}}:t_{\mathbf{R}})$, $(t_{\mathbf{N}}:t_{\mathbf{R}}))$, and $t_{\mathbf{N}}$, and where

$$\mathfrak{D}_{p,\mathfrak{R},m} \equiv_D \mathfrak{R}(p) \wedge (\forall p' \in U_n)[\mathfrak{R}(p') \equiv^{\cap} \Box \mathfrak{R}(p')]\,. \tag{9.5}$$

Now we can replace axiom scheme A9.1 with the equivalent single axiom (in MC^{ν})

$$\begin{aligned} & (\forall n, m, \mu, q, j, r, s, \sigma, \tau, p, A, \mathcal{M}).(n, m, \mu, q, j \in \mathbf{N}) \wedge \\ & \wedge (r, s \in C_{m,n}) \wedge (\sigma, \tau \in C_{\mu,n}) \wedge p \in U_m \wedge \mathfrak{D}_{p,\mathfrak{R},m} \wedge \gamma_{A,p,m,r,s} \wedge \mathcal{E}_{A,\mathcal{M},m,\mu,q,\sigma} \wedge \\ & \wedge j \leq \mu \wedge (\exists^{(1)} u_1) \ldots (\exists^{(1)} u_j)(\exists A, \mathcal{M}).(u \in U_m \wedge \\ & \wedge \gamma_{A,p,m,r,s} \wedge \mathcal{E}_{A,\mathcal{M},m,\mu,q} \wedge \delta_{A,u,\mu,\sigma,\tau}) \supset (\exists u_j)\big(A(\sigma_j) \ni_{u_j} A(\tau_j)\big)\,. \end{aligned} \tag{A9.1'}$$

10. A precise existence rule derived from A9.1. Application of it to prove some fundamental existence theorems.

In the theory $PT^* = PC^* + \text{A9.1}$ we derive the following

RULE 10.1 (*existence*). *Assume that* (i) $\mathfrak{D}_p$, $\mathcal{E}_\alpha$, γ_α, $\delta_{\alpha u}$ *and the variables* α_1 *to* α_n *and* u_1 *to* u_n *fulfil conditions* (*a*) *to* (*d*) *in* N. 9, (ii) β_1 *to* β_n ($\in \mathcal{E}_0$) *and* v_1 *to* v_n ($\in \mathcal{E}_{t_{\mathbf{R}}}$) *are other distinct variables,* (iii) $\mathcal{E}_\beta[\delta_{\beta v}]$ *are obtained from* $\mathcal{E}_\alpha[\delta_{\alpha u}]$ *by the replacement of* α_i *and* u_j *with* β_i *and* v_j *respectively* ($i = 1, \ldots, m$; $j = 1, \ldots, \mu$; $\mu \leqslant n$) *and* (iv) *under abbreviations such as*

$$(10.1) \qquad u = v \equiv_D \bigwedge_{j=1}^{\mu} u_j = v_j\,, \qquad (\exists u)\,\delta_{\alpha u} \equiv_D \bigwedge_{j=1}^{\mu} (\exists u)(\alpha_{\sigma_j} \ni_{u_j} \alpha_{\tau_j})$$

we have

$$(10.2) \qquad \mathfrak{D}_p,\ \mathcal{E}_\alpha,\ \mathcal{E}_\beta,\ \gamma_\alpha,\ \gamma_\beta,\ \delta_{\alpha u},\ \delta_{\beta v} \vdash_{CP^*} u = v\,.$$

Then

$$(10.3) \qquad \vdash_{TP^*} \gamma_\alpha\, \mathfrak{D}_p\, \mathcal{E}_\alpha \supset^{\cap} (\exists u)\,\delta_{\alpha u}\,.$$

PROOF OF RULE 10.1. A9.1 is equivalent to $(\square)\mathcal{B} \supset . \mathfrak{D}_p \gamma_\alpha \mathcal{E}_\alpha \supset \supset (\exists u)\,\delta_{\alpha u}$ where $\mathcal{B} \equiv_D (\exists^{(1)} u)(\exists \alpha_1, \ldots, \alpha_n)(\gamma_\alpha\, \mathcal{E}_\alpha\, \delta_{\alpha u})$. By the deduction theorem, (10.2) implies $\vdash_{TP^*} \mathcal{B}$.

Now we prove the existence theorems (10.4, 6) below:

$$(10.4) \qquad \vdash (\alpha \ni \beta)(\alpha \ni \beta \vee \gamma)(\alpha\beta \supset^{\cap} \sim \gamma) \supset (\alpha \ni \gamma)\,,\ \vdash (\alpha \ni \beta) \supset (\alpha \ni \sim \beta)\,.$$

PROOF. Set $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \gamma$ and $\alpha_4 = \beta \vee \gamma$. With a view to applying Rule 10.1, we set

$$(10.5) \qquad \begin{cases} \mathfrak{D}_p \equiv_D 1 =^{\cap} 1\,, & \mathcal{E}_\alpha \equiv_D (\alpha_1 \alpha_2 \supset^{\cap} \sim \alpha_3)(\alpha_4 \equiv^{\cap} \alpha_2 \vee \alpha_3)\,, \\ \gamma_\alpha \equiv_D (\alpha_1 \ni_{p_1} \alpha_2)(\alpha_1 \ni_{p_2} \alpha_4)\,, & \delta_{\alpha u} \equiv_D \alpha_1 \ni_u \alpha_3\,. \end{cases}$$

In the case $\sim \Diamond \alpha_1$ we easily obtain $\alpha \ni_1 \gamma$ by $(6.1)_1$, whence $\alpha \ni \gamma$; hence $(10.4)_1$ holds.

Now we consider the case $\Diamond \alpha_1$ and assume the antecedent of $\supset$

in $(10.4)_1$. Then by the choice rule (for some α_4, α_5, p_1, and p_2) we obtain $\mathfrak{D}_p$, $\mathcal{E}_\alpha$, γ_α. Furthermore by $\diamond\alpha_1$, $\delta_{\alpha u}$, and $(6.3)_1$ we obtain $(\exists_1 p).\alpha_1 \ni_p \alpha_3$. Then, by (10.5) and A5.6, from $\mathcal{E}_\alpha$, γ_α and $\delta_{\alpha u}$ we deduce $p_2 = p_1 + u$. Analogously, from the assumptions γ_β, $\delta_{\beta v}$ and $\diamond\beta_1$ we deduce $p_2 = p_1 + v$ whence $u = v$.

Thus (10.2) holds, and by Rule 10.1, $\mathcal{E}_\alpha$, γ_α, and $\delta_{\alpha u}$ yield $(\exists u)\delta_{\alpha u}$, which by $(10.5)_4$ is $\alpha_1 \ni \alpha_3$. Thus $(10.4)_1$ has been proved. For $\gamma \equiv \sim\beta$ it yields $(10.4)_2$.

$$(10.6) \qquad \begin{cases} \vdash (\exists p \neq 0)(\alpha \ni_p \beta)(\alpha \ni \beta\gamma) \supset (\alpha\beta \ni \gamma)\,, \\ \vdash (\exists q \neq 0)(\alpha\beta \ni_q \gamma)(\alpha \ni \beta\gamma) \supset (\alpha \ni \beta)\,. \end{cases}$$

Proof. Indeed if $\sim\diamond(\alpha\beta)$, then $\alpha\beta \ni \gamma$ by $(6.1)_1$, so that $(10.6)_1$ holds.

Now we assume $\diamond(\alpha\beta)$ and with a view to applying Rule 10.1 we set

$$(10.7) \qquad \begin{cases} \mathfrak{D}_p \equiv_D p_1 \neq 0\,, & \mathcal{E}_\alpha \equiv_D (\alpha_4 \equiv^\cap \alpha_2\alpha_3)(\alpha_5 \equiv^\cap \alpha_1\alpha_2)\,, \\ \gamma_\alpha \equiv_D (\alpha_1 \ni_{p_1} \alpha_2)(\alpha_1 \ni_{p_2} \alpha_4)\,, & \delta_{\alpha u} \equiv_D (\alpha_5 \ni_u \alpha_3)\,. \end{cases}$$

We also assume the antecedent of $\supset$ in $(10.6)_1$, hence by the choice rule, for some α_4, α_5, p_1, and p_2 we have $\mathfrak{D}_p$, $\mathcal{E}_\alpha$, γ_α, and $\delta_{\alpha u}$. Furthermore by $\diamond(\alpha\beta)$, $\delta_{\alpha u}$, and $(6.3)_1$, we have $(\exists_1 u)(\alpha_5 \ni_u \alpha_3)$. Then by A5.7 and (10.7), $\mathcal{E}_\alpha$ and γ_α yield $p_1 u = p_4$. Similarly from $\mathfrak{D}_p$, γ_β, $\delta_{\beta u}$, $\mathcal{E}_\beta$, and $\diamond\beta_5$ we deduce $p_1 v = p_4$. Then since $p_1 \neq 0$, $u = v$. Thus (10.2) holds, which by Rule 10.1 yields (10.3). Hence $\mathfrak{D}_p \mathcal{E}_\alpha \gamma_\alpha$ yields $\alpha_5 \ni_u \alpha_3$. We conclude that $(10.6)_1$ holds. We can prove $(10.6)_2$ in a quite similar way. q.e.d.

By $(10.4)_2$ we deduce the first of the theorems

$$(10.8) \qquad \begin{cases} \vdash \sim\diamond(\alpha\beta) \supset (\alpha \ni_0 \beta)\,, \\ \diamond\alpha.(\alpha \ni_p \beta) \supset p \leqslant 1\,, \\ \beta \equiv^\cap \square\beta \supset (\alpha \ni_1 \beta) \vee (\alpha \ni_0 \beta)\,, \end{cases}$$

from $(6.2)_1$, the second from A5.5, and the third from A5.4.

11. Version in ML_*^ν or ML^ν of some theorems of probability calculus. Stochastic independence.

We formulate a definition of stocastic independence in TP^* [TP] and some theorems connected with it. We do not write their formal (modal) proofs because it is easy to write them on the basis of NN. 9-10 and the corresponding ordinary mathematical proofs.

THEOR. 11.1. *Assume that* $\hat{\beta} \in \{\beta, \sim\beta\}$, $\hat{\gamma} \in \{\gamma, \sim\gamma\}$, *and that* $\mathcal{F}$ *is a matrix built by means of* β, γ, $\sim$, $\wedge$, *and parentheses; and set*

$$G_{\alpha,\beta,\gamma} \equiv_D (\exists p, q, r \in \mathbb{R})[p \neq 1 \neq q \wedge r \neq 0(\alpha \ni_p \beta)(\alpha \ni_q \gamma)(\alpha \ni_r \beta\gamma)]\,; \tag{11.1}$$

then

$$\vdash G_{\alpha,\beta,\gamma} \supset (\alpha\hat{\beta} \ni \hat{\gamma})(\alpha\hat{\gamma} \ni \hat{\beta})(\alpha \ni \mathcal{F})\,. \tag{11.2}$$

Now we define $SI(\alpha, \beta, \gamma)$, γ *is stocastically independent of* β *with respect to* α, *by*

$$SI(\alpha, \beta, \gamma) \equiv_D G_{\alpha,\beta,\gamma} H_{\alpha,\beta,\gamma} (\forall p, q \in \mathbb{R}).(\alpha\beta \ni_p \gamma)(\alpha \ni_q \gamma) \supset p = q\,, \tag{11.3}$$

where

$$H_{\alpha,\beta,\gamma} \equiv_D \Diamond(\alpha\beta\gamma) \Diamond(\alpha\sim\beta\gamma) \Diamond(\alpha\beta\sim\gamma) \Diamond(\alpha\sim\beta\sim\gamma)\,. \tag{11.4}$$

For $\hat{\beta}$ and $\hat{\gamma}$ as in Theor. 11.1, one can prove that

$$\vdash SI(\alpha, \beta, \gamma) \equiv SI(\alpha, \gamma, \beta)\,, \qquad \vdash SI(\alpha, \beta, \gamma) \supset SI(\alpha, \hat{\beta}, \hat{\gamma})\,, \tag{11.5}$$

$$\vdash SI(\alpha, \beta, \gamma) \wedge (\alpha\beta \ni_p \gamma)(\alpha\sim\beta \ni_q \gamma) \supset p = q\,. \tag{11.6}$$

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