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Piermarco Cannarsa


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# On a Maximum Principle for Elliptic Systems with Constant Coefficients.

PIERMARCO CANNARSA (*)

## 1. Introduction.

Let $\Omega \subset R^n$ be a bounded open set and let $N$ be a positive integer. Let $(\,|\,)_N$, $\|\;\|_N$ be the scalar product and the norm in $R^N$ (1). We set $D_i = \partial/\partial x_i$, $i = 1, \ldots, n$.

Let $H^1(\Omega, R^N)$ be the usual Sobolev space of vectors $u\colon \Omega \to R^N$ with norm

$$\|u\|_{H^1(\Omega,R^N)} = \left\{ \int_\Omega \|u\|^2\,dx + \int_\Omega \sum_{i=1}^n \|D_i u\|^2\,dx \right\}^{\frac12} \tag{1.1}$$

and let $H_0^1(\Omega, R^N)$ be the closure of $C_0^\infty(\Omega, R^N)$ with respect to norm (1.1).

Let $L^{2,\lambda}(\Omega, R^N)$, $0 \leqslant \lambda \leqslant n$, be the Banach space defined as follows

$$L^{2,\lambda}(\Omega, R^N) = \left\{ u \in L^2(\Omega, R^N)\colon \|u\|^2_{L^{2,\lambda}(\Omega,R^N)} = \sup_{\substack{\varrho>0\\ x\in\bar\Omega}} \varrho^{-\lambda} \int_{B(x,\varrho)\cap\Omega} \|u\|^2\,dy < +\infty \right\}$$

(here $B(x, \varrho) = \{y \in R^n\colon \|x - y\| < \varrho\}$) and

$$H^{1,(\lambda)}(\Omega, R^N) = \{u \in H^1(\Omega, R^N)\colon D_i u \in L^{2,\lambda}(\Omega, R^N),\ 1 \leqslant i \leqslant n\}$$

(*) Indirizzo dell'A.: Scuola Normale Superiore - Piazza dei Cavalieri 7 - 56100 Pisa.

(1) We shall often omit the subscript $N$ and write simply $(\,|\,)$, $\|\;\|$.

$H^{1,(\lambda)}(\Omega, R^N)$ is a Banach space with norm

$$\|u\|_{H^{1,(\lambda)}(\Omega,R^N)} = \|u\|_{L^2(\Omega,R^N)} + \sum_{i=1}^{n} \|D_i u\|_{L^{2,\lambda}(\Omega,R^N)} .$$

Let $A_{ij}(x)$ $(i, j = 1, \dots, n)$ be $N \times N$ matrices satisfying the ellipticity condition

$$(1.2) \qquad \sum_{i,j=1}^{n} \xi_i \xi_j (A_{ij}(x)\eta|\eta)_N \geq \nu \|\xi\|_n^2 \|\eta\|_N^2 , \qquad \nu > 0 , \quad \forall x \in \bar{\Omega},\ \forall \xi \in R^n,\ \forall \eta \in R^N .$$

The following regularity theorem is proved in [1] (2)

THEOREM 1.I. *Let* $\Omega \subset\subset R^n$ *be a* $C^1$ (3) *open set,* $u \in H^{1,(\lambda)}(\Omega, R^N)$, $f_i \in L^{2,\lambda}(\Omega, R^N)$ $(0 \leq \lambda < n,\ i = 1, \dots, n)$ *and let* $A_{ij}$ *be continuous in* $\bar{\Omega}$ *and satisfy* (1.2). *Then, if* $v$ *is the solution of Dirichlet problem*

$$(1.3) \qquad \begin{cases} v - u \in H_0^1(\Omega, R^N) , \\ \displaystyle\int_\Omega \sum_{i,j=1}^{n} (A_{ij}(x) D_j v | D_i \varphi)\, dx = \int_\Omega \sum_{i=1}^{n} (f_i | D_i \varphi)\, dx \qquad \forall \varphi \in H_0^1(\Omega, R^N) , \end{cases}$$

*$v$ belongs to* $H^{1,(\lambda)}(\Omega, R^N)$ *and*

$$(1.4) \qquad \|v\|_{H^{1,(\lambda)}(\Omega,R^N)} \leq C_1 \left\{ \sum_{i=1}^{n} \|f_i\|_{L^{2,\lambda}(\Omega,R^N)} + \|u\|_{H^{1,(\lambda)}(\Omega,R^N)} \right\} .$$

In this paper we prove a more specific regularity result which can be summarized as follows

THEOREM 1.II. *Let* $\Omega \subset\subset R^n$ *be a* $C^1$ *convex* (4) *open set, let* $u$ *belong to* $H^{1,(n-2)} \cap L^\infty(\Omega, R^N)$ *and let* $A_{ij}^0$ *be* $N \times N$ *constant matrices satisfying*

(2) In [1] the result is stated in the case of only one equation; it is known that it holds unchanged in the case of systems.

(3) We say that a bounded open set $\Omega \subset R^n$ is of class $C^1$ if for every point $x_0 \in \partial\Omega$ we can find an open neighbourhood $\Omega(x^0)$ and a $C^1$ homeomorfism $x \to \phi(x)$ which maps $\overline{\Omega(x^0)}$ onto $\overline{B(0, 1)}$, $\Omega(x^0) \cap \Omega$ onto the set $\{x \in B(0, 1): x_n > 0\}$ and $\Omega(x^0) \cap \partial\Omega$ onto $\{x \in B(0, 1): x_n = 0\}$.

(4) The hypothesis that $\Omega$ is convex may be eliminated.

(1.2). *Then, if $v$ is the solution of Dirichlet problem*

$$(1.5)\qquad \begin{cases} v-u\in H_0^1(\Omega, R^N)\,, \\ \int_\Omega \sum_{i,j=1}^n (A_{ij}^0 D_j v|D_i\varphi)\,dx = 0 \qquad \forall\varphi\in H_0^1(\Omega, R^N)\,, \end{cases}$$

*$v$ belongs to $H^{1,(n-2)}\cap L^\infty(\Omega, R^N)$ and*

$$(1.6)\qquad \sup_\Omega \|v\| + \sum_{i=1}^n \|D_i v\|_{L^{2,n-2}(\Omega,R^N)} \leqslant C_2\Big\{\sup_\Omega \|u\| + \sum_{i=1}^n \|D_i u\|_{L^{2,n-2}(\Omega,R^N)}\Big\}\,.$$

A trivial consequence of theorem 1.II is the following maximum principle

THEOREM 1.III. *Let $\Omega\subset\Lambda\subset\subset R^n$, be two open sets and let $\Omega$ be convex* ([4]) *and of class $C^1$. Let $u\in H^1\cap L^\infty(\Lambda, R^N)$ be such that*

$$(1.7)\qquad \begin{aligned} &D_i u\in L^{2,n-2}(\Omega, R^N)\,, \qquad 1\leqslant i\leqslant n\,, \\ &\sum_{i=1}^n \|D_i u\|_{L^{2,n-2}(\Omega,R^N)} \leqslant C_3 \sup_\Lambda \|u\|\,. \end{aligned}$$

*Then, if $v$ is the solution of Dirichlet problem* (1.5), *$v$ verifies the following inequality*

$$(1.8)\qquad \sup_\Omega \|v\| \leqslant C_4 \sup_\Lambda \|u\|\,.$$

Property (1.7) is quite usual in the study of nonlinear elliptic systems; consider, for example, the following problem.

Let $A_{ij}(x, w)$ $(i, j = 1, \ldots, n)$ be $N\times N$ bounded continuous matrices defined in $\bar\Lambda\times R^N$, satisfying the following ellipticity condition

$$(1.9)\qquad \begin{aligned} &\sum_{i,j=1}^n (A_{ij}(x, w)\xi^j|\xi^i) \geqslant \nu(K)\sum_{i=1}^n \|\xi^i\|^2\,, \qquad \nu>0\,, \\ &\forall (x, w)\in\bar\Lambda\times\{\|w\|\leqslant K\},\ \forall\xi^1, \ldots, \xi^n\in R^N\,. \end{aligned}$$

Let $f\colon \Lambda\times R^N\times R^{nN}\to R^N$ be measurable in $x\in\Lambda$ and continuous in $(w, p)$; suppose also that $f$ has quadratic growth

$$\|f(x, w, p)\|_N \leqslant a(K)\|p\|^2_{nN} + b(K)\,, \qquad \forall (x, w, p) \in \Lambda \times \{\|w\| \leqslant K\} \times R^{nN}\,. \tag{1.10}$$

Finally, let $Dw$ denote the vector $(D_1 w|\dots|D_n w)$ of $R^{nN}$.

It is then known ([3]) that every solution $u \in H^1 \cap L^\infty(\Lambda, R^N)$ of system

$$\int_\Lambda \sum_{i,j=1}^{n} (A_{ij}(x, u) D_j u|D_i\varphi)_N \, dx = \int_\Lambda (f(x, u, Du)|\varphi)_N \, dx \qquad \forall \varphi \in H^1_0 \cap L^\infty(\Lambda, R^N) \tag{1.11}$$

which satisfies the following inequality (with $M = \sup_\Lambda \|u\|$)

$$2Ma(M) < \nu(M) \tag{1.12}$$

is Hölder continuous in $\Lambda \setminus \Lambda_0$, where $\Lambda_0$ is closed in $\Lambda$ and such that $H_{n-q}(\Lambda_0) = 0$ (5) for a certain $q > 2$.

The proof given in [3] needs a boundedness result of the following kind:

*let* $u \in H^1 \cap L^\infty(\Lambda, R^N)$ *be a solution of system* (1.11) *verifying* (1.12);

*let* $A^0_{ij}$ $(i, j = 1, \dots, n)$ *be constant* $N \times N$ *matrices satisfying* (1.2);

*let* $v$ *be the solution of Dirichlet problem*

$$\begin{cases} v - u \in H^1_0(B(x_0, r), R^N) \ \textit{with} \ B(x_0, 2r) \subset\subset \Lambda \ \textit{and} \ 0 < r \leqslant 1 \\ \displaystyle\int_{B(x_0,r)} \sum_{i,j=1}^{n} (A^0_{ij} D_j v|D_i\varphi)\, dx = 0 \qquad \forall \varphi \in H^1_0(B(x_0, r), R^N)\,. \end{cases}$$

*Then*

$$v \in L^\infty(B(x_0, r), R^N) \qquad \textit{and} \qquad \sup_{B(x_0,r)} \|v\| \leqslant C_5 \sup_\Lambda \|u\| \tag{1.13}$$

*where* $C_5$ *does not depend on* $x_0$ *and* $r$.

(5) $H_\alpha$, $\alpha \geqslant 0$, is the $\alpha$-dimensional Hausdorff measure.

In order to get (1.13), the proof of [3] recalls the maximum principle proved in [2], which depends on the possibility of representing $v$ by adequate potentials.

In section 3 we prove that (1.13) may be obtained in a simpler way, showing that $u$ verifies the hypotheses of Theorem 1.III.

This method can be extended to more general situations, such as elliptic systems of order $2m \geqslant 2$, even with continuous coefficients, and $C^1$ bounded open sets $\Omega$ not necessarily convex.

I would like to thank S. Campanato for the useful discussions we had on this subject.

## 2. Proof of Theorem 1.II.

Having fixed $y \in \Omega$, we define

$$d = \operatorname{dist}(y, \partial\Omega) = \|y - z\| \qquad \text{with } z \in \partial\Omega\,.$$

As $v$ solves problem (1.5) and $A^0_{ij}$ are constant, the following inequality holds ([1], Lemma 7.I)

$$\int_{B(y,\varrho)} \|v\|^2\,dx \leqslant C(\nu)\left(\frac{\varrho}{d}\right)^n \int_{B(y,d)} \|v\|^2\,dx \qquad \forall 0<\varrho\leqslant d\,. \tag{2.1}$$

On the other hand

$$\int_{B(y,d)} \|v\|^2\,dx \leqslant \int_{B(z,2d)\cap\Omega} \|v\|^2\,dx \leqslant \leqslant C(n)\Big[d^n \sup_{\Omega} \|u\|^2 + \int_{B(z,2d)\cap\Omega} \|v-u\|^2\,dx\Big]\,. \tag{2.2}$$

As $v - u \in H^1_0(\Omega, R^N)$, Poincaré inequality is valid ([4]):

$$\int_{B(z,2d)\cap\Omega} \|v-u\|^2\,dx \leqslant C(n)\,d^2 \int_{B(z,2d)\cap\Omega} \sum_{i=1}^n \|D_i(v-u)\|^2\,dx \;(^6)\,. \tag{2.3}$$

(6) As $\Omega$ is convex the constant $C(n)$ does not depend on $y$ (in general we shall write $C(n, \nu, \ldots)$ to mean a constant that depends on the algebraic data $n, \nu, \ldots$).

From (2.1), (2.2) and (2.3) we get

$$(2.4)\qquad \frac{1}{\varrho^n}\int_{B(y,\varrho)}\|v\|^2\,dx\leqslant$$

$$\leqslant C(n,\nu)\left[\sup_{\Omega}\|u\|^2+d^{2-n}\int_{B(z,2d)\cap\Omega}\sum_{i=1}^{n}\|D_i(v-u)\|^2\,dx\right].$$

Theorem 1.I implies that

$$v\in H^{1,(n-2)}(\Omega,R^N)$$

and

$$(2.5)\qquad \|v\|_{H^{1,(n-2)}(\Omega,R^N)}\leqslant C_1\|u\|_{H^{1,(n-2)}(\Omega,R^N)}\,.$$

Combining (2.4) and (2.5) we prove (1.6) and the theorem.

## 3. Application to quasilinear systems.

Let $\Lambda\subset R^n$ be a bounded open set. Let $A_{ij}(x,u)$ $(1\leqslant i,j\leqslant n)$ be $N\times N$ bounded continuous matrices defined in $\bar\Lambda\times R^N$, satisfying the ellipticity condition

$$(3.1)\qquad \sum_{i,j=1}^{n}(A_{ij}(x,u)\xi^j|\xi^i)\geqslant\nu(K)\sum_{i=1}^{n}\|\xi^i\|^2,\qquad \nu>0\,,$$

$$\forall(x,u)\in\bar\Lambda\times\{\|u\|\leqslant K\},\ \forall\xi^1,\dots,\xi^n\in R^N.$$

Let $f:\Lambda\times R^N\times R^{nN}\to R^N$ be measurable in $x\in\Lambda$, continuous in $(u,p)$ and with quadratic growth

$$(3.2)\qquad \|f(x,u,p)\|_N\leqslant a(K)\|p\|^2_{nN}+b(K)\,,$$

$$\forall(x,u,p)\in\Lambda\times\{\|u\|\leqslant K\}\times R^{nN}.$$

Let us consider the quasilinear system in divergence form

$$(3.3)\qquad -\sum_{i,j=1}^{n}D_i\big(A_{ij}(x,u)\,D_ju\big)=f(x,u,Du)\,,\qquad \text{in }\Lambda\,.$$

The following lemma can be deduced from a « Caccioppoli inequality » proved in [3].

LEMMA 3.I. *Let* $u \in H^1 \cap L^\infty(\Lambda, R^N)$ *be a weak solution of system* (3.3) *satisfying the following inequality* $\left(\text{with } M = \sup_\Lambda \|u\|\right)$

$$(3.4) \qquad Ma(M) < \nu(M)\,.$$

*Then* $u \in H^{1,(n-2)}_{\text{loc}}(\Lambda, R^N)$ *and for every ball* $B(x^0, r) \subset B(x^0, 2r) \subset \Lambda$

$$(3.5) \qquad \sum_{i=1}^{n} \|D_i u\|_{L^{2,n-2}(B(x^0 r), R^N)} \leqslant C' \sup_\Lambda \|u\|$$

*where* $C'$ *depends on* $M$, *but neither on* $r$ *nor on* $x^0$.

PROOF. As $u \in H^1 \cap L^\infty(\Lambda, R^N)$ is a weak solution of (3.3)

$$(3.6) \qquad \int_\Lambda \sum_{i,j=1}^{n} \left(A_{ij}(x, u) D_j u | D_i \varphi\right) dx = \int_\Lambda \left(f(x, u, Du) | \varphi\right) dx$$

$$\forall \varphi \in H^1_0 \cap L^\infty(\Lambda, R^N)\,.$$

Having fixed $y \in \overline{B(x^0, r)}$ and $0 < \sigma < r/2$, we choose $\theta \in C^\infty_0(B(y, 2\sigma))$ with $0 \leqslant \theta \leqslant 1$, $\theta = 1$ in $B(y, \sigma)$ and $\|D\theta\| \leqslant 2/\sigma$.

If we substitute $\varphi = \theta^2 u$ in (3.6), we get as in [3] the following « Caccioppoli inequality »:

$$\int_{B(y,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2\, dx \leqslant C(\nu) \left\{\frac{1}{\sigma^2} \int_{B(y,2\sigma)} \|u\|^2\, dx + [b(M)]^2 \sigma^{n+2}\right\}.$$

Hence, if $\sigma$ is such that

$$[b(M)]^2 \sigma^4 \leqslant \sup_\Lambda \|u\|^2$$

we get

$$\int_{B(y,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2\, dx \leqslant C' \sigma^{n-2} \sup_\Lambda \|u\|^2\,.$$

This proves (3.5) and the lemma.

REMARK 3.I. Let $u \in H^1 \cap L^\infty(\Lambda, R^N)$ be as in Lemma 3.I and consider a ball $B(x^0, r) \subset B(x^0, 2r) \subset \Lambda$, $0 < r \leqslant 1$. Let $A^0_{ij}$ $(i, j = 1, \ldots, n)$

be $N \times N$ constant matrices satisfying the ellipticity condition

$$\sum_{i,j=1}^{n} \xi_i \xi_j (A^0_{ij}\eta|\eta)_N \geqslant \nu \|\xi\|_n^2 \|\eta\|_N^2 , \qquad \forall \xi \in R^n, \ \forall \eta \in R^N .$$

Let $v$ be the solution of the following Dirichlet problem

$$\begin{cases} v - u \in H^1_0(B(x_0, r), R^N) , \\ \int\limits_{B(x_0,r)} \sum_{i,j} (A^0_{ij} D_j v | D_i \varphi)\, dx = 0 \qquad \forall \varphi \in H^1_0(B(x_0, r), R^N) . \end{cases}$$

From Lemma 3.I and Theorem 1.III we draw the conclusion that

$$\sup_{B(x_0,r)} \|v\| \leqslant C^* \sup_{\Lambda} \|u\| .$$

Moreover, $C^*$ does not depend on $x^0$ and $r$.
The last statement can be shown by a homothetical argument.

## REFERENCES

[1] S. Campanato, *Equazioni ellittiche del II ordine e spazi* $\mathfrak{L}^{2,\lambda}$, Ann. Matem. Pura e Appl., **69** (1965).

[2] A. Canfora, *Teorema di massimo modulo e teorema di esistenza per il problema di Dirichlet relativo ai sistemi fortemente ellittici*, Ricerche di Mat., **15** (1966).

[3] M. Giaquinta - E. Giusti, *Nonlinear elliptic systems with quadratic growth*, Manuscr. Mathem., **24** (1978).

[4] G. Stampacchia, *Sur les espaces de fonctions qui interviennent dans le problèmes aux limites elliptiques*, Colloque sur l'Analyse Fonctionnelle, C.B.R.M., Louvain (1960).

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