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Giuliano Artico

Roberto Moresco

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An Answer to a Question on Hyperspaces.

Giuliano Artico - Roberto Moresco (*)

In [AM] we have raised the following question: if a topological space is either Lindelöf or paracompact or normal, does anyone of these properties hold in the hyperspace of its compact subsets? In this note we answer in the negative, supplying a hereditarily Lindelöf space (the « Sorgenfrey line ») whose hyperspace of compact subsets is not normal.

Let X denote a Tychonoff space, μ an admissible uniformity on X, $H(\mu X)$ the uniform (or topological) hyperspace as in [AM], 2^X the hyperspace equipped with the finite topology as in [M], $K(X)$ the space of the compact subsets of X equipped with the topology induced by 2^X (it is well-known that it coincides with the one induced by any $H(\mu X)$). Recall that a base of the open sets in 2^X is given by $\langle U_1, \ldots, U_n\rangle = \left\{A : A \text{ closed of } X,\ A \subseteq \bigcup_{i=1}^{n} U_i,\ A \cap U_i \neq \emptyset,\ \forall i\right\}$ where $U_1, \ldots, U_n$ is any finite collection of open subsets of X.

Lemma. X is separable if and only if $K(X)$ is separable.

The easy proof may be obtained as in [M] 4.5.1. ■

Remark. It can be shown that X is separable whenever $H(\mu X)$ is separable; the converse statement does not hold (take a uniformly discrete countable space).

Example. Let X be the set of the real numbers equipped with the topology which has for a base the family of the half-open intervals $[a, b) = \{x \in X : a \leqslant x < b\}$ where a and b are real numbers.

(*) Indirizzo degli AA.: Istituto di Matematica Applicata, Via Belzoni 7, 35100 Padova.

It is well-known that X is a hereditarily Lindelöf (hence strongly paracompact and perfectly normal) space. We shall show that $K(X)$ is not even normal.

Take the set $D = \{\{-x, x\}: 0 < x \in X\}$. The set D is a discrete subspace of $K(X)$ since $\langle[-x, 0), [x, 2x)\rangle \cap D = \{-x, x\}$ for every positive element x of X. Furthermore D is closed: first remind that $\mathcal{F}_2(X) = \{A \subseteq X: A$ has at most two elements$\}$ is closed in $K(X)$ [M], so it is enough to show that D is closed in $\mathcal{F}_2(X)$; then observe that, with the obvious significance of the symbols, for any $y > x \geqslant 0$ the following open sets: $\langle(-\infty, 0)\rangle$, $\langle[0, +\infty)\rangle$, $\langle[-x, 0), [y, +\infty)\rangle$, $\langle[-x, -x/2), [0, x/2)\rangle$, cover $\mathcal{F}_2(X)\setminus D$ and none of them meets D.

Since X is separable, $K(X)$ is separable too. It follows that the set of the real-valued continuous functions on $K(X)$ has power c, hence the discrete closed set D is not C-embedded in it since the power of D is c: therefore $K(X)$ is not normal. ■

REFERENCES

[AM] G. Artico - R. Moresco, *Notes on the Topologies and Uniformities of Hyperspaces*, Rend. Sem. Mat. Univ. Padova, **63** (1980).

[M] E. Michael, *Topologies on Spaces of Subsets*, Trans. Amer. Math. Soc., **71** (1951), pp. 152-182.

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