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On Some Algebraic and Geometric Extension of the Theory of Adjoints.

LUCA CHIANTINI (*)

In the classical literature there are various definitions of adjoint divisor to a plane curve over an algebraically closed field (see [B-N], [Go], [K], [W]). In [G-V] there is an ample discussion on the relations between the concepts of adjoint to a curve on a smooth surface with an investigation on the conductor sheaf and on effective passage through neighbouring points.

In the present paper we extend the idea of adjoint to a more general situation. We consider an excellent curve on a regular surface without any assumption on the base field (it may be not algebraically closed, or even missing). Our investigation shows that all the classical definitions may be easily given and compared. This is more or less the content of the first section where we prove that some relations established in [G-V] again hold, but in this general case we have two ways to define the order of a branch, which give rise to two distinct definitions of adjoint. Only the « algebraic » way leads to a « good » definition.

The second section is concerned with the permanence of adjonction in some class of morphisms; more precisely we consider a faithfully flat morphism $\varphi: Y \to X$ of regular surfaces and the curve $\varphi^*(D)$

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on Y. We can show that if φ is normal at a point $P \in D$ then a divisor H is adjoint (of any kind excluded the « wrong » one) to D at P if and only if $\varphi^*(H)$ is such to $\varphi^*(D)$ at every point of $\varphi^{-1}(P)$.

Preliminaries.

1) A scheme X is « excellent » if it has an open cover of affine subsets $\{U_i\}$ where $U_i = \operatorname{Spec} A_i$ and A_i is an excellent ring (for properties of excellent rings see [M] ch. XIII).

The « points » of X are the closed points.

2) A scheme X is a surface if it is locally noetherian and has dimension 2; when X is a regular surface, an effective Cartier divisor of X is called a « curve on X » if its associated closed subscheme is reduced. Note that a regular surface need not be smooth, i.e. it is not necessarily geometrically regular over a base field, unless it is perfect.

We consider only excellent curves on regular surfaces.

3) Words as « singular points », « normalization », « blowing up », etc. are used in the sense of [G-V]. We recall that the « conductor sheaf » of a curve D is the sheaf $\operatorname{Ann}_{O_{\bar{D}}}(O_{\bar{D}}/O_D)$, where $\bar{D}$ is the normalization.

4) It is well known that given any excellent curve D on a regular surface, there is a chain of blowing up's of X along singular points of D, of finite length and such that the induced chain on D: $D = D_0 \to D_1 \to \dots \to D_n = \bar{D}$ normalizes the curve. A point which belongs to some strict transform of D in this chain is called « neighbouring point of D » while a point of D is called also an « actual » point. A point Q of the normalization of D which lies over the actual point $P \in D$ is called « place » over P. Since the local ring of Q in $\bar{D}$ is a D.V.R., we have a canonical valuation associated with Q. If P_i is the center of the blowing up of D_i, the set of points of D_{i+1} lying over P_i is called « first neighbour of P_i in D_i ». If Q is a place over P, a chain $P = P_0 - P_1 - \dots - P_s = Q$, where $\forall i\ P_{i+1}$ is in the first neighbour of P_i, is called « branch of D with center P ». For more details, see [G-V], § 2, or [V], § 4.

5) The «multiplicity» of a Cartier divisor H at a point P of X is 0 if $P \notin H$, otherwise it is the algebraic multiplicity of the local ring of H at P (see [Z-S], vol. 2, ch. VIII, p. 294). If P is a *non* actual point of X, the multiplicity of H at P is the multiplicity at P of a strict transform of H on a strict transform of X where P is actual.

I) DEFINITIONS OF ADJOINT DIVISOR

Through this section X is always a regular surface, D is an excellent curve on X, H is an effective Cartier divisor of X and P an actual point of D.

1. Order of a banach.

Le (A, m) be the local ring of D at P; it is well known that the places of D with center P correspond one-to-one both to the maximal ideals of the normalization $\bar{A}$ of A and to the minimal prime ideals of the completion $A^\wedge$ of A (see [G₁], Th. 2.1).

DEFINITION 1.1. *Let Q be a place of D over P, n the corresponding maximal ideal of $\bar{A}$ and $\mathfrak{p}$ the corresponding minimal prime of $A^\wedge$. We define*:

1) *«Algebraic order» of the branch $P-Q$ the number*

$$e_{\bar{A}_n}(m\bar{A}_n) = l_{\bar{A}_n}(\bar{A}_n/m\bar{A}_n)\,.$$

2) *«Analytic order» of the branch $P-Q$ the number* $e(A^\wedge/\mathfrak{p})$. ($e =$ *multiplicity*, $l =$ *length, see* [Z-S], *vol.* 2, *ch.* VIII).

PROPOSITION 1.2. *Let $P-Q$ be a branch of D, z_1 its algebraic order and z_2 its analytic order. Then $z_1 \leqslant z_2$ and the equality holds if and only if $K(P) = K(Q)$ ($K(\cdot) =$ residue field).*

PROOF. We write B for $\bar{A}_n$ and C for $A^\wedge/\mathfrak{p}$. It is well known that $B^\wedge$ is isomorphic to $\bar{C}$ (see [G-V], Remark at p. 6). B is a D.V.R., hence principal; put $\bar{n} = nB = (x)$ and $mB = (x)^q = (x^q)$; we have:

$l_B(B/mB) = q$ and $q = z_1$; moreover $B^\wedge$ is local, its maximal ideal is $xB^\wedge$ hence:

$$mB^\wedge = x^q B^\wedge = (xB^\wedge)^q \quad \text{and} \quad l_{B^\wedge}(B^\wedge/mB^\wedge) = q = z_1 .$$

On the other hand we have: $z_2 = e(C) = e_C(mC)$ (see [Z-S], vol. 2, p. 294).

Now, $\bar{C} = B^\wedge$ is a local domain and $m\bar{C} = x^q B^\wedge$ is $nB^\wedge$-primary, thus we may apply the equation 8′ at p. 300 of [Z-S] getting:

$$z_2 = e_C(mC) = [B^\wedge/nB^\wedge : C/mC]\, e_{B^\wedge}(mB^\wedge) = \varepsilon z_1 ,$$

where $\varepsilon = [B^\wedge/nB^\wedge : C/mC]$ is the dimension of $B^\wedge/nB^\wedge = B/nB = K(Q)$ as a $C/mC = A/m = K(P)$ vector space.

Since obviously $\varepsilon \geqslant 1$, we have $z_2 \geqslant z_1$ and $\varepsilon = 1$ if and only if $K(Q) = K(P)$.

If $K(P)$ is algebraically closed, then automatically $K(P) = K(Q)$. The previous proposition is well known in the classical theory of projective curves over an algebraically closed field. It has been also more or less proved in [C].

We give an example of a branch such that $z_1 \neq z_2$.

EXAMPLE 1.3. Let X be the affine plane over the real field $\mathbb{R}$, D the curve defined by $x^2 + y^2$. It is easy to check that D has multiplicity 2 at $P \equiv (0, 0)$. The local ring $O_{D,P}$ is $(\mathbb{R}[x, y]/(x^2 + y^2))_{(x,y)}$ and it is isomorphic to $A = \mathbb{R}[T, iT]_{(T,iT)}$.

We have $\bar{A} = \mathbb{C}[T]_{(T)}$ and $A^\wedge = \mathbb{R}[[T, iT]]$ ($\mathbb{C}$ = complex field). Since $A^\wedge$ is a domain, we have only a place Q of D over P, which corresponds to the null ideal of $A^\wedge$ and to the maximal ideal of $\bar{A}$.

Look at the orders of the branch $P - Q$. We have:

$$z_1 = l_{\bar{A}}(\bar{A}/m\bar{A}) = l_{\bar{A}}(\mathbb{C}) = 1 ,$$

while, using the Jordan-Hölder equality (see [Z-S], vol. 1, p. 160), a straightforward computation shows that $z_2 = 2$.

2. Adjoint divisors.

Through this section we discuss various definitions of adjoint divisor to a curve D, extending to the case of an excellent curve on a regular surface concepts which are classically well known for algebraic curves over an algebraically closed field.

First we start with «local definitions», involving a single singular point P of D; then we shall try to globalize our concepts.

DEFINITION 2.1. i) *H is «A1 to D at P» (P actual point of D) if it passes through every r-fold actual or neighbouring point of D lying over P with multiplicity $r-1$ (at least). If the multiplicity of H at every such point is exactly $r-1$, we say that H is a «special adjoint (AS)» to D at P.*

ii) *H is «A2 to D at P» if its local equations in $O_{D,P}$ belongs to the stalk of the conductor sheaf of the curve at P.*

iii) *Let $\Gamma(P = P_0 - P_1 - \ldots - P_n = Q)$ be a branch of D with center P. We say that the number $d_Q = \sum (s_i - 1) r_i$, where $s_i = e_{P_i}(D)$ and $r_i =$ algebraic (resp. analytic) order of the branch $P_i - Q$, is the «algebraic (resp. analytic) coefficient of the branch».*

We define on the normalization $\bar{D}$ of D the «algebraic (resp. analytic) divisor of double points of D over P» as $\sum d_Q Q$ where Q ranges over the places of D with center P.

iv) *We say that H is «algebraic (resp. analytic) A3 to D at P» if and only if for every branch $P - Q$ with center P, we have: $v_Q(h) \geqslant d_Q$, where $v_Q =$ valuation associated with Q, $h =$ local equation of H in $O_{D,P}$ and $d_Q =$ algebraic (resp. analytic) coefficient of the branch $P - Q$.*

PROPOSITION 2.2. *If H is A1 to D at P, then it is A2 to D at P. If D is desingularizable at P with one blowing up, then also the converse holds.*

PROOF. The proof of the first statement is faithfully equal to the proof of [G-V] Th. 4.3 i), we only need to make a suitable use of Prop. 3.11 of [G-V].

To see the converse, put $(A, m) =$ local ring of D at P; then by our hypothesis and by Prop. 3.12 of [G-V], the conductor of A is m^{s-1} where $s = e_P(D)$. The claim follows easily from the definitions.

There are divisors A2 but no A1 to a curve D at a point P even in the case of the affine complex plane. See [G-V], Ex. 4.6.

The next statement easily follows by Prop. 1.1.

PROPOSITION 2.3. *If H is A3An to D at P, then it is A3Alg to D at P. If for every place Q of D over P we have $K(Q) = K(P)$ (e.g. if $K(P)$ is algebraically closed) then also the converse holds.*

EXAMPLE 2.4. We give an example of a divisor which is A3Alg but not A3An to a curve at a point. We use the curve of Ex. 1.3. If Q is the (unique) place of D over $P \equiv (0, 0)$, then the algebraic divisor of double points of D over P is Q itself, while the analytic one is $2Q$. Now, it is immediate that the divisor H defined by $x-y$ is A3Alg but not A3An to D at P.

THEOREM 2.5. *H is* A2 *to D at P if and only if it is* A3Alg *to D at P.*

PROOF. Put $(A, m) = O_{D,P}$ and let $(\bar{A}, n_1, \ldots, n_k)$ be its normalization; put $\gamma_A =$ conductor of A and $h =$ local equation of H at P.

Since $\bar{A}_{n_i}$ is a D.V.R., we have:

$$\gamma_A \bar{A}_{n_i} = (n_i \bar{A}_{n_i})^{a_i}$$

moreover by [G-V] 3.12:

$$\gamma_A = \prod m_r^{s_r - 1} \bar{A}$$

where the m_r's are the maximal ideals corresponding to the center of some blowing up in the desingularization of D and $s_r = e(m_r)$. It follows:

$$\gamma_A \bar{A}_{n_i} = \prod m_{r,i}^{s_{r,i}-1} \bar{A}_{n_i}$$

where $m_{r,i} =$ ideal of the r-th point of the branch $P - Q_i$ (Q_i is the place corresponding to n_i). Again by the fact that $\bar{A}_{n_i}$ is a D.V.R.:

$$a_i = l(\bar{A}_{n_i} / \prod m_{r,i}^{s_{r,i}-1} \bar{A}_{n_i}) = \sum (s_{r,i} - 1) l(\bar{A}_{n_i} / m_{r,i} \bar{A}_{n_i}) = d_i$$

the algebraic coefficient of the branch $P - Q_i$, hence:

$$\gamma_A \bar{A}_{n_i} = (n_i \bar{A}_{n_i})^{d_i}.$$

If H is A2 to D at P, then $h \in \gamma_A$, hence $\forall i$ the image h_i of h in $\bar{A}_{n_i}$ belongs to $(n_i \bar{A}_{n_i})^{d_i}$; this is just as to say that $v_i(h) \geqslant d_i$, where v_i is the valuation associated with Q_i.

Conversely, if H is A3Alg to D at P, then we have $\forall i$:

$$h_i \in (n_i \bar{A}_{n_i})^{d_i} = \gamma_A \bar{A}_{n_i}$$

and by [B], p. 111-112, h belongs to $\gamma_A \bar{A} = \gamma_A$.

LEMMA 2.6. *Let A be a non normal local ring with finite normalization $\bar{A}$; then the conductor γ_A is not contained in any principal ideal generated by a regular element.*

PROOF. If $\gamma_A \subseteq aA$, where a is a regular element, then $\forall x \in \gamma_A$, $x = ax'$ and by definition of conductor we have $(ax')\bar{A} \subseteq \gamma_A$; since normalization preserves regular elements, a is regular in $\bar{A}$ too; it follows that $x'\bar{A} \subseteq \gamma_A$ thus $x' \in \gamma_A$ and $x' = aA$. Repeating the argument, we obtain $x \in \bigcap a^n A = (0)$ that is $\gamma_A = (0)$, absurd.

PROPOSITION 2.7. *Let X be a regular surface and P a point of X such that $K(P)$ is infinite. Let D be a curve on X through P and H a divisor which is A2 to D at P. Then there are divisors $H_1, \ldots, H_t$ which are AS to D at P and such that if $h, h_1, \ldots, h_t$ are local equations in $O_{D,P}$ for them, then $h = \sum h_i$.*

PROOF. Put $(A, m) = O_{D,P}$ and let $A = A_0 \to A_1 \to \ldots \to A_n = \bar{A}$ be a desingularization of A; put $m_i =$ center of the blowing up $A_i \to$ $\to A_{i+1}$, $r_i = e_{m_i}(A_i)$ and $\forall i$ choose a regular element $x_i \in A_i$ such that $m_i A_{i+1} = x_i A_{i+1}$ (it is possible, see [Mt], Th. 12.2).

By [G-V] 4.11 our claim is proved if we show by induction on the length of the desingularization, that we have $h = \sum h_j$ (finite sum) where h is a local equation of H at P and the h_j's are elements of A such that:

i) $h_j \in m_0^{r_0-1} - m_0^{r_0}$;

ii) $\forall s \leqslant n-2$, $h_j / x_0^{r_0-1} \ldots x_s^{r_s-1} \in m_{s+1}^{(r_{s+1}-1)} - m_s^{r_{s+1}}$.

The step $n = 0$ is obvious and the step $n = 1$ follows at once by Prop. 2.2, thus suppose $n > 1$. By our assumption we have (see [G-V], 3.12): $h \in \gamma_A = x_0^{r_0-1}\gamma_{A_1}$ and $h = x_0^{r_0-1}h'$ where $h' \in \gamma_{A_1}$, hence by induction $h' = \sum h'_j$ where the h'_j's fulfil conditions i) and ii). If $\forall j$ $h'_j \notin x_0 A_1$ we put $h_j = x_0^{r_0-1}h'_j$ and we are done.

Assume $h'_1 \in x_0 A_1$. By Lemma 2.6 there is $f' \in \gamma_{A_1} - x_0 A_1$ and by induction again, write $f' = \sum f'_j$ where the f'_j's fulfil conditions i) and ii). At least one of the f'_i's, say f'_1, does not belong to $x_0 A_1$.

Put $h'_{1,i} = h'_1 / x_1^{r_1-1} \ldots x_i^{r_i-1}$ and $z_{1,i} =$ initial form of $h'_{1,i}$ in the associated graded ring $gr_{m_i}(A_i)$; similarly we define the elements $f'_{1,i}$ and their initial forms $w_{1,i}$; since $\forall i$, $gr_{m_i}(A_i)$ ia an A/m vector space, using [G-V] 4.12 we find $a' \in A/m$ such that $\forall i$, $a' w_{1,i} \neq z_{1,i}$. Lift a' to an element $a \in A$ then put $g'_1 = af'_1$. Now, $h'_1 - g'_1$ and g'_1 both fulfil condition i) and ii), do not belong to x_0A_1 and they can replace h_1'. Repeating the argument we see that we may suppose $\forall j$, $h_j \notin x_0A_1$, hence we are done.

3. Global statements.

We can restate our theory in a global form simply defining H adjoint of any kind *to* D if and only if it is such at every singular point of the curve. The following proposition is an obvious consequence of the statements of the previous section.

PROPOSITION 3.1. i) *If* H *is* A1 *to* D, *then it is* A2 *to* D. *The converse holds when only a blowing up at every singular point is needed to normalize* D.

i) H *is* A2 *to* D *if and only if it is* A3Alg *to* D.

iii) *If* H *is* A3An *to* D *then it is also* A3Alg *to* D; *the converse holds if for every singular point* P *of* D *and every place* Q *of* D *over* P *we have* $K(P) = K(Q)$.

The following example shows that Prop. 2.7 is not true in a global form.

EXAMPLE 3.2. Put $Y = \mathbb{A}^2_{\mathbb{C}}$ and let C be a cuspidal cubic curve on it. Let X denote the regular surface obtained glueing two copies Y_1 and Y_2 of Y along $Y - C$ (we use the glueing of [H], p. 80, Ex. 2.12; namely we obtain an affine complex plane with the curve C « doubled »). Let C_1 and C_2 be the two unglued copies of C; let C' be the image on X of another cuspidal cubic curve on Y, whose cusp is a regular point of C. Put $D = C' + C_1$ (sum of divisors, i.e. local product of the equations), thus D is a curve on X and it is singular at the cusp P_1 of C_1 (double point), at the cusp P_2 of C' lying over C_1 (triple point) and at the other cusp of C' (double point).

We have the two following easy facts:

a) every divisor of X comes from both a divisor of Y_1 and of Y_2;

b) if a divisor H of X does not contain C_1, then $e_{P_2}(H) \leqslant e_{P_3}(H)$.

A divisor AS to D must pass twice through P_2 and only once through P_3 thus by *b*) a divisor which does not contain C_1 cannot be AS to D while for every divisor H which contains C_1 we have

$e_{P_1}(H) \geqslant 2$ hence no divisor is AS to D. On the other hand there are divisors A2 to D and this shows that the divisors of Prop. 2.7 cannot in general be chosen globally AS to the curve.

This Example however is rather pathological: we have a non-separated surface and there is not an affine open subset of X containing all the singular points of D.

With a slight modification of the proof of Prop. 2.7 one can easily check that the following statement holds:

If D is an excellent curve on a regular surface X and:

a) there is an open affine subset of X which contains all the singular points of D;

b) the residue fields at the points of $\operatorname{Sing} D$ are infinite;
then for every divisor H A2 to D there are divisors $H_1, \ldots, H_t$ globally AS to D which satisfy the claim of 2.7 at every point of $\operatorname{Sing} D$.

(To prove it, start with the semilocal ring A of the singular points of D then follow the proof of 2.7, using the fact that the desingularization trees of two distinct points do not overlap.)

II) ADJOINTS AND MORPHISMS

Our purpose is to look for morphisms which preserve adjoint divisors. We begin pointing out some (already known) algebraic result.

LEMMA 1.1. *Let A, B be reduced local rings and $\varphi\colon A \to B$ a faithfully flat morphism; let γ_A, γ_B be the conductors. Then:*

i) $\gamma_B \cap A \subseteq \gamma_A$;

ii) *if moreover $\bar{A}$ is finite over A and $\bar{B} = \bar{A} \otimes_A B$ then $\gamma_B = \gamma_A B$ hence $\gamma_B \cap A = \gamma_A$.*

PROOF. i) Let $h \in \gamma_B \cap A$; then $h\bar{A} \subseteq h\bar{B} \cap \bar{A} \subseteq B \cap \bar{A}$ but by [N] 18.4 we have $B \cap \bar{A} = A$.

ii) $\gamma_A B = \gamma_A \otimes_A B = (A : \bar{A}) \otimes_A B$ and by [N] 18.1, this is equal to $(A \otimes_A B) : (\bar{A} \otimes_A B) = \gamma_B$; using i) we see that $\gamma_B \cap A = \gamma_A$.

REMARK 1.2. If φ is normal and faithfully flat, then $\bar{A} \otimes_A B = \bar{B}$ (see [G_2], Prop. 1). If $\dim A = \dim B \leqslant 1$ and φ is faithfully flat and reduced, then its fibers are reduced artinian hence it is also regular.

Lemma 1.3. *Let* (A, m), (B, n) *be reduced local rings and* $\varphi\colon A \to B$ *a faithfully flat morphism such that* $mB = n$. *Then* $e(A) = e(B)$.

Remark. The hypothesis $mB = n$ is satisfied if $\dim A = \dim B$ and φ is faithfully flat and reduced, namely in this case, by the going down theorem, the fiber over m is reduced and artinian.

Proof. φ is dominant, injective and by [N] 19.1 the theorem of transition holds for A and B, thus, since $mB = n$, we have $l_B(B/n^r) = l_B(B/m^rB) = l_B(B/mB)\,l_A(A/m^r) = l_A(A/m^r)$ and A and B have the same Hilbert function.

If $\varphi\colon Y \to X$ is a morphism of schemes and F is an O_X-module we define φ^*F as the O_Y-module $O_Y \otimes_{\varphi^{-1}O_X} (\varphi^{-1}F)$ (see [H], p. 110). On the stalks it corresponds to consider $\forall Q \in Y$ the $O_{Y,Q}$-module $O_{Y,Q} \otimes_{O_{X,P}} F_P$, where $P = \varphi(Q)$.

If F is an O_X-ideal we may also consider the O_Y-ideal $FO_Y = (\varphi^{-1}F)O_Y$ which is defined by F through the canonical morphism $\varphi^{-1}O_X \to O_Y$ (see [H], p. 163); on the stalks it corresponds to consider the extended ideal $F_P O_{Y,Q}$ in the canonical morphism $O_{X,P} \to O_{Y,Q}$.

There is a canonical morphism $\varphi^*F \to FO_Y$ and it is obvious that when φ is flat, $\varphi^*F = FO_Y$.

If H is a divisor and φ is faithfully flat, then $\varphi^*(H)$ is still a divisor, namely our previous argument shows that its stalk is generated at every point $Q \in Y$ by the image of a local equation of H at $\varphi(Q)$.

Lemma 1.4. *Let* $\varphi\colon Y \to X$ *be a flat morphism of locally noetherian schemes,* I *a quasi-coherent* O_X*-ideal. Put:*

$\tilde{X}$ = *blowing up of* X *along* I; $\tilde{Y}$ = *blowing up of* Y *along* $J = \varphi^*I$.
Then $\tilde{Y} \simeq Y \times_X \tilde{X}$.

Proof. Look at the canonical diagram

$$\begin{tikzcd} Y' \arrow[r, "\varphi'"'] \arrow[d] & \tilde{X} \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

where $Y' = Y \times_X X$. Since it is commutative, we have: $JO_{Y'} = (IO_Y)O_{Y'} = (IO_X)O_{Y'}$. Since φ and φ' are flat, we have: $\varphi^* I = I \otimes O_Y = IO_Y$ and similarly $\varphi'^* IO_{\tilde{X}} = (IO_{\tilde{X}})O_{Y'}$; since $IO_{\tilde{X}}$ is invertible, $\varphi'^* IO_{\tilde{X}}$ is invertible too (see [S], § 2).

Let Z be another scheme with a morphism $\omega: Z \to Y$ such that JO_Z is invertible; then IO_Z is invertible and by the universal property of the blowing up we have a morphism $\sigma: Z \to X$ and a commutative diagram

$$\begin{tikzcd}
 & \tilde{X} \times_X Y \arrow[r] \arrow[d] & \tilde{X} \arrow[d] \\
Z \arrow[urr, "\sigma"] \arrow[r, "\omega"'] & Y \arrow[r] & X
\end{tikzcd}$$

which, by the universal property of the fibred product, is closed with a morphism $\psi: Z \to \tilde{X} \times_X Y$. This works for every such Z, hence $X \times_X Y$ fulfils the universal property of the blowing up.

Using again the universal property of the blowing up, it is easy to check the following

LEMMA 1.5. *Let X be a locally noetherian scheme and let $I_1, \dots, I_n$ be a finite collection of pairwise comaximal O_X-ideal. If X is the blowing up of X along $\prod I_i$ and $X \xrightarrow[f_1]{} X_1 \to \dots \xrightarrow[f_n]{} X_n$ is a chain of blowing up*'s, *where f_i is a blowing up along I_i, then $X = X_n$.*

2. Ascent and descent of adjoint divisors.

Through this section X and Y are regular surfaces, D is an excellent curve on X, H is an effective Cartier divisor of X, $\varphi: X \to Y$ is a faithfully flat morphism such that $D' = \varphi^* D$ is still reduced (hence a curve on Y) and excellent. Let φ_D be the induced map $D' \to D$.

We shall show that A1 and A2 are stable under faithfully flat normal morphisms while A2 descends if the morphism is only faithfully flat.

REMARK 2.1. We are strongly interested in the case in which φ or φ_D are normal at a point $P \in D$ (that is, when $\forall Q \in \varphi^{-1}(P)$, the fibers of the morphism $O_{X,P} \to O_{Y,Q}$ are geometrically normal and the morphism is flat).

If $\forall Q \in \varphi^{-1}(P)$, $\dim O_{D,P} = \dim O_{D',Q}$ then, as in Remark 1.2, φ_D has artinian fibers at P, hence if it is reduced, it is also normal (even regular). Note that, since $O_{D,P}$ is the quotient of $O_{X,P}$ by a regular element, we have $\dim O_{D,P} = \dim O_{X,P} - 1$ and the same happens to $O_{D',Q}$; it follows that the equality $\dim O_{D,P} = \dim O_{D',Q}$ holds if and only if $\dim O_{X,P} = \dim O_{Y,Q}$ (hence it does not depend on the divisor D but only on the point and on the surface).

By faithful flatness, we have always $\dim O_{X,P} \leqslant \dim O_{Y,Q}$ hence when $\dim O_{X,P} = 2$ the equality holds, however a surface may have points P of codimension less than two (take for example on $X = \operatorname{Spec} K[\![x]\!][y]$ the point associated with the maximal ideal $(1 - xy)$).

We have the two following basic situations:

i) $\dim O_{D,P} = \dim O_{D',Q}$; if φ_D is reduced at P, it is automatically normal (even regular).

ii) $\dim O_{D,P} = 0$ and $\dim O_{D',Q} = 1$; if φ_D normal is needed, it is not enough to require φ_D reduced. In fact in this case $K = O_{D,P} =$ $=$ field, $A = O_{D',Q} =$ 1-dimensional geometrically reduced K-algebra, and it is well known that A may be not geometrically regular (see also Ex. 3.4).

When $\dim O_{D,P} = 0$, P is a component of D (not embedded since D is reduced) and since a reduced subscheme of X must be regular at the generic point of every component, P is a regular point of D; in any case if D has such a point, it has a component of dimension 0 and this shows that $\dim O_{D,P} \neq \dim O_{D',Q}$ may happen only in a somewhat pathological situation.

Observe also that when $\dim O_{X,P} = 2$, then the fiber of φ over P is artinian, hence $\varphi^{-1}(P)$ is a finite set of points (never empty by faithful flatness).

THEOREM 2.2. *Let $\varphi^*(H)$ be* A2 *to D' at a point $Q \in \varphi^{-1}(P)$; then H is* A2 *to D at P. If moreover φ_D is normal at P, then the following are equivalent*:

a) *H is* A2 *to D at P*;

b) *$\varphi^*(H)$ is* A2 *to D' at every point $Q \in \varphi^{-1}(P)$*;

c) *$\varphi^*(H)$ is* A2 *to D' at a point $Q \in \varphi^{-1}(P)$*.

PROOF. Let h be a local equation for H in $O_{D,P}$ and γ_P and γ_Q be the conductors of $O_{D,P}$ and $O_{D',Q}$ respectively. By our hypothesis and by Lemma 1.1 $h \in \gamma_Q \cap O_{D,P}$ hence $h \in \gamma_P$ and the first claim follows.

To prove the second, we only need to show that $a) \Rightarrow b)$, but if $h \in \gamma_P$ again by Lemma 1.1, for every $Q \in \varphi^{-1}(P)$ h belongs to the conductor of $O_{D',Q}$.

REMARK 2.3. Let D be singular at P (hence by Remark 2.1, $\dim O_{D,P} = 1$), and suppose φ_D normal at P. Take a desingularization of D at P: $D = D_0 \to D_1 \to \ldots \to D_n = \bar{D}$ where $P_0, \ldots, P_n$ are the centers of the blowing up's. Put $D'_i = D_i \times_D D'$ and look at the induced chain: $D' = D'_0 \xrightarrow[f_1]{} D'_1 \to \ldots \xrightarrow[f_n]{} D'_n$. By Lemma 1.4 each f_i is a blowing up along the finite set of points of D'_i lying over P_i, then by Lemma 1.5 each f_i may be viewed as a product of consecutive blowing up's, everyone along a point, and we obtain a «refined» chain which is a desingularization of D' at $\varphi^{-1}(P)$, namely the canonical morphism φ_n: $D'_n \to D_n$ is normal moreover Lemma 1.3 tells us that we blow up only singular points of D'.

LEMMA 2.4. *Every neighbouring point Q of D' is actual in some D'_i.*

PROOF. Same notation as in the previous Remark. Q is actual in some curve of the «refined» chain, say Q actual in D^0 and let D^0 lie between D'_i and D'_{i+1}, that is, we have a chain $D'_i \xrightarrow[g_1]{} D^0 \xrightarrow[g_2]{} D'_{i+1}$ where $g_1 \circ g_2 = f_i$. g_1 and g_2 are blowing up's along finite sets of points, say T_1 and T_2 moreover by construction every point of T_2 is actual in D'_i. Now, if $Q \notin T_2$ it is actual in D'_{i+1}, otherwise it is actual in D'_i.

THEOREM 2.6. *If the induced morphisms φ_H: $\varphi^*(H) \to H$ and φ_D are normal at P, then the following are equivalent*:

a) *H is* A1 (*resp.* AS) *to $\bar{A}$ at P*;

b) *$\varphi^*(H)$ is* A1 *to D'* (*resp.* AS *to D'*) *at every point $Q \in \varphi^{-1}(P)$*;

c) *$\varphi^*(H)$ is* A1 *to D'* (*resp.* AS *to D'*) *at a point $Q \in \varphi^{-1}(P)$*.

PROOF. If $\dim O_{X,P} = 1$ then D is normal at P and the claim is trivially true. Thus suppose $\dim O_{X,P} = 2$. Let Q' be an s-fold point of a branch of D' with center Q. By Lemma 2.4 we may suppose Q' actual in some D'_i. Put $P' = \varphi_i(Q')$ where φ_i: $D'_i \to D_i$ is the canonical morphism. By Lemma 1.3 and Remark 2.3, P' is an s-fold point of a branch of D with center P, hence $e_{P'}(H) \geqslant s - 1$ (resp. $e_{P'}(H) = s - 1$) and by the same argument $e_{Q'}(\varphi^*(H)) = e_{Q'}(\varphi_i^*(H)) = e_{P'}(H)$; this shows that $a) \Rightarrow b)$.

$b) \Rightarrow c)$ is obvious

$c) \Rightarrow a)$ If $P' \in D_i$ is an s-fold point of a branch of D with center P, then by Remark 2.3 there is a point $Q' \in \varphi_i^{-1}(P')$ which is an s-fold point of a branch of D' with center Q; thus $e_{Q'}(\varphi^*(H)) \geqslant \geqslant s-1$ (resp. $e_{Q'}(\varphi^*(H)) = s-1$) but again $e_{P'}(H) = e_{Q'}(\varphi^*(H))$.

REMARK. When $\dim O_{X,P} = 2$, by Remark 2.1, the hypothesis « φ_D and φ_H normal » are satisfied if φ is reduced.

DEFINITION 2.7. *We say that H is a «true adjoint» (TA) to D at P if its local equation h in $O_{D,P}$ generates the conductor γ_P of $O_{D,P}$* (see [A] or [O]).

PROPOSITION 2.8. *If H is* TA *to D' at a point $Q \in \varphi^{-1}(P)$ then it is* TA *to D at P. If moreover φ_D is regular at P, then the following are equivalent*:

a) H is TA *to D at P;*

b) $\varphi^*(H)$ is TA *to D' at every point $Q \in \varphi^{-1}(P)$;*

c) $\varphi^*(H)$ is TA *to D' at a point $Q \in \varphi^{-1}(P)$.*

PROOF. If $hO_{D,P} = \gamma_P$ then $hO_{D',Q} = \gamma_P O_{D',Q} = \gamma_Q$ (by Lemma 1.1) and this proves the first claim; it remains to show that $a) \Rightarrow b)$; in this case we have $hO_{D',Q} = \gamma_Q$ hence by faithful flatness we have $hO_{D,P} = hO_{D',Q} \cap O_{D,P} = \gamma_Q \cap O_{D,P} = \gamma_P$ by Lemma 1.1 again.

Obviously, the behaviour of A3Alg under our morphism is the behaviour of A2; we study the behaviour of A3An.

PROPOSITION 2.9. *Let $P - \bar{P}$ be a branch of D and $Q \in \varphi^{-1}(P)$. Let $\bar{Q}_1, \dots, \bar{Q}_t$ be the places of D' with center Q, lying over $\bar{P}$ in the canonical morphism $\bar{D}' \to \bar{D}$ induced by φ_D, and suppose φ_D normal at P. Then: analytic order of $P - \bar{P} = \Sigma_t$ analytic order of $Q - \bar{Q}_t$.*

PROOF. Put $\delta =$ local equation of D in $O_{X,P}$; then $P - \bar{P}$ corresponds to a prime factor δ' of δ in $\hat{O}_{X,P} = R$ (which is local and regular, hence U.F.D.) while the branches $Q - \bar{Q}_1, \dots, Q - \bar{Q}_t$ correspond to the prime factors $\delta'_1, \dots, \delta'_t$ of δ' in $R' = \hat{O}_{Y,Q}$.

The analytic order of $P - \bar{P}$ is $e(R/(\delta'))$ and since the morphism $R/(\delta) \to R'/\delta R'$ is normal (it is the completion of the normal morphism $O_{D,P} \to O_{D',Q}$ and $O_{D',Q}$ is excellent) we have that also the morphism $R/(\delta') \to R'/\delta' R'$ is normal, and by Lemma 1.3 and [Z-S], p. 294 $e(R/(\delta')) = e(R'/\delta' R') = \sum_i e(R'/\delta'_i)$, but note that $\forall i$, $e(R'/\delta'_i)$ is just the analytic order of the branch $Q - \bar{Q}_i$.

COROLLARY 2.10. *Let γ_D, P, $\bar{P}$, Q be as above and $\bar{Q}$ be a place of D' with center Q lying over $\bar{P}$. Then:*

$$\textit{analytic coefficient of } P - \bar{P} \geqslant \textit{analytic coefficient of } Q - \bar{Q}.$$

PROOF. Write down the two branches $P = P_0 - P_1 - \dots - P_n = \bar{P}$ and $Q = Q_0 - Q_1 - \dots - Q_n = \bar{Q}$ where $\forall i$, Q_i lies over P_i (see Remark 2.3), then we have $e_{P_i}(D) = e_{Q_i}(D')$ and by the previous proposition analytic order of $Q_i - \bar{Q} \leqslant$ analytic order of $P_i - \bar{P}$ hence our claim follows by the definitions.

THEOREM 2.11. *If φ_D is normal at P and H is* A3An *to D at P, then $\varphi^*(H)$ is* A3An *to D' at every point $Q \in \varphi^{-1}(P)$.*

PROOF. Put $O_{D,P} = (A, m)$, $O_{D',Q} = (B, n)$. Let $\bar{Q}$ be a place of D' with center Q and put (B', n') = local ring of $\bar{Q}$ on the normalization of D', (A', m') = local ring of the place $\bar{P}$ of D with center P over which $\bar{Q}$ lies; put d_1 = analytic coefficient of $P - \bar{P}$ and d_2 = analytic coefficient of $Q - \bar{Q}$, h = local equation of H in A.

By our assumption, $h \in (m')^{d_1}$ hence $h \in (m'B')^{d_1}$ but by the regularity of the morphism $A' \to B'$, $m'B = n'$ and by Corol. 2.10 $d_1 \geqslant d_2$ hence $h \in (n')^{d_2}$. This just means that $v(h) \geqslant d_2$, where v is the valuation associated with $\bar{Q}$, and the claim follows.

Example 3.6 shows that the converse of this theorem does not hold in general.

We give a global version of the main resulties of this section.

THEOREM 2.12. i) *If $\varphi^*(H)$ is* A2 *to $\varphi^*(D)$, then H is* A2 *to D.*

ii) *If φ_D is normal, then the following are equivalent:*

a) *H is* A2 (*resp.* TA) *to D,*

b) *$\varphi^*(H)$ is* A2 (*resp.* TA) *to $\varphi^*(D)$.*

Moreover if H is A3An *to D, then $\varphi^*(H)$ is* A3An *to $\varphi^*(D)$.*

iii) *If φ_D and φ_H are normal, then* ii) *holds with* A1 *or* AS *instead of* A2.

REMARK 2.13. Our statements work even if $\varphi\colon Y \to X$ is only flat and $P \in \varphi(Y)$, namely in this case we may restrict our attention to $O_{X,P}$.

3. Examples.

3.1. If D is an excellent curve on a regular surface and P is a point of D, then the canonical morphism $\varphi: A \to A^\wedge$ is regular and faithfully flat; hence our theorems show that to text whether or not a divisor is A2 or A1 to D at P, we may work in $O^\wedge_{D,P}$.

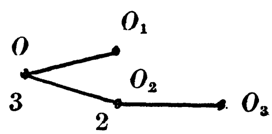
3.2. A flat NR_2 morphism of ring (see [G_3], it is the case for instance, of absolutely flat morphisms, hence etale morphisms too) is regular, thus if φ is moreover faithfully flat, our statements work in this situation.

As a consequence: if K is a field and L is a separable extension of K, for every K-algebra A, the morphism $\varphi: A \to A \otimes_K L$ is faithfully flat and regular. In particular, if K is a perfect field and L is its algebraic closure, we may pass from $\mathbb{P}^2_K$ to $\mathbb{P}^2_L$ to text if a divisor is adjoint to a curve.

3.3. Let φ be the automorphism of $\mathbb{A}^2_{\mathbb{C}}$ induced by the inclusion of $\mathbb{C}[x, t]$, where $t = y^2$ in $\mathbb{C}[x, y]$. $\mathbb{C}[x, y]$ is free, hence faithfully flat over $\mathbb{C}[x, t]$.

Clearly every divisor is A1 or A2 or A3An to the smooth curve $D \equiv x^3 - t$ but this is no longer true for $\varphi^*(D) \equiv x^3 - y^2$ and this shows that adjoints of any kind are not stable under φ.

Take now the curve $D \equiv xt^2 + x^4 + t^6$ and put $H \equiv x^7$. Then $O \equiv \equiv (0, 0)$ is a triple point of D with the following normalization tree:



and H does not pass through O_2 hence it is not A1 to D at O. On the other hand, we have $D' = \varphi^*(D) \equiv xy^4 + x^6 + y^{14}$ and the singularity in O is solvable with one blowing up, hence $\varphi^*(H)$ is A1 to D' at O; this shows that A1 does not descend through faithfully flat morphism.

Note that $\varphi^*(H)$ is also A2 to D' at O, hence H is A2 to D at O by Theorem 2.1 (this is not obvious at a first sight).

3.4. If φ_D has discrete fibers and is reduced, then it is also normal, but the following example shows that the hypothesis « φ_D reduced » does not suffice if points of strange codimension are involved.

Put $X = \mathrm{Spec}\,(K[x, y] \oplus K[u^2 - v^3])$ and $Y = \mathrm{Spec}\,(K[x, y] \oplus K[u, v])$; the canonical morphism $\varphi\colon Y \to X$ is faithfully flat, indeed $K[u, v]$ is free over $K[u^2 - v^3]$. The curve D defined on X by $(1, u^2 - v^3)$ is regular, but its inverse image is the cuspidal cubic curve on a copy of $\mathbb{A}^2_K$, hence it is singular at the origin. It is clear that φ_D is not normal even if it is reduced and every divisor A2 to D' is also A2 to D but there are divisors A2 to D, whose inverse image is not A2 to D'.

3.5. In the three-space there are non flat morphisms (see [M], p. 24) from a smooth surface onto a plane, which send regular curves to non regular ones. Obviously these morphisms do not preserve adjoints of any kind.

3.6. We show that in general A3An does not descend even through faithfully flat regular morphisms of surfaces; namely in *I*, Example 1.3 we provided a divisor H A3Alg but not A3An to a curve D on $\mathbb{A}^2_{\mathbf{R}}$; passing to $\mathbb{A}^2_{\mathbf{C}}$, by Theorem 2.2 $\varphi^*(H)$ is A3Alg to D', but note that in $\mathbb{A}^2_{\mathbf{C}}$ A3An$\Leftrightarrow$A3Alg.

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