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A sufficient condition for existence of real analytic solutions of P.D.E. with constant coefficients, in open sets of $\mathbb{R}^2$

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A Sufficient Condition for Existence of Real Analytic Solutions of P.D.E. with Constant Coefficients, in Open Sets of $\mathbb{R}^2$.

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0. – Given a differential operator P with constant coefficients, and an open set $\Omega \subset \mathbb{R}^2$, Hörmander and Ehrenpreis found a sufficient (and necessary) condition in order that:

(1). $$PC^\infty(\Omega) = C^\infty(\Omega)\,.$$

They proved that (1) holds if (and only if):

(2) every characteristic line of P intersects Ω in an open interval.

Here we prove that (2) is also sufficient in order that:

(3) $$PA(\Omega) = A(\Omega)\,,$$

where $A(\Omega)$ is the space of real analytic functions in Ω.

We mention that Kawai proved that (2) is sufficient when Ω is bounded [5]; he constructed « good » elementary solutions for locally hyperbolic operators i.e. solutions with analytic singular supports contained in the union of the positive or negative local propagation cones i.e. in the cones along which the singularities of the solutions to the homogeneous equations propagate.

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Using flabbiness of the sheaf of germs of hyperfunctions as well as that of Sato's sheaf $\mathcal{C}$ he gave the analytic solution of the equation by means of a convolution with the fundamental solution. Of course such a method breaks down when the open set Ω is unbounded.

On the contrary we tackle the problem from the point of view of the resolvability of an overdetermined system ($Pu = f, Qu = 0$) in an open neighbourhood of Ω in $\mathbf{R}^3$ (see for instance [3]) to give Kawai's result also when Ω is unbounded. We emphasize that our tools of proof are much simpler than those of Kawai. In fact we only use: resolvability of Cauchy's problems for operators with hyperbolic principal part; Holmgren's uniqueness theorem; and by De Giorgi first and fifth conjectures that have been proved by us in [6].

1. – Lemma. *Let E be a homogeneous polynomial of n variables which never vanishes in $\mathbf{R}^n \sim \{0\}$. Let $\bar{E}$ be the polynomial whose coefficients are the conjugated of those of E and let t be a new variable. Then $E\bar{E} + t^{2\deg E}$ never vanishes in $\mathbf{R}^{n+1} \sim \{0\}$.*

Proof. $E\bar{E}$ is obviously real and since it never vanishes in $\mathbf{R}^n \sim \{0\}$, there it has constant (positive) sign.

Theorem. *Let Ω be an open set of $\mathbf{R}^2$. If $PC^\infty(\Omega) = C^\infty(\Omega)$ then $PA(\Omega) = A(\Omega)$.*

Proof. As already mentioned one needs prove that (2) implies (3). If m is the order of P let $P = P_m + R$, where P_m is homogeneous and $\deg R < m$, and substitute $\big(-i(\partial/\partial x), -i(\partial/\partial y)\big)$ with (x, y). Suppose that every real zero of P_m is proportional to some element in the set $\{(\gamma_i, -1)\}_{i=1,\dots,r}$ where every zero is repeated as many times as its multiplicity. Since P_m is homogeneous one has: $P(x, y) = \prod_{i=1}^{r}(x + \gamma_i y)E + R$ where E never vanishes in $\mathbf{R}^2 \sim \{0\}$. Consider, $\forall n \in N$, the set $\Omega_{1/n} \cap S(n)$ where $\Omega_{1/n} = \{x \in \Omega\colon d(x, \mathbf{R}^2 \sim \Omega) > 1/n\}$ and $S(n) = \{x \in \mathbf{R}^2\colon |x| < n\}$. Given a decreasing succession of positive reals $\{T_n\}$ set $\Omega_T = \bigcup_{n=1}^{\infty} C\big(\Omega_{1/n} \cap S(n), T_n\big)$ where $C\big(\Omega_{1/n} \cap S(n), T_n\big) = \{(x, y, t) \in \mathbf{R}^3\colon (x, y) \in \Omega_{1/n} \cap S(n), |t| < T_n\}$. As $\{T_n\}$ varies in the set of the decreasing successions of positive reals, the opens Ω_T describe a fundamental system of neighbourhoods of Ω in $\mathbf{R}^3$. If $\deg E > 0$, setting $P' = -\prod_{i=1}^{r}(D_x +$

$+ \gamma_i D_y) D_t^{2\deg E} + R\bar{E}$ (1) and $Q = E\bar{E} + D_t^{2\deg E}$ which is elliptic by the previous lemma, it is enough to prove that, $\forall \{T_n\}$, $(\Omega_T, \Omega_T, P', Q)$ is compatible i.e. that $\forall w \in \mathcal{E}_Q(\Omega_T)$ $(= \{w \in C^\infty(\Omega_T): Qw = 0\})$ there exists $u \in \mathcal{E}_Q(\Omega_T)$ s.t. $P'u = w$ or, since $D_t^{2\deg E} u = -E\bar{E}u$, s.t.

$$\prod_{i=1}^{r} (D_x + \gamma_i D_y) E\bar{E}u + R\bar{E}u = w\,.$$

If this is so, $\forall f \in A(\Omega)$ there exists, in view of the Cauchy-Kowalevsky theorem, a Ω_T and a function $\tilde{f} \in \mathcal{E}_Q(\Omega_T)$ s.t. $\tilde{f} = f$ in Ω; then if $u \in \mathcal{E}_Q(\Omega_T)$ resolves $P'u = \tilde{f}$ it follows that the restriction of $\bar{E}u$ to Ω is an analytic solution of the equation $Pu = f$ (2).

Fix $\{T_n\}$ and choose a point in $\mathbf{R}^3 \sim \Omega_T$, that, by translation of the coordinate system, can be 0. Consider a closed and convex angular domain H in the plane $t = 0$, with vertex in 0, contained in $\mathbf{R}^3 \sim \Omega_T$, whose boundary are characteristic lines respect to $\prod_{i=1}^{r} (x + \gamma_i y)$ (thought of as a polynomial in two variables), and such that no characteristic line of $\prod_{i=1}^{r} (x + \gamma_i y)$ intersects the domain only in 0. Such a domain exists because $\Omega_T \cap \{\xi: \xi_3 = 0\}$, if it is not empty, coincides with $\Omega_{1/n} \cap S(n)$ for some n; since Ω verifies (2), $\Omega_{1/n} \cap S(n)$ also verifies it; finally use theorem 3.7.2 of [4]. One can also suppose that $H \times \{t: t \geqslant 0\} = H'$ is contained in $\mathbf{R}^3 \sim \Omega_T$ (otherwise one reasons on $H \times \{t: t \leqslant 0\}$). We want to prove that $(\mathbf{R}^3 \sim H', \mathbf{R}^3 \sim H', P', Q)$ is compatible since then, because of the arbitrariness of the chosen point, we can conclude in view of the De Giorgi's fifth conjecture proven by us, in the elliptic case, in theorem 3 of [6] (3).

To this end it is enough to prove, in view of the first conjecture (partially proved in theorem 1 of [6]), that for every B, relatively compact open of $\mathbf{R}^3 \sim H'$, $(\mathbf{R}^3 \sim H', B, P', Q)$ is compatible.

Note that P'_m, i.e. the principal part of P', is hyperbolic with respect to every non-characteristic cotangent vector. In fact, $\forall i$, $x + \gamma_i y$ is hyperbolic with respect to $(1, \gamma_i, 0)$ and thus by theorem 5.5.5

(1) D_x denotes $-i\,\partial/\partial x$; the same for D_y and D_t.

(2) If $\deg E = 0$ we set $P' = P$ and $Q = \Delta$; nothing changes in the following.

(3) In fact a consequence of that theorem is that, if P and Q are relatively prime with Q elliptic, the (infinite) intersection of (P, Q)-convex open sets is (P, Q)-convex if it is open (Ω is (P, Q)-convex if and only if (Ω, Ω, P, Q) is compatible).

of [4], with respect to every vector of $\Gamma(x+\gamma_i y, (1,\gamma_i, 0))$ which is the component of $(1,\gamma_i,0)$ in the open $\{\xi\in\mathbf{R}^3: \xi_1+\gamma_i\xi_2\neq 0\}$ (we identify a vector N with the point $0+N$). Since it is hyperbolic also with respect to $(-1,-\gamma_i,0)$, then it is with respect to all vectors in $\mathbf{R}^3\sim\{\xi: \xi_1+\gamma_i\xi_2=0\}$. On the other hand $t^{2\deg E}$ is hyperbolic with respect to the vectors of $\mathbf{R}^3\sim\{\xi: \xi_3=0\}$.

We conclude noting that the product of polynomials that are hyperbolic with respect to a vector, is hyperbolic with respect to that vector.

Therefore if N is a non-characteristic vector of H' s.t. $\{\xi: \langle\xi, N\rangle<0\}\subset \subset \mathbf{R}^3\sim H'$ then P'_m is hyperbolic with respect to N and $H'=\Gamma^*(P'_m, N)$ ($=\Gamma^*$ in short) where $\Gamma^*(P'_m, N)$ is the closed dual cone of $\Gamma(P'_m, N)$ (see [4] pg. 137).

We are now able to use the classical existence theorem for the Cauchy problem with data in the classes of functions γ^δ (4). Fix ε, real positive, and consider, $\forall f\in\mathcal{E}_Q(\mathbf{R}^3\sim\Gamma^*)$, the function $f\cdot(\chi*\varphi)$ where χ is the characteristic function of the set $\mathbf{R}^3\sim(-(\varepsilon/2)N+\Gamma^*)$ and $\varphi\in\gamma^\delta\cap C_0^\infty$ (with $1<\delta<m/(m-1)$ and $\int\varphi\,dx=1$) has so small support that $\chi*\varphi=1$ in $\mathbf{R}^3\sim((-\varepsilon/2-\varepsilon/4)N+\Gamma^*)$ (5). Then $\chi*\varphi$ is in γ^δ and so $f\cdot(\chi*\varphi)$ since f is analytic and $\operatorname{supp}(\chi*\varphi)\subset\mathbf{R}^3\sim\Gamma^*$; finally $f\cdot(\chi*\varphi)$ coincides with f in $\mathbf{R}^3\sim((-\varepsilon/2-\varepsilon/4)N+\Gamma^*)$. Let u be a solution in the half-space $\langle\xi, N\rangle<-(\varepsilon/2+\varepsilon/4)|N|^2$ of the system:

$$\begin{cases} P'u=f=f\cdot(\chi*\varphi)\\ Qu\ =0\end{cases}$$

which exists by the theorem of resolvability in convex regions (see theorem A in [4]). Since the principal part of P' is hyperbolic with respect to N we can solve in the class γ^δ the following Cauchy problem:

$$\begin{cases} P'\tilde u=f\cdot(\chi*\varphi)\\ \tilde u-u=0(\langle\xi, N\rangle+\varepsilon|N|^2)^m,\end{cases}$$

(4) For the definition and the properties see [4], p. 146; here we recall that if Γ^δ is the δ-th Gevray class then $\gamma^\delta\subset\Gamma^\delta\subset\gamma^{\delta+\eta}$ for every positive real η and that if $\delta>1$ there exist functions $\varphi\in\gamma_0^\delta$ $(=\gamma^\delta\cap C_0^\infty)$ with support in an arbitrary neighbourhood of 0 s.t. $\varphi\geqslant 0$ and $\int\varphi\,dx=1$; finally γ^δ is a ring and the convolution $u*\varphi$ is in γ^δ if $u\in\mathcal{D}'(\mathbf{R}^n)$ and $\varphi\in\gamma_0^\delta$.

(5) If H consists of one characteristic line we consider the sets $(\mathbf{R}^3\sim\Gamma^*)_\varepsilon$ instead of $\mathbf{R}^3\sim(-\varepsilon N+\Gamma^*)$; in this case we must solve a Cauchy problem for every plane $\langle\xi, N\rangle=-\eta$ with $\eta\geqslant\varepsilon|N|$.

or equivalenty, by the uniqueness of the solution to the non-characteristic Cauchy problem, we can find a solution of $P'\tilde{u} = f\cdot(\chi * \varphi)$ which coincides with u in $\langle \xi, N\rangle < -\varepsilon|N|^2$. By Holmgren's uniqueness theorem, given in the form of Corollary 5.3.2 of [4], we have $Q\tilde{u} = 0$ in $\mathbb{R}^3 \sim (-\varepsilon N + \Gamma^*)$; in fact $P'Q\tilde{u} = 0$ in $\mathbb{R}^3 \sim (-\varepsilon N + \Gamma^*)$ and $Q\tilde{u} = 0$ in $\langle \xi, N\rangle < -\varepsilon|N|^2$. It is therefore proved that $\forall f \in \mathcal{E}_Q(\mathbb{R}^3 \sim \Gamma^*)$ the system $(P'u = f, Qu = 0)$ is resolvable on the open $\mathbb{R}^3 \sim (-\varepsilon N + \Gamma^*)$ and thus, with ε tending to zero, on all relatively compact open subsets of $\mathbb{R}^3 \sim \Gamma^*$.

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