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Contributions to foundations of probability calculus on
the basis of the modal logical calculus $MC^\nu$ or $MC^\nu*$

Rendiconti del Seminario Matematico della Università di Padova,

<http://www.numdam.org/item?id=RSMUP_1981__64__109_0>
Contributions to Foundations of Probability Calculus
on the Basis of the
Modal Logical Calculus $MC^r$ or $MC^r_*$.

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Summary - In [3] an axiom system $\Sigma$ for probability calculus, based on the
modal logical calculus $MC^r$—cf. [2]—was proposed; and the same was done
with a more natural (equivalent) version $\Sigma_*$ of it, based on the extension
$MC^*_r$ of $MC^r$, which includes propositional variables. Furthermore a
precise axiom, [3, A12.8] was proposed in [3] as a substitutum for the
existence rule, used e.g. by Reichenbach in [15] and some probabilists
—cf. [7]. These axioms have no set-theoretic character and in this respect
comply with Freudenthal and De Finetti’s views—cf. [8], [6]. In [3]
both $\Sigma$ and [3, A12.8] were said to need checking. In the present work,
devided in three parts, first $\Sigma$ (as well as $\Sigma_*$) and [3, A12.8] are positively
checked in Parts 1 and 2 respectively: the main theorems expected to
be provable on their basis have been effectively proved. In Part 3 the
two main notions of random variables—so to say the physical one and
the mathematical notion—are analysed (and defined) by means of modal
concepts: extensional and absolute relations—cf. [2]. In addition pro-
bability spaces are defined and some existence theorems for them are
proved.

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PART 1

Basic Theorems of a Recent Modal Version of the Probability Calculus, Based on $MC^*$ or $MC^*_c$.

1. Introduction (**).

Most actual approaches to probability calculus introduce a probability space at the outset, which is a particular measured space. Then the axioms are simply the purely mathematical conditions defining these spaces. These approaches, started by Kolmogorov, and the corresponding wide application of measure theory to probability calculus, were substantially (practically) very useful; and progress in this direction is still being made and divulged—cf. recent treatises such as [9] and [10], that deal with measures also on (infinite dimensional) functional spaces.

The afore-mentioned approaches are not much interested in the theoretical systematization of the conditions under which in a given real situation a given probability space can be used or how it can be used. Likewise random variables are usually introduced in a purely

(**) This work is based on the dissertation (thesis) of A. Montanaro, made for his degree in mathematics in the research group of logic, under the direction of A. Bressan. Bressan’s contribution to this work belongs to the sphere of activity of the CNR (Consiglio Nazionale delle Ricerche) in the academic years 1977-78 and 1978-79.

The new results presented in Part 1—i.e. mainly the modal theorems of the probability calculus $PC^*$ started and proved in NN. 6, 7—are substantially due to A. Montanaro. Bressan re-exposed the whole matter in a more concise way; in particular in some cases he selected theorems, slightly changed some of them, and especially made some of their proofs shorter. The same can be said of the theorems connected with the existence axiom presented in Part 2. However Bressan showed how to reduce his existence axiom (scheme), already proposed in [3] as A12.8, to a single axiom. Both axioms were simplified later by Montanaro into A9.1 and A9.1′ respectively.

In Part. 3 the analysis of the notion of (casual or) random variables is due to Bressan, whereas the definition of the probability spaces relative to a proposition and the theorems connected with it are due to Montanaro.
mathematical way as functions on the above spaces. Some authors such as Freudenthal and De Finetti criticize this a priori set-theoretic approach, admittedly from the didactical point of view—cf. the introduction of Part 3—and prefer an approach that takes a probability function $\mathcal{P}$ of propositions as a primitive—cf. [8], [6].

The present work happens to be in agreement with the above views in that, not by didactical purposes but by theoretical reasons of completeness, we prefer approaches to probability calculus that (i) take as a primitive the probability $\mathcal{P}_{\alpha,\beta}$ of the proposition $\beta$ relative to the proposition $\alpha$, or something similar, and (ii) state existence axioms (1). Such a theory is proposed by Reichenbach in [15] on the basis of von Mises’s theory [14], and is accepted also by some probabilists—cf. e.g. [7]. Reichenbach assumes as a primitive $(\forall i)(x_i \in A \equiv y_i \in B)$—abbreviated by $A \equiv B$—where $A$ is a class of trials, $B$ a class of results, $x_i$ a trial, and $y_i$ the corresponding result ($i = 1, 2, \ldots$). Let $N_{A,n}$ be the number of those among $x_1$ to $x_n$ that are in $A$ and let $N_{A,\mathcal{P},n}$ be the number of the $x_i$’s such that $x_i \in A$, $y_i \in B$. Then $A \equiv B$ means that $\mathcal{P}$ is the limit of $N_{A,\mathcal{P},n}/N_{A,n}$ for $n \to +\infty$.

Reichenbach’s frequentistic theory is based on extensional logic so that it is natural to assume the above limit to exist when the sequence $x_i$ includes all trials in $A$ and is ordered in a natural way (respecting time order).

The reference to extensional logic, and hence the restriction of the probability calculus to the real world, causes severe limitations on this theory, which are usually not spoken of (2). They are similar

(1) Our existence axioms for probabilities are conditional assertions stating the existence of certain probabilities in case certain other probabilities exist. These axioms do not at all compel us to believe that absolute objective probabilities exist. In certain situations (subjective or even objective) probabilities can be regarded to exist in that this is satisfactory relative to certain purposes. Likewise, for certain mechanical purposes it is satisfactory to admit the existence of Euclidean inertial spaces, according to classical physics, whereas for other purposes, related with general relativity, this admission is not possible.

(2) Reichenbach’s frequentistic version of the probability theory, based on extensional logic is incompatible with the existence for every $\delta \in 1^{-1}, 2$, of the probability $p_0$ that a given ball $S$ launched in a given way in a region where the air has the density $\delta(A_0)$, on a plane $\Pi$, may fall within the circle $\Gamma$ in $\Pi(B_0)$.

Indeed by the use of extensional logic the existence of $p_0$ implies the existence of a sequence $x_i(\in A_\delta)$ of the above launches of $S$ in air of density $\delta$. Every experiment $x_i(\in A_\delta)$ is localized in a 3-dimensional region $R_{i,\delta}$; and
to the unacceptable defects of Hermes’s extensional axiomatizations of particle mechanics according to Mach and Painlevé—cf. [11] (and [12]) criticized by Rosser—cf. [16]. These defects disappear when modalities are used—cf. [12], [1], and [2].

By the reasons above it was natural to use the recent constructions of general modal calculi, in particular $MC^r$, to formulate a modal probability theory, the more so as probability is often asserted intuitively to be a degree (quantity) of possibility; and a set of axioms for such a theory was proposed in [3]; more, for the sake of simplicity $MC^r$ was extented to a calculus $MC^*_r$ having propositional variables and variables for relations and functions among propositions and other entities, and so on, so that a probability implication $\alpha \vDash_\rho \beta$ similar to Reichenbach’s can be introduced naturally. In [3] also a precise existence axiom to be substituted for the so called existence rule used by Reinchenbach and some probabilists such as Dore—cf. [15], p. 53, [7]—was proposed. Furthermore, there it is said that the whole theory, which contains essentially modal axioms (having no extensional analogues), ought to be tested, especially in connection with the existence axiom. This demand is met in the Part 1 and 2 of the present work, in connection with the main consequences of the axioms different from the existence axioms, and those of all axioms respectively. In Part 3 the concept of random (or casual) variables is analyzed and a notion of probability spaces relative to a proposition $\alpha$ is defined. Especially Part 3 happens to agree to a certain extent with the afore-mentioned views of Freudenthal and De Finetti.

The present work differs considerally from the paper [17], which is based on an extensional infinitary language.

Now we describe the content of Part 1 in more detail. The analogues for Parts 2 and 3 will be done in the introductions of these parts. In NN. 2-4, the modal language $ML^*_r$ and the logical calculus $MC^*_r$ which is based on it, and extends the calculus $MC^r$ presented in [2], is briefly remembered from [3].

In N. 4 the class $El$ of certain analogues defined in $MC^r$ for the elementary possible cases, and some theorems holding for it (also in $MC^*_r$), are remembered from [2].

$$R_{i,\delta} \cap R_{j,\delta'} = \emptyset$$ for $i \neq j$ or $\delta \neq \delta'$. Then if $p_3$ existed for all $\delta \in 1 \parallel 2$, the 3-dimensional space could be divided into a class of nonvanishing 3-dimensional regions, that has the continuum power, which is absurd.
In N. 5 axioms A5.1 to A5.7 for the probability calculus $PC^*$, [PC] are substantially taken from [3]. Axiom A5.8 and the considerations on it are added. In N. 6 basic modal theorems of probability calculus on $\exists_\sigma$ are stated and proved. The same is done in N. 7 with the probability function $\mathcal{F}_{\alpha, \beta}$.

2. Formation rules for $ML^*_\sigma$.

We want to consider the modal calculus $MC^*_\sigma$ introduced in [3] and based on the modal language $ML^*_\sigma$, an extension of $ML^r$—cf. [2]. To this purpose we denote the $n$-tuple formed with $a_1$ to $a_n$ by $\langle a_1, ..., a_n \rangle$ and we define recursively $\tau^*_\sigma$, the set of types for $ML^*_\sigma$, by the conditions (a) and (b) below, where $n$ runs over $Z^+ = \{1, 2, ...\}$:

(a) $\{0, 1, ..., \nu\} \subset \tau^*_\sigma$, and

(b) if $t_0, t_1, ..., t_n \in \tau^*_\sigma$, then $\langle t_1, ..., t_n, t_0 \rangle \in \tau^*_\sigma$.

We call 0 the sentence type and 1 to $\nu$ the individual types. For $t_0, t_1, ..., t_n \in \tau^*_\sigma$ and $t_0 = 0$ [$t_0 \neq 0$] we say that $\langle t_1, ..., t_n, t_0 \rangle$ is a relation [function] type. Of course these names suggest how the various types will be used. Following [5] we use, for $t_0, ..., t_n \in \tau^*_\sigma$, the notations

\[
\begin{cases}
(t_1, ..., t_n) =_D \langle t_1, ..., t_n, 0 \rangle, \\
(t_1, ..., t_n : t_0) =_x \langle t_1, ..., t_n, t_0 \rangle \quad \text{for } t_0 \neq 0 \ (\text{v}).
\end{cases}
\]

We also define the set $\tau^r$ recursively by the conditions (a) and (b) below, where $n$ runs over $Z^+$:

(a) $\{1, ..., \nu\} \subset \tau^r$,

(b) if $t_0, ..., t_n \in \tau^r$ and $t_0 \in \tau^r \cup \{0\}$, then $\langle t_1, ..., t_n, t_0 \rangle \in \tau^r$.

Of course

\[
\tau^r \subset \tilde{\tau} \subset \tau^*_\sigma \quad \text{where} \quad \tilde{\tau} = \tau^r \cup \{0\}.
\]

(\text{v}) Hence in harmony with [5] $\langle 0, 0 \rangle$ is (0)—the assertion $\langle 0, 0 \rangle = (0; 0)$ is meaningless. Thus e.g. negation is a property of propositions (and not a function among propositions).
The symbols of $ML^*[ML^r]$ are "(,)", "~", "\wedge", "\Box" (necessity), "=" (identity), the reversed iota "1" (for descriptions), the variables $v_{tn}$ and constants $c_{tn}$ where $n \in Z^+$ and $t \in \tau^*_t$. The class $\mathcal{E}_t (= \mathcal{E}_t^{\tau^*_t})$ of the designators or wffs (well-formed expressions) of type $t \in \tau^*_t$ for $ML^*_t$ can be defined by conditions (f1) to (f8) below (formation rules) where $n$ runs over $Z^+$ and $t_0, t_1, \ldots, t_n$ run over $\tau^*_t$.

(f1) $v_{tn}, c_{tn} \in \mathcal{E}_t$.

(f2) If $\Delta, \Delta' \in \mathcal{E}_t$, then $(\Delta = \Delta') \in \mathcal{E}_0$.

(f3) If $\Delta_1 \in \mathcal{E}_{t_1} (i = 0, \ldots, n)$ and $\Delta \in \mathcal{E}_{t_0, \ldots, t_n, t_0}$, then $\Delta(\Delta_1, \ldots, \Delta_n) \in \mathcal{E}_{t_0}$.

(f4) If $\Delta \in \mathcal{E}_0$, then $(v_{tn}) \Delta \in \mathcal{E}_t$.

(f5-8) If $\Delta, \Delta' \in \mathcal{E}_t$, then $\sim \Delta, (\Delta \wedge \Delta'), (v_{tn}) \Delta, \Box \Delta \in \mathcal{E}_0$.

In case $\Delta \in \mathcal{E}_t$, we say that $\Delta$ is a matrix or wff [term] for $t = 0 \ [t \neq 0]$; and if $t \neq 0$, $\Delta$ is said to be an individual expression, attribute, or functor according as $t$ is a type for individuals, relations, or functions respectively.

The expressions of type $t \in \tau^*_t$ for $ML^r$ are those for $ML^*_r$ where only symbols of $ML^r$ occur, and where "=" occurs only between terms as its arguments—cf. e.g. [2], p. 12.

The connectives $\wedge, \vee, \Box$, and $\equiv$, and the existence [possibility] sign $\exists [\Diamond]$ are understood to be introduced in the usual way. The cohesive powers of $(v_{tn})$ (universal quantifier), $(v_{tn})$ (descriptor operator), $\Box, \sim, \wedge, \lor, \exists, \Box$, and $\equiv$ are considered to decrease in the written order; e.g.:

(2.3) $(x)\alpha \wedge \beta \lor \gamma$ stands for $((x)\alpha \land \beta) \lor \gamma$ ($\alpha, \beta, \gamma \in \mathcal{E}_0$).

In addition we write $\wedge_{i=1}^n \alpha_i$, $\vee_{i=1}^n \alpha_i$, and $(\forall x_1, \ldots, x_n)\alpha$ for $\alpha_1 \wedge \ldots \wedge \alpha_n$, $\alpha_1 \lor \ldots \lor \alpha_n$, and $(x_1) \ldots (x_n)\alpha$ respectively; sometimes we drop $\land$; furthermore a dot may replace a left [right] handed parenthesis, the mate of which is to be restored at the farthest possible place within all pairs of parentheses that include the dot itself:

(2.4) $[(x)\alpha \land \beta \lor \gamma] \circ \beta$ stands for $\{(x)[\alpha \land (\beta \lor \gamma)]\} \circ \beta$.

We shall use $\epsilon, \epsilon^c, \epsilon^\cup, =^c, \land^c$ to $=^c$, and $\land^c$ to $=^c$ in accordance
with the definitions

(2.5) \( \Lambda \in \Lambda' \equiv_D \Lambda'(\Lambda) \) if \( \Lambda \in \mathcal{E}_t \) and \( \Lambda' \in \mathcal{E}_{(t)} \) for \( t \in \tau_* \),

(2.6) \[
\begin{align*}
\Lambda_1 = \wedge \Lambda_2 \equiv_D \Box \Lambda_1 = \Lambda_2, \\
\Lambda \in \wedge \Lambda' \equiv_D \Box \Lambda \in \Lambda', \\
\wedge \beta \equiv_D (\wedge \beta),
\end{align*}
\]

(2.7) \[
\begin{align*}
\Lambda_1 = \vee \Lambda_2 \equiv_D \Diamond \Lambda_1 = \Lambda_2, \\
\Lambda \in \vee \Lambda' \equiv_D \Diamond \Lambda \in \Lambda', \\
\vee \beta \equiv_D (\vee \beta).
\end{align*}
\]

Bound variables are understood to be defined in the usual way in connection with both \((v_n)\) and \((v_n)\) \((t \in \tau_*, n \in \mathbb{Z}^+)\); and the same holds for free variables and for terms that are free for a variable in a wff—cf. e.g. [13], pag. 47-48, closed matrices are called sentences.

Convention 2.1. If we denote a matrix by \( \Phi(x, y) \) where \( x \) and \( y \) express variables, and then we use \( \Phi(\Lambda, \Lambda') \), it is understood that \( \Lambda'[\Lambda'] \) is a term free for \( x[y] \) in \( \Phi(x, y) \) and that \( \Phi(\Lambda, \Lambda') \) is obtained from \( \Phi(x, y) \) by substituting \( \Lambda \) for \( x \) and \( \Lambda' \) for \( y \).

Def. 2.1. A matrix \( \alpha \) is said to be modally closed if it is constructed starting out from matrices of the form \( \Box \beta \) by means of \( \neg, \wedge, \Box, \) and \((v_n)\).

We define \( \exists^{(1)}[\exists^{(1)}] \) (there is [is strictly] at most one) and \( \exists_1[\exists_1^{(1)}] \) (there is exactly [strictly] one) by

\[
\begin{align*}
\exists^{(1)}_x \Phi(x) =_D (\forall x, y)[\Phi(x)\Phi(y)] \supset \begin{cases} x = y \\
x = \neg y \end{cases}, \\
\exists^{(1)}_x \Phi(x) =_D (\exists x)\alpha \supset \begin{cases} (\exists^{(1)}_x)\alpha \\
(\exists_1^{(1)}_x)\alpha \equiv_D (\exists^{(1)}_x)\alpha \wedge \begin{cases} (\exists^{(1)}_x)\alpha \\
(\exists_1^{(1)}_x)\alpha \end{cases} \end{cases},
\end{align*}
\]

the non-existing object by

\( a^* =_D a^*_t =_D (v_t)_1 \neq v_{tt} \),

and the relational and functional lambda expressions by

(2.10) \( (\lambda x_1, ..., x_n) \alpha =_D (\iota R)(\forall x_1, ..., x_n)R(x_1, ..., x_n) = \alpha \),

(2.11) \( (\lambda x_1, ..., x_n) \Lambda =_D (\iota f)(\forall x_1, ..., x_n)f(x_1, ..., x_n) = \Lambda \),
where \( \alpha \in \mathcal{E}_0, \Delta \in \mathcal{E}_t \) with \( t_0 \neq 0 \), \( x_1 \) to \( x_n \) are distinct variables of the respective types \( t_1 \) to \( t_n \), and \( R[f] \) is the first variable of type \( (t_1, \ldots, t_n) \) \([t_1, \ldots, t_n : t_0]\) that does not occur in \( \alpha[D] \).

We can now define the modal product \([\text{sum}]\ R \cap[R \cup]\) of the relation \( R \) of type \( (t_1, \ldots, t_n)\):

\[
\begin{align*}
R \cap &= _D (\lambda x_1, \ldots, x_n) \square R(x_1, \ldots, x_n), \\
R \cup &= _D (\lambda x_1, \ldots, x_n) \Diamond R(x_1, \ldots, x_n).
\end{align*}
\]

(2.12)

3. An axiom system for the logical calculus \( MC^* \) [\( MC^{\prime}\)] based on \( ML^* \) [\( ML^{\prime}\)].

Modus ponens is the only inference rule for \( MC^{\prime} \) and \( MC^* \). We now write a set of axioms for \( MC^{\prime} \) which is taken substantially from [2, N. 12] and holds also for \( MC^* \). Then we add one axiom for \( MC^* \). Below \( \alpha, \beta, \) and \( \gamma \) are arbitrary matrices and \( x, y, \) and \( z \) are distinct variables; furthermore types are always understood to be such that the written expressions are well formed.

An arbitrary strings of universal quantifiers and \( \square \)s is denoted by \((\square)\), or by \((\ )\) if the \( \square \)'s are lacking. In any case it has the least cohesive power.

A3.1-3 \((\square) \alpha \supset \alpha \), \((\square) \alpha \supset \supset \alpha \), \((\square) (\alpha \supset \beta) \supset \supset (\beta \supset \gamma) \supset \supset (\gamma \supset \alpha)\),

A3.4-5 \((\square) (x)(\alpha \supset \beta) \supset (x) \alpha \supset (x) \beta\), \((\square) (\alpha \supset \beta) \supset \supset (\square \alpha \supset \square \beta)\),

A3.6 \((\square) \alpha \supset (x) \beta \) if \( x \) is not free in \( \alpha \),

A3.7 \((\square) \alpha \supset \square \alpha \) if \( \alpha \) is modally closed [Def. 2.1],

A3.8 \((\square) (x) \Phi(x) \supset \Phi(\Delta)\) see convention 2.1,

A3.9 \((\square) \square \alpha \supset \alpha \).

The next three axioms concern identity and descriptions.

A3.10-11 \((\square) x = x\), \((\square) x = y \supset z = x\),

A3.12 \((\square) x = \supset y \supset \Phi(x) \supset \Phi(y)\) cf. Convention 2.1,

A3.13 \((a) \alpha(\exists x \alpha) \supset (\exists x \alpha) \alpha = x\), \((b) \supset (\exists x \alpha) \alpha \supset (\exists x \alpha) \alpha = a^*\).
The next two axioms concern quasi extensionality and A3.16-18 concern existence of relations and functions. We assume that \(x_1, \ldots, x_n, R, S, f, \text{ and } g\) are distinct variables of the respective types \(t_1, \ldots, t_n, t = (t_1, \ldots, t_n), t, \theta = (t_1, \ldots, t_n; t_0), \text{ and } \theta (t_0 \neq 0), \text{ that } \Delta \in \mathcal{E}_t, \text{ and that } R[f] \text{ is not free in } \alpha[\Delta].

A3.14  
\((\square) \ R = S \equiv (\forall x_1, \ldots, x_n).R(x_1, \ldots, x_n) \equiv S(x_1, \ldots, x_n),\)

A3.15  
\((\square) \ f = g \equiv (\forall x_1, \ldots, x_n)f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n),\)

A3.16  
\((\square) \ (\exists R)(\forall x_1, \ldots, x_n).R(x_1, \ldots, x_n) \equiv \alpha,\)

A3.17  
\((\square) \ (\exists f)(\forall x_1, \ldots, x_n)f(x_1, \ldots, x_n) = \Delta,\)

A3.18  
\((\square) \ (\exists R)R' = R \lor = S.\)

That we are dealing with an effectively modal language, or that at last two \(\Gamma\)-cases exist, is substantially asserted by

A3.19  
\((\exists x, y).\Box x = y \Diamond x \neq y.\)

In the following conventional axioms on the non-existing object \(x_i\) means \(v_{t_i+1} (i = 0, \ldots, n).\)

A3.20  
\((\square) \ a^*_t(x_1, \ldots, x_n) \supset \bigwedge_{i=1}^n x_i = a^*_t \text{ for } t = (t_1, \ldots, t_n),\)

A3.21  
\((\square) \ a^*_\theta(x_1, \ldots, x_n) \supset x_0 = a^*_\theta \text{ for } \theta = (t_1, \ldots, t_n; t_0) \text{ with } t_0 \neq 0.\)

The axiom of choice—where e.g. \((\forall F, G \in \mathcal{P})\alpha\) stands for \((\forall F, G)(F \in \mathcal{P} \land G \in \mathcal{P} \supset \alpha)—reads as in extensional logic:

A3.22  
\((\exists F)(F \in \mathcal{P}) \land (\forall F, G \in \mathcal{P})(\exists x \in F)[(\exists y \in G)y \in F \supset F = G] \supset \supset (\exists G)(\forall F \in \mathcal{P})(\exists x)(x \in F \land x \in G).\)

One can postulate the existence of infinitely many individuals—see A8 45.1 in [2]—or one can be contented with the following

A3.23  
\((\exists v_{t_1}) \Box v_{t_1} \neq a^*_t \text{ for } (t = 1, \ldots, v).\)
Before stating the next axiom, which is AS 25.1 in [2], we must remember the definitions of modally constant relations of type $t = (t_1, ..., t_n)$ (MConst, or briefly MConst), modally separated relations (MSep), and absolute relations (Abs) of type $t$:

\begin{align}
(3.1) \quad R \in \text{MConst} & \equiv \forall x \quad R^\circ = R^\wedge — \text{cf. (2.12)}, \\
(3.2) \quad R \in \text{Msep} & \equiv (\forall x_1, y_1, ..., x_n, y_n) R(x_1, ..., x_n) \land \\
& \land R(y_1, ..., y_n) \land \diamond \bigwedge_{i=1}^{n} x_i = y_i, \\
(3.3) \quad R \in \text{Abs} & \equiv \exists \forall x \in \text{MConst} \land R \in \text{Msep}.
\end{align}

Mathematical classes are absolute concepts. In order to use them in connection with the physical world, it is important to define the extensionalization $R^{(e)}$ of any relation $R$ (of type $t$), and the property of extensionality (Ext)

\begin{align}
R^{(e)} & = (\forall x_1, ..., x_n)(\exists y_1, ..., y_n) R(y_1, ..., y_n) \land \bigwedge_{i=1}^{n} x_i = y_i, \\
R \in \text{Ext} & \equiv \exists \forall x \in \text{MConst} = R^{(e)},
\end{align}

where $x_1, y_1, ..., x_n, y_n$ are suitable distinct variables.

\begin{align}
\text{A3.24} \quad (\exists F) \quad F \in \text{Abs} \land a^x \in F \land (x \in F^{(e)}).
\end{align}

Those among the above axioms for $MC^*_v$, that belong to $ML^r$ are the axioms for $MC^r$. The only remaining axiom of $MC^*_v$ is the assertion (outside $ML^r$) that two propositions coincide iff they are equivalent.

\begin{align}
\text{A3.25} \quad (ML^v) \quad (x)(\beta) x = \beta \equiv ^\wedge (x \equiv \beta) \quad (x, \beta \in \varepsilon).
\end{align}

Remark that by essential uses of AA3.8, 23 the, so to say, quasi extensional axioms A3.16-17 on classes and functions can be strengthened into the following theorems—cf. [2, (46) on p. 166]—in both $MC^*_v$ and $MC^r$ (and in the same way)

\begin{align}
(3.5) \quad (\square) \quad (\exists R)(\forall x_1, ..., x_n) R(x_1, ..., x_n) \equiv ^\wedge x, \\
(3.6) \quad (\square) \quad (\exists f)(\forall x_1, ..., x_n) f(x_1, ..., x_n) = ^\wedge \Lambda.
\end{align}
4. On the analogue $E_l$ of elementary possible cases, defined in $ML^*_e$ itself, and indicators of propositions.

In [2], p. 203 the analogue $E_l$ of the class $I$ of elementary (possible) cases is defined in $ML^*_e$ itself. Hence this definition belongs to $ML^*_e$ too. The same holds for the matrix $|u|$ where $u$ is a particular variable—cf. [2], p. 204—which substantially means: the elementary case $u$ occurs. Let us remember from [2], pp. 203, 4, 6, 8 the following theorems on $E_l$ and $|u|$, where $x$, $\beta$ are matrices (and hence they may be variables if $MC^*_e$ is referred to) and where $u$ is a variable that does not occur free in $x$ or $\beta$.

\begin{align*}
(4.1) & \vdash E_l \in \text{Abs}, \quad \vdash (\exists_1 u)|_u, \quad \vdash (\exists_1^0 u)|_u, \quad \vdash u \in E_l \equiv \Diamond |_u, \\
(4.2) & \vdash \Diamond (|_u x) \equiv u \in E_l \land (|_u \triangledown x), \quad \vdash \Diamond [ |_u (x \sim \beta) ] \equiv u \in E_l \land \\
& \quad \land \Diamond (|_u x) \land \sim \Diamond (|_u \beta), \\
(4.3) & \vdash (u)[u \in E_l \triangledown \Diamond (|_u x)] \equiv \Box x \equiv (u)|_u \triangledown x, \\
(4.4) & \vdash (\exists u)[u \in E_l \land (|_u \triangledown x) \equiv \Diamond x \equiv (\exists u) \Diamond x|_u].
\end{align*}

The extensionalization $\Phi(x)_{(ex)}$ of any matrix $\Phi(x)$ is defined—cf. [2], p. 36—by

\begin{align*}
(4.5) & \quad \Phi(x)_{(ex)} = \square \exists y.\Phi(x) \equiv y \text{ where } y \text{ is not free in } \Phi(x).
\end{align*}

The theorems below on $r$ will be used (cf. [2], pp. 155, 164

\begin{align*}
(4.6) & \quad \vdash (\exists x)x = (\exists x)x_{(ex)}, \quad \Box (\exists x)x \supset (\exists x)x = (\exists x) \Box x_{(ex)}, \\
(4.7) & \quad \vdash (\exists_1 x) \Phi(x) \supset \Psi[(\exists x) \Phi(x)] \quad \text{where} \quad \Psi(x) \equiv \exists_0 \Phi(x)_{(ex)}, \\
(4.8) & \quad \vdash \Box (\exists_1 x) \square \Phi(x) \supset \square \Phi[(\exists x) \square \Phi(x)], \\
& \quad \vdash (\exists_1 x) \exists y (x).x = (\exists x) x \equiv x_{(ex)}.
\end{align*}

(4) Formulas (4.1) to (4.4) are (82)\textsubscript{3}, (83)\textsubscript{3,4}, (89), (92), (90), (91) (pp. 203-8 of [2]) respectively.

(5) Formulas (4.6) to (4.8) are (33), Ths. 39.5 (II) and 38.2, (35)\textsubscript{4}, (31)\textsubscript{1}, (pp. 155-164 of [2]), respectively.
After [4], we consider the indicator \( \text{ind} (\alpha) \), briefly \( \alpha \), for any proposition \( \alpha \); substantially it is the range of \( \alpha \), i.e. the set of elementary possible cases in which \( \alpha \) holds:

\[
\alpha = \bar{\lambda}u \Diamond (\alpha|u) \quad \text{for } u \text{ not free in the matrix } \alpha.
\]

Set theoretic relations or operations such as \( \subseteq, \cap, \cap, \cup \) and \( \cup \) are understood to be introduced in \( ML^r \) or \( ML^r_\ast \) as usually—cf. [2], p. 66. Then by (4.1) and (4.9)

\[
(4.10) \quad \vdash \alpha \subseteq \text{El}, \quad \vdash \alpha \in \text{Abs}.
\]

Now we briefly prove formally that

\[
(4.11) \quad \vdash \iota(\sim \alpha) = \bar{\alpha}, \quad (\bar{\alpha} = \bar{\lambda} \text{El} - \alpha), \quad \vdash \iota(\alpha \cap \beta) = \alpha \cup \iota \beta,
\]

\[
(4.12) \quad \vdash (\beta \cap \gamma) \equiv \iota \beta \subseteq \gamma, \quad \vdash (\beta \equiv \gamma) \equiv \iota \beta = \gamma \equiv \iota \beta = \bar{\gamma}.
\]

By (4.9) and (4.2) \( \vdash u \in \iota(\sim \alpha) \equiv u \in \text{El} (\iota \sim \alpha) \).

Furthermore we deduce likewise that \( \vdash u \in \overline{\alpha} \equiv u \in \text{El} (\iota \sim \alpha) \equiv u \in \text{El} \sim \Diamond (\iota \alpha) \). Hence \( \vdash u \in \text{El} \sim \Diamond (\iota \alpha) \). By (4.9) and (4.10)_2 yields (4.11)_1.

By (4.9) \( \vdash u \in \alpha \cup \iota \beta \equiv u \in \text{El} \sim \Diamond (\alpha|u) \cup \Diamond (\beta|u) \equiv u \in \text{El} \sim \Diamond (\alpha \cap \beta)|u \equiv u \in \iota(\alpha \cap \beta) \). Hence (4.11)_2 holds.

By (4.11)_1, \( \vdash \iota(\alpha \beta) = \iota (\alpha \cap \beta) = \bar{\alpha} \cup \bar{\beta} = \alpha \cap \iota \beta \), whence (4.11)_2.

Let \( u \) not occur free in \( \beta \) or \( \gamma \); then, by (4.9), (a) \( (\iota \beta \subseteq \gamma) \) is equivalent to (b) \( (\iota \gamma) \cap (\iota \beta)|u \), which is obviously implied by (c) \( \beta \cap \gamma \). Furthermore \( \sim (\beta \cap \gamma) \) yields \( \Diamond (\beta \sim \gamma) \), whence by (4.4)_2 \( \exists u \cap (\beta \sim \gamma)|u \). By (4.2)_1 this yields \( (\exists \gamma). (\beta|u) \sim (\gamma|u) \) which contrasts to (b). Hence \( \vdash (b) \cap (c) \). We conclude that \( \vdash (a) \equiv (b) \equiv (c) \). Hence (4.12)_1 holds. This and (4.10)_2 imply (4.12)_2.

5. Notions and axioms of probability expressed in \( ML^r \) and \( ML^r_\ast \).

To deal with Reinchenbach's implication \( \exists \) in any \( L \) of the languages \( ML^r \) and \( ML^r_\ast \) after [3], we consider (i) the constant \( \text{R} \)—denoted by Real in [2]—that expresses the class of real numbers in \( L \),

\[
\text{R}(\alpha) \equiv \alpha(\text{R}).
\]
(ii) the types $t_R$ and $t_{El}$ of $R$ and $El$, and (iii) the first constant $c^0 [c^*_0]$ of $ML^r [ML^s]$ of type $(t_{El}, t_{El}, t_R) [(0, 0, t_R)]$. Then we set
\[
\alpha \ni \beta \equiv \{ \begin{array}{ll}
\phi^0(\alpha, \psi, p) & \text{in } ML^r, \\
\phi^*_0(\alpha, \psi, p) & \text{in } ML^s,
\end{array}
\]
and, following substantially Reichenbach, we accept
\[
\alpha \ni \beta \equiv \exists p \alpha \ni \beta
\]
as a metalinguistic definition, so that (if $\alpha$ can be realized) $\alpha \ni \beta$ says that the probability of $\beta$ relative to $\alpha$ exists.

By their interpretations it is natural to consider $\ni$ and $\exists$ as more cohesive than $\supset$ and less cohesive than $\land$ and $\lor$.

By $A3.25$ in $ML^*$ the substitution theorem
\[
\vdash (\alpha \equiv \gamma)(\beta \equiv \delta) \supset \exists \alpha \ni \beta \equiv \gamma \ni \delta
\]
is a trivial consequences of $A3.12$. In $ML^r$ (5.3) holds by (4.12)$_s$ and $A3.12$. By (5.3)
\[
\beta \ni \exists \alpha \equiv \begin{cases} 
\alpha \ni \beta \equiv \alpha \ni \beta \gamma, \\
\beta \ni \alpha \equiv \beta \gamma \ni \beta.
\end{cases}
\]

Following substantially [3], we take as proper axioms (axiom schemes) of the probability calculus $PC_*[PC]$ based on $MC^*[MC^r]$ the assertions $A5.1-8$ of $ML^*[ML^r]$ below ($^*$), where $R$ is understood to express the natural concept of real numbers (denoted by $Real$ in [2]) and to be defined in an obvious way in terms of $N$ which is a constant of type $t_N$ that expresses the natural concept of natural numbers in $\mathcal{L}$ ($^*$). One can prove
\[
\vdash N \in Abs, \quad \vdash R \in Abs, \quad \vdash a^* \notin R.
\]

**Convention 5.1.** The instance of $(\ )A$ or $(\Box)A$ with $\Delta \in \mathcal{E}_0$, that is the closure or modal closure of $\Delta$, can be denoted by $[ ]\Delta$ or $[\Box]\Delta$

($^*$) In [3] only A1 to A7 are stated.
($^*$) In [2], NN. 27, 28, 45, $N$—denoted by $N_n$—is defined on purely logical grounds; and the basic theorems for it are stated.
respectively.

A5.1 absoluteness \[ [\ ] \Diamond (x \exists_{p} \beta) \supset \Box x \exists_{p} \beta, \]

A5.2 absoluteness \[ [\ ] x \exists_{p} \beta \supset p \in \mathbb{R}, \]

A5.3 uniqueness \[ [\ ] p, q \in \mathbb{R} \land p \neq q \supset (x \exists_{p} \beta)(x \exists_{q} \beta) \equiv \sim \Diamond x, \]

A5.4 normalization \[ [\ ] \Diamond (x \exists_{p} \beta) \supset x \exists_{1} \beta, \]

A5.5 normalization \[ [\Box \ ] \Diamond (x \exists_{p} \beta) \supset p \geq 0, \]

A5.6 sum \[ (x \exists_{p} \beta)(x \exists_{q} \gamma)(x \exists_{p+q} \beta \sim \gamma) \supset x \exists_{p+q} \beta \psi, \]

A5.7 product \[ (x \exists_{p} \beta)(x \exists_{q} \gamma) \supset x \exists_{pq} \beta \psi, \]

A5.8 \[ [(\forall r \in \mathbb{N})(x \exists_{p} B) \land (x B_{r+1} \supset \sim B_{r}) q = \sup \{ p_r \} \supset x \exists_{q} (\exists r \in \mathbb{N}) B_r, \]

where B[p] is a variable of type \((t_{\mathbb{N}}; 0)\) \(\{t_{\mathbb{R}}; t_{\mathbb{R}}\}\) and e.g. \(B, [p_{r}]\) stands for \(B(r) [p(r)]\)—cf. the definition of \((\forall x \in \mathcal{P})\) above A3.22.

By reasonings that are essentially mathematic and substantially known—and hence omitted here—the conjunction of A5.6 and A5.8 is equivalent to

A5.9 \[ [(\forall r, s \in \mathbb{N})(x \exists_{p} B_r) (r < s \land x B_r \supset \sim B_s) \left( q = \sum_{r-1}^{\infty} p_r \right) \supset \]

\[ \sup x \exists_{q} (\exists r \in \mathbb{N}) B_r, \]

where B and p are as is A5.8.

Incidentally A5.9 is strictly more powerful than any of A5.6 and A5.8; and by theorem (6.1) below, A5.4 can be replaced with the equivalent axiom

A5.4' \[ \alpha \supset \beta \supset x \exists_{1} \beta. \]

In \(ML'\) propositional variables can be replaced with the variables of type \((t_{\mathbb{N}})\) that are restricted to the class \(PR_1:\)

\[ F \in PR_1 \equiv_B (x). x \in F \supset \sup x = ^1. \]
Thus, briefly speaking, every proposition $\alpha$ is associated with the $F \in PR_1$ for which $\alpha \equiv^* F(1)$ holds; and this correspondence is a bijection. For instance the counterparts of $A5.2$ and $A5.6$ in $MC^*$ are

\[(5.7) \quad \Box [F_1(1) \ni \exists \beta \in R, \quad \square [F_1(1) \ni \exists \beta \in R]\wedge
\quad \wedge [F_1(1) \ni \exists \beta \in R] \wedge [F_1(1) \ni \exists \beta \in R] \wedge [F_1(1) \ni \exists \beta \in R] \wedge [F_1(1) \ni \exists \beta \in R] .\]

The metalinguistic formulation $A5.2$ and $A5.6$ can be used also for their counterparts $(5.7)_1,2$ provided $\alpha_1$ to $\alpha_n$ are meant as metalinguistic variables standing for $F_1(1)$ to $F_n(1)$ respectively. The same holds for the others among axioms $A5.1-7$ and $A5.4'$. Obviously $A5.8$ and $A5.9$ are meaningful also in $ML^*$, so that they can be included into the $PC$.

6. Some theorems on $\exists x$ in $MC^*$ and $MC^*$.  

The theorems (6.1) to (6.3) below on $\exists x$ differ from their usual analogues in that most of them are explicitly modal and all of them belong to the modal calculus $PC^*$ or $PC$. We think that to prove them explicitly is interesting especially because these proofs involve some modal axioms without any ordinary analogues.

\[(6.1) \quad \vdash \sim \diamond \alpha \vdash \exists x (\forall \beta \in R) \alpha \exists x \beta , \quad \vdash \alpha \vdash \alpha , \quad \vdash \alpha \vdash \beta \wedge \sim r .\]

\[(6.2) \quad \vdash \sim \diamond (\alpha \beta) \vdash \exists x \vdash \beta , \quad \vdash \beta \vdash \exists x \vdash \beta , \quad \vdash \beta \vdash \exists x \vdash \beta .\]

\[(6.3) \quad \vdash (\alpha \exists x) \vdash (\exists x) \vdash (\exists x p) \alpha \exists x \beta \vdash (\exists x p) \alpha \exists x \beta .\]

Proof. Assume $\vdash \sim \diamond \alpha$ and $\beta \in R$. Then, for $q = p + 1$ $A5.3$ easily yields $\alpha \exists x \beta$. We conclude that (6.1) holds; and hence $A5.4$ yields $A5.4'$. Since $\vdash \sim \alpha \vdash \alpha [\alpha \vdash \beta \wedge \sim r ]$, (6.1) holds by $A5.4$. Since $\vdash \sim \diamond (\alpha \beta) \equiv (\alpha \vdash \beta)$, $A5.4$ yields (6.2). By $A5.4$ (6.2) also hold.

Now we assume that (a) ($\alpha \exists x \beta$) $\diamond \alpha$ holds and (b) ($\exists x p) \alpha \exists x \beta$ does not. By (a) and $A5.2$ $\alpha \exists x \beta$ and $\beta \in R$ for some $p$, or more explicitly by the choice rule; and if $\sim (b)$ holds, then for some $q \neq p$, $\alpha \exists x \beta$, whence $q \in R$ by $A5.2$. Then $\sim \diamond \alpha$ by $A5.3$, which contrasts to (a).

We conclude that $\vdash (a) \vdash (b)$.

Assume (b). In order to deduce (c) ($\exists x p) \alpha \exists x \beta$, we also assume
\[ \alpha \equiv_{p} \beta \ (i = 1, 2), \text{whence} \ p_{1}, p_{2} \in \mathbb{R} \text{and} \ p_{1} = p_{2} \text{by} \ (b). \text{Then} \ p_{1} = \neg p_{2} \text{by} \ (5.5), (3.3), \text{and} \ (3.2). \text{We conclude that} \ (c) \text{holds by} \ (2.8)_{5,4}. \text{Thus} \ \vdash (b) \supset (c); \text{and by} \ (2.8)_{5,4}, \ \vdash (c) \supset (b). \]

By \ (2.8)_{3}, \ (b) \text{and} \ \sim (a) \text{yield} \ \sim \diamond \alpha \text{and} \ (b') \ (\exists \iota)_{p} \alpha \equiv_{p} \beta. \text{By} \ \sim \diamond \alpha \text{and} \ A5.3 \ (\alpha \equiv_{p} \beta)(\alpha \equiv_{1} \beta) \text{which contrasts to} \ (b'). \text{Hence} \ (b) \vdash (a). \text{We conclude that} \ \vdash (a) \equiv (b) \equiv (c), \text{i.e.} \ (6.3) \text{holds.} \]

By A5.1 \text{for any} \ \gamma \text{of the members} \ (a) \text{to} \ (c) \text{of the equivalences in} \ (6.3), \ \vdash \gamma \equiv \Box \gamma. \]

7. The probability function \( \mathcal{F} \) and some theorems on it in \( MC_{*} \) and \( MC' \).

The probability \( \mathcal{F}(\alpha, \beta) \) of \( \text{(the event) } \beta \text{ relative to (the trial) } \alpha \text{ can be defined in both } MC_{*} \text{ and } MC' - \text{cf. def. (5.1)---by} \)

\[
(7.1) \quad \mathcal{F}_{\alpha, \beta} = D \mathcal{F}(\alpha, \beta) = D (\iota p)(\alpha \equiv_{p} \beta). \]

However in \( ML' \) this definition must be regarded as metalinguistic, whereas in \( ML_{*} \) it can be thought of as a contextual one that introduces a constant \( \mathcal{F} \) of type \( (0, 0 : t_{R}) \).

We now prove the following basic theorems on \( \mathcal{F} \), which involve modalities and the non-existing objects.

\[
(7.2) \quad \vdash (\exists_{1} p)(\alpha \equiv_{p} \beta) \iff \mathcal{F}_{\alpha, \beta} \neq a^{*}, \quad \vdash \mathcal{F}_{\alpha, \beta} \neq R \iff \mathcal{F}_{\alpha, \beta} = a^{*} \iff \mathcal{F}_{\alpha, \beta} = \neg a^{*},
\]

\[
(7.3) \quad \vdash \mathcal{F}_{\alpha, \beta} \neq a^{*} \supset p = \neg \mathcal{F}_{\alpha, \beta} \equiv \alpha \equiv_{p} \beta, \quad \vdash \mathcal{F}_{\alpha, \beta} = a^{*} \equiv \alpha \equiv_{p} \beta,
\]

\[
(7.4) \quad \vdash \diamond \alpha \cdot \mathcal{F}_{\alpha, \beta} = a^{*} \diamond (\alpha \equiv_{p} \beta), \quad \vdash (\alpha \equiv \gamma)(\beta \equiv \delta) \supset \mathcal{F}_{\alpha, \beta} = \neg \mathcal{F}_{\gamma, \delta}.
\]

**Proof.** Assume \( (a) \ (\exists_{1} p) \alpha \equiv_{p} \beta. \text{Then by} \ (4.8)_{1} \text{with} \Phi(x) \text{identified with} \ \alpha \equiv_{p} \beta \text{and} \ A5.1, \)

\[
(7.5) \quad \vdash \Box (\exists_{1} p) \alpha \equiv_{p} \beta \supset \Box \Box \alpha \equiv_{p} \beta \text{ where} \ q = D (\iota p) \alpha \equiv_{p} \beta. \]

By the last remark in N. 6 \( (a) \) yields \( \Box (a) \) and hence \( \alpha \equiv_{p} \beta \) by \( (7.5)_{1}. \text{Then, by} \ A5.2, q \in \mathbb{R}, \text{which by} \ (7.5)_{2} \text{is} \ (b) \mathcal{F}_{\alpha, \beta} \in \mathbb{R}. \text{By} \ (5.5)_{3} \text{this yields} \ (c) \mathcal{F}_{\alpha, \beta} \neq a^{*}. \)

On the other hand \( \sim (a) \) yields \( \mathcal{F}_{\alpha, \beta} = a^{*} \) by \( (7.1) \) and A3.13 \( (b). \text{We conclude that} \ (7.2)_{1} \text{holds.} \)
We saw that (a) \( \vdash (b) \) and (b) \( \vdash (c) \); and (c) \( \vdash (a) \) by (7.2). Then
\( \vdash (b) \equiv (c) \), i.e. the first equivalence in (7.2)_2 is a theorem. The second
equivalence holds by (5.5)_2, (3.3) and (3.2).

Now assume (c), whence (a) follows by (7.2), We showed on the basis of (7.5), that (a) yields \( \exists \equiv \beta \). Furthermore, by the second equivalence in (6.3), (a) yields \( (\exists p) \equiv \exists \equiv \beta \).

Then by (2.8)_4 and \( \exists \equiv \beta \) we obtain \( \exists \equiv \beta \supset p \equiv q \); hence by
\( A5.2 \) and (7.5)_3, \( \exists \equiv \beta \supset p \equiv \exists \equiv \beta \). The converse implication is a the-
orem by \( A3.12 \). We conclude that (7.3)_1 holds.

Now assume \( \diamond \alpha \) and \( \exists \equiv \beta \), whence (a) follows by (6.3). Then,
by the instance of \( A3.13 \) (a) and by \( A5.2 \), we
obtain \( p \equiv \exists \equiv \beta \). Lastly (a) and (7.2)_1 yield \( \exists \equiv \beta \equiv a^* \). We conclude that
\[ (7.6) \]
\( \vdash \diamond \alpha \wedge (\exists \equiv \beta) \supset p \equiv \exists \equiv \beta \equiv a^* \).

Let us conversely assume \( \diamond \alpha \wedge p \equiv \exists \equiv \beta \equiv a^* \). Then (a) holds by
(7.2)_1, which yields \( \square (a) \), as we saw. Then we have \( \exists \equiv \beta \) by (7.5).
This, \( p \equiv \exists \equiv \beta \), and \( A3.12 \) yield \( \exists \equiv \beta \). We conclude that
\( \vdash \diamond \alpha \wedge p \equiv \exists \equiv \beta \equiv a^* \), which by (7.6) yields (7.3)_2.

Assume \( \diamond \alpha \) and \( (\exists p). \equiv \beta \), whence \( \exists \equiv \beta \) by the choice rule. Then
\( \exists \equiv \beta \equiv a^* \) by (7.5)_3: Thus we can assert theorem (7.4)_1. It yields (7.4)_2
by (7.1) and (5.3).

q.e.d.

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of the Philosophy of Science Association, edited by K. F. Schaffner and


Manoscritto pervenuto in redazione il 18 febbraio 1980.