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Existence of Extremal Solutions and Comparison Results for Delay Differential Equations in Abstract Cones.

A. S. VATSALA and R. L. VAUGHN (*)

1. - Introduction.

The study of Cauchy problem for ordinary differential equations in a Banach space has been extensive [2, 4, 10]. It is of interest to look at the corresponding problem for delay differential equations since such equations occur in many physical problems. Existence of solutions of such equations are considered in [8, 9, 6] using monotonicity conditions and dissipative conditions.

In this paper our objective is to prove the existence of extremal solutuions for the delay differential equation

$$x'(t) = f(t, x, x_t), \qquad x_{t0} = \varphi_0 \tag{1.1}$$

relative to a cone k of the Banach space E. For this purpose, we begin by proving an existence result under a simple set of conditions without assuming uniform continuity on f, we then develop needed theory of differential inequalities and utilize this to show the existence of extremal solutions for (1.1). Several useful comparison theorems are then proved including a flow invariance result. Our results generalize some of the recent results for equations without delay [5, 7].

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2. – Preliminaries and definitions.

Let $\tau > 0$ be a given real number and let E be a Banach space with norm $\|\cdot\|$. Let $\mathcal{C} = C[[-T, 0], E]$ denote the Banach space of continuous functions with the norm of $\varphi \in \mathcal{C}$ given by

$$\|\varphi\|_0 = \max_{-\tau \leqslant s \leqslant 0} \|\varphi(s)\| .$$

If $t_0 \in R^+$ and $x \in C[[t_0 - \tau, \infty), E]$, then for any $t \in [t_0, \infty)$, we let $x_t \in \mathcal{C}$ be defined by

$$x_t(s) = x(t+s) , \qquad -\tau \leqslant s \leqslant 0 .$$

Also let

$$\mathcal{C}_\varrho = \{\varphi \in \mathcal{C}\colon \|\varphi\|_0 < \varrho\} .$$

Similarly if $A \subset C[[t_0 - \tau, \infty), E]$ and for $J \subset [t_0, \infty)$ we will let

$$A_J = \{x_t\colon x \in A, t \in J\} .$$

To establish existence criteria for the Cauchy problem (1.1) we require that f satisfies a compactness condition. The compactness condition for this paper will be given in terms of Kurtowski measure of noncompactness α. The measure of noncompactness $\alpha(s)$ is defined by

$$\alpha(s) = \inf\{d > 0\colon s \text{ can be covered by a finite number of sets of diameter } d\}$$

for each bounded subset of E. We denote $\alpha_E, \alpha_B, \alpha_{\mathcal{C}}$ to denote the Kurtowski measure of noncompactness defined relative to the Banach spaces E, B and $\mathcal{C}$ respectively.

A cone k is a proper subset of E such that if $v, \omega \in k$, $\lambda \in R^+$ then $v + w$, $\lambda v \in k$. Throughout this paper we will consider a closed cone k and its interior k^0. These cones induce orderings on E defined by

$$x \leqslant y \quad \text{if } y - x \in k$$

$$x < y \quad \text{if } y - x \in k^0 .$$

For some of the fundamental properties of α and the cone k see [2, 3, 4, 10].

Let k^* be the set of all continuous linear functionals c on E such that $c(x) \geqslant 0$ for all $x \in k$, and let k_0^* be the set of all continuous linear functionals c on E such that $c(x) > 0$ for $x \in k^0$.

A function $f \in C[J \times E \times \mathcal{C}, E]$ is said to be quasimonotone nondecreasing in x for fixed t, φ if $x \leqslant y$ and $c(x) = c(y)$ for $c \in k_0^*$ then $c(f(t, x, \varphi)) \leqslant c(f(t, y, \varphi))$.

A function $f \in C[J \times E \times \mathcal{C}, E]$ is said to be monotone nondecreasing in φ for fixed t, x, if for $c \in k_0^*$ $\varphi_1(s) \leqslant \varphi_2(s)$ then $c(f(t, x, \varphi_1)) \leqslant$ $\leqslant c(f(t, x, \varphi_2))$.

A function $f \in C[J \times E \times \mathcal{C}, E]$ is said to satisfy one sided Lipschitz's condition if there exists an $L > 0$ such that

$$f(t, x, \varphi_1) - f(t, y, \varphi_2) \leqslant L(x - y) + \sup_{s \in [-\tau, 0]} \{\varphi_1(s) - \varphi_2(s)\}$$

whenever $x \geqslant y$ and $\varphi_1 \geqslant \varphi_2$.

The function $f(t, x, \varphi)$ is said to be quasinonnegative if $x \geqslant 0$, $\varphi \geqslant 0$, $c(x) = 0$ for $c \in k_0^*$ implies $c(f(t, x, \varphi)) \geqslant 0$. If in the above definition if the inequalities are reversed then f is said to be quasinonpositive.

In what follows when we say $x_t \leqslant y_t$ we mean $x_t(s) \leqslant y_t(s)$, $-\tau \leqslant s \leqslant 0$.

A closed set $F \subset E$ is said to be flow invariant relative to the system (1.1) if for every solution $x(t)$ of (1.1), we have

$$x_{t0} \in F\,, \quad \text{implies } x(t) \in F \text{ for } t \geqslant t_0\,.$$

We state below the Darbo fixed point theorem [1] and Mazur's theorem [5] which are needed in our existence theorem and on the results on differential inequalities respectively.

THEOREM 2.1 (Darbo). Let E be a Banach space and A be a closed, bounded convex, nonvoid subset of E. If $T \in C[A, A]$ is such that $\alpha(T(B)) \leqslant k\alpha(B)$ where $k < 1$ for each bounded subset B of A, then T has a fixed point.

THEOREM 2.2 (Mazur's). Let k be a cone with nonempty interior k_0. Then

(i) $x \in k$ is equivalent to $c(x) \geqslant 0$ for all $c \in k^*$,

(ii) $x \in \partial k$ implies that there exists a $c \in k_0^*$ such that $c(x) = 0$.

3. – Existence.

In this section we prove the existence of a solution of the Cauchy problem (1.1)

THEOREM 3.1. Let $f \in C[J \times E \times \mathcal{C}_\varrho, E]$ where $J = [t_0, t_0 + a]$, $a > 0$ and suppose the following conditions are satisfied.

(a_1) $\|f(t, x, \varphi)\| \leqslant M$ for all $t, x, \varphi \in J \times E \times \mathcal{C}_\varrho$

(a_2) $\alpha_E(f(J \times A_1 \times A_2) \leqslant \beta \max\left(\alpha_E(A_1), \alpha_{\mathcal{C}}(A_2)\right)$

where $\beta > 0$ and A_1 and A_2 are any bounded subsets of E and $\mathcal{C}$ respectively. Then given an initial function $\varphi_0 \in \mathcal{C}_\varrho$ at $t = t_0 \geqslant 0$ there exists a $\gamma > 0$ such that there is a solution $x(t, t_0, \varphi_0)$ of (1.1) existing on $[t_0 - \tau, t_0 + \gamma)$.

PROOF. Define $y \in C\big[[t_0 - \tau, t_0 + a], E\big]$ as follows:

$$y(t) = \begin{cases} \varphi_0(t - t_0) & t_0 - \tau \leqslant t \leqslant t_0 \\ \varphi_0(0) & t_0 \leqslant t \leqslant t_0 + a\,. \end{cases}$$

Then $f(t, y, y_t)$ is a continuous function of t on $[t_0, t_0 + a]$ and hence $\|f(t, y, y_t)\| \leqslant M$ by (a_1). We can show [11] that there exists a constant

$$b \in \big(0, \varrho - \|\varphi_0(0)\|\big)$$

such that

$$\|f(t, x, \psi) - f(t, y, y_t)\| \leqslant 1$$

whenever $t \in [t_0, t_0 + a]$, $x_t = \psi \in \mathcal{C}_\varrho$ and $\|\psi - y_t\|_0 \leqslant b$.

If now follows that $\|f(t, x, \psi)\| \leqslant M + 1$ whenever $t \in [t_0, t_0 + a]$, $x_t = \psi \in \mathcal{C}_\varrho$ and $\|\psi - y_t\|_0 \leqslant b$. Choose $\gamma = \min\big(a, (b/M + 1), 1\big)$.

Let B denote the space of continuous functions from $[t_0 - \tau, t_0 + \gamma]$ into E. For an element $x \in B$ define the norm

$$\|x\|_0 = \max_{t_0 - \tau \leqslant t \leqslant t_0 + \gamma} \|x(t)\|\,.$$

Then B is a Banach space with respect to this norm. Let $S \subset B$ be defined as follows

$$S = \begin{cases} x \in B & \text{(i) } x(t) = \varphi_0(t-t_0) \qquad t_0-\tau \leqslant t \leqslant t_0 \\ & \text{(ii) } \|x(t_1)-x(t_2)\| \leqslant M|t_1-t_2| \quad t_1, t_2 \in [t_0, t_0+\gamma]\,. \end{cases}$$

clearly S is closed, bounded, convex and equicontinuous.

We now define a mapping on S as follows. For an element $x \in S$, let

$$\text{(i) } T(x_{t0}) = \varphi_0$$

$$\text{(ii) } T(x(t)) = \varphi_0(0) + \int_{t_0}^{t} f(s, x(s), x_s)\, ds, \quad t_0 \leqslant t \leqslant t_0+\gamma.$$

If $x \in S$, $T(x_{t0}) = \varphi_0$

$$\|(Tx)(t_1) - T(x)(t_2)\| \leqslant \int_{t_1}^{t_2} \|f(s, x(s), x_s)\|\, ds \leqslant M|(t_2-t_1)| \quad t_2 > t_1\,.$$

Consequently $TS \subseteq S$ and T is bounded.

To show T is continuous let $\{x_n\} \subseteq S$ be a sequence converging to x. By the continuity of f we have

$$f(t, x_n(t), x_{n_t}) \to f(t, x(t), x_t)\,.$$

Further $\|f(t, x_n(t), x_{n_t}\| \leqslant M$. By applying the bounded convergence theorem $Tx_n \to Tx$.

Now let $\Sigma \subset S$. Then using the properties of α for $t > t_0$,

$$\begin{aligned} \alpha_E(T\Sigma(t)) &= \alpha_E\Big(\Big\{\varphi_0(0) + \int_{t_0}^{t} f(s, x(s), x_s)\, ds : x \in \Sigma\Big\}\Big) \\ &\leqslant \alpha_E\Big(\Big\{\int_{t_0}^{t} f(s, x(s), x_s)\, ds : x \in \Sigma\Big\}\Big) \\ &\leqslant |t-t_0|\,\alpha_E\Big(\overline{\text{co}}\, f\big([t_0, t], \Sigma([t_0, t]), \Sigma_{[t_0,t]}\big)\Big) \\ &\qquad \leqslant |t-t_0| \max\Big\{\alpha_B\big(\Sigma([t_0, t]), \alpha_{\mathcal{C}}(\Sigma_{[t_0,t]})\big)\Big\}\,. \end{aligned}$$

Since S is equicontinuous and $\Sigma \subset S$, then Σ is also equicontinuous.

Furthermore $\Sigma_{[t_0,t]}$ is an equicontinuous subset of $\mathcal{C}$. Hence we have

$$\alpha_{\mathcal{C}}(\Sigma_{[t_0,t]}) = \alpha_E\Big(\Sigma_{[t_0,t]}([-\tau, 0])\Big) = \sup_{\sigma\in[-\tau,0]} \alpha_E(\Sigma_{[t,t_0]}(\sigma)) .$$

But $\Sigma_{[t_0,t]}(\sigma) = \Sigma([t_0 + \sigma, t + \sigma])$, and so

$$\alpha_{\mathcal{C}}(\Sigma_{[t_0,t]}) \leqslant \alpha_B\Big(\Sigma([t_0 - \tau, t])\Big) = \alpha_E\Big(\Sigma([t_0, t])\Big) .$$

We may then write

$$\alpha_E(T\Sigma(t)) \leqslant |t - t_0|\alpha_B\Big(\Sigma([t_0, t])\Big) \leqslant \tfrac{1}{2}\alpha_E\Big(\Sigma([t_0, t])\Big) .$$
$$\leqslant \tfrac{1}{2}\alpha_B(\Sigma)$$

Consequently,

$$\alpha_B(T\Sigma) = \sup_{t\in[t_0,t]} \alpha_E(T\Sigma(t) \leqslant \tfrac{1}{2}\alpha_B(\Sigma)) .$$

By the Darbo fixed point Theorem there exists a fixed point x of T. Such a fixed point is a solution of (1.1).

4. – Differential inequalities.

In this section we develop the theory of differential inequalities which is used as a tool to prove the existence of extremal solutions and a comparison result. All the inequalities are relative to a cone k of the Banach space E.

THEOREM 4.1. Let $f \in C[J \times E \times \mathcal{C}, E]$ (where $J = (t_0, \infty)$) and f is quasimonotone nondecreasing in x for fixed (t, φ) and monotone nondecreasing in φ for fixed (t, x). Let $x, y \in C[[t_0 - \tau, \infty), E]$ and

$$x_{t_0} < y_{t_0} .$$

Assume further that

$$D_- x(t) \leqslant f(t, x(t), x_t) \tag{4.1}$$

$$D_- y(t) \geqslant f(t, y(t), y_t) \tag{4.2}$$

for $t \in (t_0, \infty)$. Then

$$x(t) < y(t) \qquad t \in [t_0, \infty)$$

provided one of the above inequalities is taken strict.

PROOF: Suppose that the assertion of the Theorem is false. Then there exists a $t_1 > t_0$ such that $y(t_1) - x(t_1) \in \partial k$ and $y(t) - x(t) \in k_0$, $t \in [t_0 - \tau, t_1)$. Thus by Mazur's Theorem, there exists a $c \in k_0^*$ with $c\big(y(t_1) - x(t_1)\big) = 0$. Setting $m(t) = c[y(t) - x(t)]$, we see that $m(t) > 0$ for $t_0 - \tau \leqslant t < t_1$ and $m(t_1) = 0$. Consequently $D_- m(t_1) \leqslant 0$.
Also

$$(4.3) \qquad \begin{aligned} D_- m(t_1) &= C[D_- y(t_1) - D_- x(t_1)] \\ &> C[f(t_1, y(t_1), y_{t_1}) - f(t_1, x(t_1), x_{t_1})] \end{aligned}$$

by (4.1), (4.2). Using the quasimonotonicity of f in x and monotonicity of f in φ it follows that

$$(4.4) \qquad c\big(f(t_1, y(t_1), y_{t_1})\big) \geqslant c\big(f(t_1, x(t_1), x_{t_1})\big) .$$

Now (4.3), (4.4) together imply

$$D_- m(t_1) > 0 .$$

Hence a contradiction and the theorem is proved.

REMARK 4.1. The above result is valid if in the inequalities (4.1), (4.2) D_- is replaced by any other diniderivative. See [11] for details.

REMARK 4.2. The conclusion of the above theorem is not valid if one of the inequalities in (4.1), (4.2) is not taken strict. However the conclusion is valid if further f satisfies a onesided Lipschitz condition.

THEOREM 4.2. Let $f \in C[J \times E \times \mathrm{C}, E]$ satisfy the assumptions of Theorem 4.1. Further let f satisfy the onesided Lipshitz's condition. Then $x_{t_0} \leqslant y_{t_0}$ together with (4.1) (4.2) imply $x(t) \leqslant y(t)$, $t \in J$.

PROOF. Consider $\tilde{x}(t) = x(t) - \varepsilon \exp[3L(t - t_0)]$ where $\varepsilon \in k^0$, certainly $\tilde{x}(t) < x(t)$, $\tilde{x}_t < x_t$ for $t \geqslant t_0$.

Also consider

$$D_-\tilde{x}(t) = D_- x(t) - 3L\,\varepsilon \exp[3L(t-t_0)]$$
$$\leqslant f(t, x, x_t) - 3L\,\varepsilon \exp[3L(t-t_0)]$$

by (4.1). Using the onesided Lipschitz's condition it follows that

$$f(t, x, x_t) \leqslant f(t, \tilde{x}, \tilde{x}_t) + 2L\,\varepsilon \exp[3L(t-t_0)]\,.$$

Now we can conclude that

$$D_-\tilde{x}(t) < f(t, \tilde{x}, \tilde{x}_t) \quad \text{and} \quad \tilde{x}_{t_0} < x_{t_0} \leqslant y_{t_0}\,, \tag{4.5}$$

Now applying Theorem 4.1 to (4.5) and (4.2), it follows that

$$\tilde{x}(t) < y(t)\,. \tag{4.6}$$

Taking the limit as $\varepsilon \to 0$ in (4.6) we get the required result.

5. – Existence of extremal solution.

In this section we prove the existence of maximal solution of 1.1 only. The existence of minimal solutions can be proved on similar lines.

THEOREM 5.1. Let $f \in C[J \times E \times \mathcal{C}_\varrho, E]$ and suppose the assumptions (a_1) and (a_2) of Theorem 3.1 are satisfied and also further $f(t, x, \varphi)$ is quasimonotone nondecreasing in x for fixed (t, φ) and monotone nondecreasing in φ for fixed (t, x) then given an initial function $\varphi_0 \in \mathcal{C}_\varrho$ at $t = t_0 \geqslant 0$ there exists a $\gamma > 0$ such that there is a maximal solution $r(t, t_0, \varphi_0)$ of (1.1) existing on $[t_0 - \tau,\ t_0 + \gamma)$.

PROOF. Consider the delay differential equation

$$x'(t) = f(t, x, x_t) + \frac{1}{n}\,y_0\,, \qquad x_{t_0} = \varphi_0 + \frac{1}{n}\,y_0 = \varphi_0^n \tag{5.1}$$

where $y_0 \in k^0$ and $\|y_0\| = 1$ for $n = 1, 2, 3, \ldots$.

The solutions of (5.1) for $n = 1, 2, 3, \ldots$ exists on some interval $[t_0 - \tau, t_0 + \gamma)$, $\gamma > 0$ from Theorem 3.1.

Let $\{x_n\}$ be any sequence of solutions for (5.1) for $n = 1, 2, 3, \ldots$ respectively. That is x_n is a solution of

$$x'(t) = f(t, x, x_t) + \frac{1}{n} y_0 , \qquad x_{t_0} = \varphi_0 + \frac{1}{n} y_0 = \varphi_0^n$$

and x_{n+1} is a solution of

$$x'(t) = f(t, x, x_t) + \frac{1}{n+1} y_0 , \qquad x_{t_0} = \varphi_0 + \frac{1}{n+1} y_0 = \varphi_0^{n+1} .$$

Using Theorem 4.1 we can conclude $x_n > x_{n+1}$ i.e. $\{x_n\}$ is a monotone decreasing sequence bounded by $x(t)$ the solution of (1.1). It can be easily shown that $\{x_n\}$ is equicontinuous, uniformly bounded and also $\alpha(\{x_n\}) = 0$. That is $\overline{\{x_n(t)\}}$ is compact. Hence by Asoli's lemma there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges. Suppose $\{x_{n_k}\}$ converges to $r(t)$. By assumption (a_1), $\|f(t, x_n(t), x_{n_t})\| \leqslant M$ and thus the bounded convergence theorem implies

$$\int_{t_0}^{t} \left\{ f(s, x_{n_k}(s), x_{n_{ks}}) + \frac{1}{n} y_0 \right\} ds \to \int_{t_0}^{t} f(s, r(s), r_s)\, ds .$$

This proves that $r(t)$ is a solution of (1.1).

If $x(t)$ is any solution of (1.1) then by theorem 4.1 it follows that

$$x_{n_k}(t) > x(t) \qquad \text{for } t \in [t_0, t_0 + \gamma] .$$

This implies that

$$x(t) \leqslant \lim_{n_k \to \infty} x_{n_k}(t) = r(t) \qquad \text{for } t \in [t_0, t_0 + \gamma] .$$

Thus $r(t)$ is the maximal solution of (1.1) on $[t_0, t_0 + \gamma]$.

6. – Comparison results.

In this section we give some comparison theorems related to the system 1.1.

THEOREM 6.1. Assume that $f \in C[J \times E \times \mathrm{C}, E]$ be quasinonnegative and that f satisfies the onesided Lipschitz's condition. Then the closed set $\bar{Q}$ is flow invariant relative to the system (1.1) where $Q = [x \in E, x > 0]$.

PROOF. Set $m(t) = x(t) + \varepsilon \exp [3Lt] y_0$, where $x(t)$ is any solution of (1.1) such that $x_{t_0} \in \bar{Q}$, $y_0 \in k^0$ and $\varepsilon > 0$ is arbitrarily small number.

Suppose the conclusion is false, then there exists a least t_1 and $c \in k_0^*$ such that $m(t_1) \geqslant 0$ and $c(m(t_1)) = 0$ and $m(t) \in k^0$ for $t \in [t_0 - \tau, t_1)$. This implies $c(m(t_1) - m(t_1 - h)) < 0$ for small $h > 0$ and therefore $c(m'(t)) \leqslant 0$.

Also

$$\begin{aligned} c(m'(t_1)) &= c(x'(t_1) + 3\varepsilon \exp [3Lt] y_0) \\ &= c\big(f(t_1, x(t_1), x_{t_1}) + 3\varepsilon \exp [3Lt] y_0\big) \\ &\geqslant c\big(f(t_1, m(t_1), m_{t_1}) + \varepsilon \exp [3Lt] y_0\big) \end{aligned}$$

using the Lipschitz's condition. Further using the quasimonotonicity of f we can conclude

$$c\big(f(t_1, m(t_1), m_{t_1})\big) \geqslant 0$$

which proves that

$$c\big(m'(t_1) \geqslant c(\varepsilon \exp [3Lt] y_0)\big) > 0 \,.$$

This leads to a contradiction. Thus $m(t) > 0$ for $t \in [t_0 - \tau, \infty)$. Taking the limit as $\varepsilon \to 0$ the conclusion follows.

REMARK 6.1. Theorem 4.2 can be obtained as a consequence of Theorem 6.1.

For this purpose set $d = y - x$ so that

$$d' = F(t, d, d_t) \equiv f\big(t, x + d, (x + d)_t\big) - f(t, x, x_t) + P(t)$$

where $P(t) = y' - f(t, y, y_t) - x' + f(t, x, x_t)$.

Clearly $d_{t_0} \geqslant 0$ and F satisfies the onesided Lipschitz's condition. Furthermore if $d(t) \geqslant 0$ $c(d(t)) = 0$, $d_t \geqslant 0$ for $c \in k_0^*$ then

$$c\big(F(t, d, d_t)\big) \geqslant c\big(f(t, x + d, (x + d)_t) - f(t, x, x_t)\big) \,.$$

Using the form of $P(t)$ and also using the quasimonotoicity property of f in x and monotonicity of f in φ we can conclude

$$c\big(f(t, x + d, (x + d)_t) - f(t, x, x_t)\big) \geqslant 0$$

whenever $d_t \geqslant 0$ $c(d(t)) = 0$.

This proves that F is quasinonpositive. The claim now follows from Theorem 6.1.

COROLLARY 6.1. Assume that f is quasinonpositive and that f satisfies the onesided Lipschitz's condition. Then the closed set $\bar{Q}$ is flow invariant relative to the system 1.1 where $Q = [x \in E, x < 0]$.

COROLLARY 6.2. Assume $f \in C[J \times E \times \mathcal{C}, E]$ satisfies the onesided Lipschitz condition. Assume also that the following condition holds: If

$$x \leqslant b\,,\ \varphi(s) \leqslant b\,,\ c(x) = c(b) \text{ for } c \in k_0^* \text{ for } s \in [-\tau, 0]\,,$$

$$\text{then } c\big(f(t, x, \varphi)\big) \leqslant 0$$

and if

$$a \leqslant x\,,\ a < \varphi(s)\,,\ c(a) = c(x) \text{ for } c \in k_0^* \text{ for } s \in [-\tau, 0]\,,$$

$$\text{then } c\big(f(t, x, \varphi)\big) \geqslant 0.$$

Then the closed set $\overline{W}$, where $W = [x \in E, a < x < b, a, b \in E]$ is flow invariant relative to the system 1.1.

We shall next give a comparison result which yields upper and lower bounds for the solutions of 1.1.

THEOREM 6.2. Assume that

(i) $g_1, g_2 \in C[J \times E \times \mathcal{C}, E]$ are quasimonotone nondecreasing in x and monotone nondecreasing in φ relative to k and for $t, x, \varphi \in J \times E \times \mathcal{C}$,

$$g_2(t, x, \varphi) \leqslant f(t, x, \varphi) \leqslant g_1(t, x, \varphi)\,.$$

(ii) $r(t)$, $\varrho(t)$ are solutions of

$$r' = g_1(t, r, r_t)\,, \quad r_{t_0} = \psi_0\,, \quad \varrho' = g_2(t, \varrho, \varrho_t)\,, \quad \varrho_{t_0} = \chi_0$$

respectively existing on $[t_0, \infty)$.

(iii) f satisfies the onesided Lipschitz's condition, then if $x(t)$ is any solution of 1.1 existing on $[t_0, \infty)$ we have

$$\varrho(t) \leqslant x(t) \leqslant x(t) , \qquad t \geqslant t_0$$

whenever $\chi_0 \leqslant \varphi_0 \leqslant \psi_0$.

PROOF. Setting $m(t) = x(t) - \varrho(t)$, we see $m(t)$ satisfies the differential equation

$$(6.1) \qquad m'(t) = F\big(t, m(t), m_t\big) , \qquad m_{t_0} = \varphi_0 - \chi_0 \geqslant 0 .$$

where

$$F(t, m, m_t) = f\big(t, x(t), x_t\big) - g_2(t, \varrho, \varrho_t)$$
$$= f\big(t, m + \varrho, (m + \varrho)_t\big) - g_2(t, \varrho, \varrho_t) .$$

It is enough to verify that F satisfies the assumptions of Theorem 6.1. so that the closed set $\bar{Q}$ is flow invariant relative to system (6.1). Let $m \geqslant 0$ $m_t \geqslant 0$, $c(m) = 0$ for some $c \in k_0^*$. Then

$$c\Big(F(t, m, m_t) = c\big(f(t, m + r, (m + r)_t\big) - g_2(t, \varrho, \varrho_t)\Big) .$$

Since g_2 is quasimonotone nondecreasing in ϱ and monotone nondecreasing in χ, we have

$$c(g_2(t, \varrho, \varrho_t)) \leqslant c\Big(g_2\big(t, m + \varrho, (m + \varrho)_t\big)\Big)$$

and this implies together with assumption (i) that

$$c\big(F(t, m, m_t)\big) \geqslant 0$$

proving that F is quasinonnegative. Clearly F satisfies the onesided Lipschitz's condition and hence by Theorem 6.1 we get $x(t) \geqslant \varrho(t)$ or $\varrho(t) \leqslant x(t)$, $t \geqslant t_0$.

A similar argument with $m(t) = x(t) - r(t)$ would yield $x(t) \leqslant r(t)$, $t \geqslant t_0$ completing the proof.

COROLLARY 6.3. If $\overline{W}$ is flow invariant relative to system 1.1 then there exist functions g_1, g_2 which are quasimonotone nondecreasing in x for $x \in E$ and monotone decreasing in φ for $\varphi \in \mathcal{C}$ provided $E = R^n$, $k = R^n_+$.

PROOF. We construct g_1, g_2 as follows: for each i, $1 \leqslant i \leqslant n$

$$g_{1i}(t, x, \varphi) = \sup [f_i(t, v, \psi) : a_j \leqslant v_j \leqslant x_j,\ v_i = x_i \text{ and } a_i \leqslant \psi_i(s) \leqslant \varphi_i(s)]$$

$$g_{2i}(t, x, \varphi) = \inf [f_i(t, v, \psi) : x_j \leqslant v_j \leqslant b_j,\ x_i = v_i \text{ and } \varphi_i(s) \leqslant \psi_i(s) \leqslant b_i] .$$

Then clearly the functions g_1, g_2 are quasimonotone nondecreasing in x and monotone decreasing in φ and satisfy (i) of Theorem 6.2.

Finally we give a comparison theorem which is a consequence of the results in sect. 4 and 5.

THEOREM 6.3. Let $f \in C[J \times E \times \mathcal{C}, E]$ satisfy assumptions (a_1), (a_2) and f be quasimonotone nondecreasing in x for fixed (t, φ) and monotone decreasing in φ for fixed (t, x). Let $r(t)$ be the maximal solution of (1.1) on $[t_0, t_0 + \gamma)$. Then if $u \in C[J \times E \times \mathcal{C}, E]$ is such that

$$D_- u \leqslant f(t, u, u_t) , \qquad u_{t_0} \leqslant \varphi_0 \tag{6.2}$$

Then $u(t) \leqslant r(t)$ for $t \in [t_0, t_0 + \gamma)$.

PROOF. Consider

$$x'(t) = f(t, x, x_t) + \frac{1}{n} y_0 , \qquad x_{t_0} = \varphi_0 + \frac{1}{n} y_0 . \tag{6.3}$$

If $x_n(t)$ is any solution of (6.3) for $n = 1, 2, 3, \ldots$. Then using Theorem 4.1 we can conclude

$$u(t) < x_n(t) , \qquad \text{for } n = 1, 2, 3, \ldots .$$

Then taking the limit as $n \to \infty$ we get

$$u(t) \leqslant \lim_{n \to \infty} x_n(t) .$$

But by Theorem 5.1 we know $\lim\limits_{n \to \infty} x_n(t) = r(t)$ the maximal solution of (1.1) and hence the conclusion follows.

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