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A Note on Products of Infinite Cyclic Groups.

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Introduction.

In his book [2], Fuchs introduces the notion of a subgroup X of a Specker group P being a *product* and goes on to establish a Lemma [2, Lemma 95.1] which yields a useful characterization of the quotient P/X and enables an easy derivation of Nunke's characterization of epimorphic images of the Specker group [4]. Unfortunately this Lemma is incorrect as we show in section 1. In section 2 by suitably strengthening the hypothesis we regain a characterization of the quotient. Throughout, all groups are additively written Abelian groups and our notation follows the standard works of Fuchs [1], [2].

§ 1. Suppose $P = \prod_{n=1}^{\infty} \langle e_n \rangle$ is a Specker group, then Fuchs defines a subgroup X of P to be a *product* $\prod_{n=1}^{\infty} \langle x_n \rangle$ if for every m, the m-th coordinates of almost all x_n are 0 and X consists of all the formal sums $\sum s_n x_n$. To avoid confusion with the usual meaning of product (i.e. X is a product if it is isomorphic to a cartesian product of infinite cyclic groups) we denote a *product* (in the sense of Fuchs) by $\sum^*$ and reserve the symbol $\prod$ for the more usual meaning.

LEMMA 1. If Y is an endomorphic image of P then Y is a *product* (in the sense of Fuchs).

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Proof. Let $\alpha: P \to P$ be an endomorphism with $\operatorname{Im}\alpha = Y$. Let π_n denote the projection of P onto $\langle e_n \rangle$, and set, for each n, $y_n = e_n\alpha$. Since $\langle e_n \rangle$ is slender, the map $\alpha\pi_n: P \to \langle e_n \rangle$ maps almost all e_i to zero i.e. $y_i\pi_n = e_i\alpha\pi_n = 0$ for all but a finite number of indices i. Thus for every n, the n-th co-ordinates of almost all y_i are zero and so the set of sums $\left\{ \sum s_i y_i \right\}$ is a *product* $\sum_{i=1}^{\infty}{}^{*}\langle y_i \rangle$ in P.

Now define $\beta: P \to P$ by $(\dots, n_i e_i, \dots)\beta = \sum n_i y_i$. For each $n = 1, 2, \dots, \alpha\pi_n$ and $\beta\pi_n$ map P into a slender group and agree on $S = \bigoplus_{i=1}^{\infty} \langle e_i \rangle$. Hence $\alpha\pi_n = \beta\pi_n$ for all n. So $\alpha = \beta$ and $Y = \operatorname{Im}\alpha = \operatorname{Im}\beta = \sum_{i=1}^{\infty}{}^{*}\langle y_i \rangle$, and thus Y is a *product* (in the sense of Fuchs).

Counter-example. With $P = \prod_{i=1}^{\infty} \langle e_i \rangle$, set $Y = \prod_{i=1}^{\infty} \langle 2^i e_i \rangle$. Then $P/Y \cong \prod_{i=1}^{\infty} \mathbb{Z}(2^i)$ and this is a complete module over the ring J_2 of 2-adic integers. Moreover the torsion submodule of this quotient is not dense in the 2-adic topology. Hence it has a direct summand $H \cong J_2$ and if $\langle x \rangle$ is dense in H then $H/\langle x \rangle$ is divisible. Choose $y \in P$ such that y maps onto x modulo Y and let $X = \langle y, Y \rangle$. Then certainly X is isomorphic to P and hence is an endomorphic image of P. By Lemma 1 X is a *product* (in the sense of Fuchs). However $P/X \cong \prod_{i=1}^{\infty} \mathbb{Z}(2^i)/\langle x \rangle$ which contains the divisible subgroup $H/\langle x \rangle$. However if the conclusion of Lemma 95.1 in [2] were correct then P/X would be reduced. So X is clearly a counter-example to the quoted Lemma.

Acknowledgement. The above arguments arose from interesting discussions with Peter Neumann and Adolf Mader. The main idea in the counter-example is essentially due to the former.

§ 2. In this section by introducing an appropriate topological concept we can regain some information about quotients. Let $P = \prod_{i=1}^{\infty} \langle e_i \rangle$ and topologize P with the product topology of the discrete topology on each component. We refer to this topology simply as the product topology on P. The subgroups $P_n = \prod_{i=n}^{\infty} \langle e_n \rangle$ are a basis of neighbourhoods of zero.

Proposition 2. If X is a subgroup of P which is closed in the product topology then

(i) X is a *product* $\sum_{i=1}^{\infty} {}^*\langle x^i\rangle$;

(ii) P/X is isomorphic to a cartesian product of cyclic groups.

Proof. Part (i) is a well-known result due to Nunke [3]. He shows that there are elements x^n in X with (a) $x_i^n = 0$ for $i < n$; (b) $x_n^n = 0$ if and only if $x^n = 0$; (c) x_n^n divides u_n for all u in $X \cap P_n$. (Subscripts denote components in the product P.) Moreover if X is closed, $X = \sum {}^*\langle x^n\rangle$.

In establishing (ii) we let $d_n = x_n^n$ in order to simplify notation. Notice that it follows easily from the properties (a), (b), (c) that if $X = \sum {}^*\langle y^n\rangle$ also, then $y_n^n = d_n$ and $x_{n+1}^n - y_{n+1}^n$ is a multiple of d_{n+1}.

Suppose $a \in P$ is given by $a = (a_1, a_2, \dots)$ then we may write

$$a_1 = r_1 d_1 + s_1 \quad \text{where} \quad 0 \leqslant s_1 < d_1 .$$

Also

$$a_2 - r_1 x_2^1 = r_2 d_2 + s_2 \quad \text{where} \quad 0 \leqslant s_2 < d_2 ,$$

$$a_3 - r_1 x_3^1 - r_2 x_3^2 = r_3 d_3 + s_3 \quad \text{where} \quad 0 \leqslant s_3 < d_3 , \qquad \text{etc.}$$

Define a map φ from P onto the cartesian product of the cyclic groups of order d_i by $\varphi(a) = (s_1, s_2, \dots, s_n, \dots)$. We must verify that φ is a well-defined homomorphism. Suppose $X = \sum {}^*\langle y^n\rangle$ then since $y_1^1 = x_1^1 = d_1$ we get that r_1 and s_1 are uniquely defined.

Now $a_2 - r_1 y_2^1 = a_2 - r_1(x_2^1 + kd_2) = (r_2 - r_1 k) d_2 + s_2$ (some $k \in \mathbb{Z}$) and so s_2 is defined as before. Note that $x^1 - y^1 - kx^2 \in X$ and so by property (c) $x_3^1 - y_3^1 - kx_3^2$ is a multiple of d_3. Making this substitution one easily obtains that $a_3 - r_1 y_3^1 - (r_2 - r_1 k) y_3^2 \equiv s_3 \bmod d_3$ and so s_3 is defined as before. Repeating this type of argument easily gives that φ is well defined. Moreover φ is easily seen to be a homomorphism.

Finally $\operatorname{Ker}\varphi = \{a \in P | s_1 = s_2 = \dots = 0\}$ i.e. if $a \in \operatorname{Ker}\varphi$ then

$$a_1 = r_1 d_1$$

$$a_2 = r_2 d_2 + r_1 x_2^1$$

$$a_3 = r_3 d_3 + r_2 x_3^2 + r_1 x_3^1 , \qquad \text{etc.}$$

i.e. $a = \sum_{i=1}^{\infty} r_i x^i$ and so $\operatorname{Ker}\varphi = X$.

Hence $P/X \cong \prod \mathbb{Z}(d_i)$ where $\mathbb{Z}(d_i)$ is to be interpreted as $\mathbb{Z}$ if $d_i = 0$.

Given Proposition 2 one can easily recover the characterization of homomorphic images of P (Nunke [4] or Fuchs [2, Prop. 95.2]).

COROLLARY 3. Every epimorphic image of P is the direct sum of a cotorsion group and a direct product of infinite cyclic groups.

PROOF. Let K be a subgroup of P and let $\bar{K}$ be the closure of K in the product topology. From Proposition 2, $\bar{K}$ is a product, say $\bar{K} = \sum_{i=1}^{\infty} {}^* \langle x_i \rangle$ and $P/\bar{K}$ is a product of cyclic groups. Let $P = P_1 \oplus P_2$ where P_1, P_2 are the products of the $\langle e_n \rangle$ with $d_n \neq 0$ and $d_n = 0$ respectively. Then $\bar{K} \leqslant P_1$ and $P_1/\bar{K}$ is algebraically compact since it is a product of finite cyclic groups. Since $\bigoplus_{i=1}^{\infty} \langle x_i \rangle$ is contained in K, the quotient $\bar{K}/K$ is cotorsion and this combined with $P_1/\bar{K}$ being cotorsion implies P_1/K is also cotorsion [1, 54 (D)].

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