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On Pocket and Empirical Temperatures. An Alternative Choice for the Heat Flux Vector in Eckart's Relativistic Thermodynamics.

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SUMMARY - Some results related with pocket temperature are presented within a certain general version $\mathcal{T}_E$ of Eckart's relativistic thermodynamics. They allow us to propose an (equally acceptable) alternative choice for the heat flux vector $\overset{P}{q}{}^\alpha$ that, unlike Eckart's $\overset{E}{q}{}^\alpha$, does not involve intrinsic acceleration. It is shown that $\overset{P}{q}{}^\alpha = [1 + O(c^{-4})]\overset{E}{q}{}^\alpha$ for non-viscous fluids. By the above results in $\mathcal{T}_E$, the general solution of a fundamental differential relation, stated within Alts and Müller's theory $\mathcal{T}_{AM}$, among empirical temperature ϑ, absolute temperature T, and mass density is found. An agreement between $\mathcal{T}_{AM}$ and the Chernikov's kinetic relativistic theory $\mathcal{T}_C$ is shown to hold up to $O(c^{-4})$. It is shown that $\mathcal{T}_E$ and $\mathcal{T}_{AM}$ are supported by $\mathcal{T}_C$ equally well.

1. Introduction.

In [1] Alts and Müller consider a relativistic theory of thermodynamics, say $\mathcal{T}_{AM}$, in which the usual *absolute temperature* T is replaced by the *empirical temperature* ϑ; this temperature is given

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an operational meaning only in certain equilibrium processes called *E-equilibria*. In these processes $d\vartheta$ is shown in $\mathcal{T}_{AM}$ to equal a certain differential form $\alpha\, dT + \beta\, dk$ where k is the conventional mass density—cf. [3], § 21; and by a natural physical assumption this relation, $d\vartheta = \alpha\, dT + \beta\, dk$, can be regarded as a completely determined differential link among ϑ, T, and k.

Within the general version $\mathcal{T}_E$ of Eckart's thermodynamics, that is presented in [3], I define *e-equilibrium* (N. 3)—a notion stronger then the analogue of E-equilibrium (used in $\mathcal{T}_{AM}$)—, by requiring the absence of heat flux ($q^\alpha \equiv 0$) and Born rigidity ($u_{(\alpha/\overset{\perp}{\beta})} \equiv 0$); furthermore I show that for a viscous fluid $\mathcal{F}$ that is capable of heat conduction and satisfies the 2nd principle—cf. (2.10)—the scalar field $\Theta = T(1 + c^{-2}\phi)^{-1}$, where ϕ is (Gibbs's) free enthalpy, is constant on regions of e-equilibrium ($\Theta_{/\alpha} \equiv 0$). Hence Θ appears as a generalization of the *pocket temperature* $T_{(p)} = T\sqrt{-g_{00}}$ studied by Tolman and Ehrenfest—cf. [3], § 45. This first results presented in this paper belongs to $\mathcal{T}_E$ and are in a tight agreement with some results obtained in [8] within a theory of relativistic thermostatics based on a variational version of the second principle (1).

The relation between the metric tensor $g_{\alpha\beta}$ (precisely g_{00}) and ϕ, or the Newtonian potential U in the case of weak gravitation, is briefly shown in N. 4.

In N. 5 an alternative to Eckart's choice $\overset{E}{q}_\alpha = -\varkappa(T_{/\overset{\perp}{\alpha}} + TA_\alpha)$ for the heat flux vector q_α is presented: a vector $\overset{P}{q}_\alpha = -\bar{\varkappa}\Theta_{/\overset{\perp}{\alpha}}$, which I call *pocket flux*. It leads to a non equivalent but equally acceptable relativistic thermodynamics for heat conducting fluids. Indeed it can be shown (N. 5) that (i) $\overset{P}{q}_\alpha = [1 + O(c^{-r})]\overset{E}{q}_\alpha$ with $r = 2$ [$r = 4$] for viscous [non-viscous] fluids, where c is the speed of light in vacuum and « $O(c^{-r})$ » means terms of the order of c^{-r}, and (ii) $\overset{P}{q}_\alpha \equiv \overset{E}{q}_\alpha = 0$ in e-equilibria. By (i) the alternative $\mathcal{T}_P$ to $\mathcal{T}_E$ obtained from $\mathcal{T}_E$ by substituting $\overset{P}{q}_\alpha$ for $\overset{E}{q}_\alpha$ is in agreement with experiments and also complies with the theorical considerations made to support Eckart's choice $\overset{E}{q}_\alpha$ for q_α, which just required (ii) to hold—cf. [3], § 45.

(1) The above result of mine in $\mathcal{T}_E$ concerning Θ and $T_{(p)}$ is taken from my thesis for the degree in physics in July 1978, which thesis was presented at the national competition for a Grant of the C.N.R. Bando no. 201.1.89 (term: 22th July 1978). After [8] appeared in 1979 the deduction of the aforementioned result in $\mathcal{T}_E$ is still interesting because of the difference between $\mathcal{T}_E$ and the thermostatic theory used in [8].

In N. 6 $\mathcal{T}_{AM}$ and $\mathcal{T}_E$ are compared in connection with the fluids dealt within $\mathcal{T}_{AM}$, the non-viscous ones. First this is done in the case of E-equilibrium; and then in connection with processes near those equilibrium processes. More in detail, for the above equilibrium differential relation $d\vartheta = \alpha\, dT + \beta\, dk$ obtained in $\mathcal{T}_{AM}$, in [1] integrability conditions are written. Here (N. 6) the general solution of this relation is shown to be $\vartheta = f(\Theta)$, where f is any mapping of $\mathbb{R}$ in $\mathbb{R}$ of class $C^{(1)}$.

Lastly in [1], N. 5, the authors assume (in $\mathcal{T}_{AM}$) that along processes near E-equilibria the constitutive functions considered there have the same form as in equilibrium processes—i.e. do not involve $\vartheta_{/\alpha}$. This assumption is compatible with $\mathcal{T}_{AM}$'s axioms up to $O(c^{-4})$—cf. fnt (6) in N. 6. Under the above assumption the heat flux q_α in $\mathcal{T}_{AM}$—where non-viscous fluids are treated—is shown to be a vector $\overset{L}{q}_\alpha$ parallel with the analogous heat flux $\overset{C}{q}_\alpha$ obtained within Chernikov's relativistic kinetic theory $\mathcal{T}_C$—see [5] to [7]—, and $\overset{L}{q}_\alpha$ can be identified with $\overset{C}{q}_\alpha$. On the other hand, by (i) above, $\overset{E}{q}_\alpha$ can be identified with $\overset{L}{q}_\alpha$ and $\overset{C}{q}_\alpha$. Thus Alts and Müller's assertion on $\mathcal{T}_{AM}$—see [1], N. 7—that the theories $\mathcal{T}_{AM}$ and $\mathcal{T}_C$ support each other, also holds for $\mathcal{T}_E$; and in either case the agreement occurs up to $O(c^{-4})$.

2. Some basic notions and theorems of Eckart's relativistic thermodynamics in its general version $\mathcal{T}_E$ presented in [3].

A) Preliminaries on space-time.

The notions and notations introduced in [3] are presupposed in this paper. Let $\mathcal{E}$ be an event point of the space-time $\mathcal{S}_4$ of general relativity, and let x^α ($\alpha = 0, \ldots, 3$) be its co-ordinates (2) in a given (admissible) reference frame—cf. [3], p. 37. For the metric at $\mathcal{E}$ we have

$$(2.1) \qquad ds^2 = -\, g_{\alpha\beta}\, dx^\alpha\, dx^\beta\,, \qquad g_{00} < 0\,, \qquad \operatorname{sign}[g_{\alpha\beta}] = +\,2\,.$$

Assume that $\mathcal{C}$ is a continuous body, P^* is any material point of it, and $x^\alpha = x^\alpha(s)$ describes the world line $\mathcal{W}_{P^*}$ of P^*. Then for

(2) Greek [Latin] letters run from 0 [1] to 3.

the 4-velocity u^α and intrinsic acceleration A^α of P^* we have

$$(2.2)\qquad u^\alpha = \frac{Dx^\alpha}{Ds}\left(=\frac{dx^\alpha}{ds}\right), \qquad A^\alpha = \frac{Du^\alpha}{Ds}, \qquad u^\alpha u_\alpha = -1\,,\ A^\alpha u_\alpha = 0\,.$$

For any tensor field $T^{\cdots}_{\cdots}$, $T^{\cdots}_{\cdots/\alpha}$ denotes its covariant derivatives based on the metric (2.1) while

$$(2.3)\qquad \dot{T}^{\cdots}_{\cdots} = \frac{DT^{\cdots}_{\cdots}}{Ds} = T^{\cdots}_{\cdots/\alpha}u^\alpha$$

is its *material derivative* (in Römer units). Let us set

$$(2.4)\qquad \overset{\perp}{g}_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta\,, \qquad T^{\cdots}_{\cdots\overset{\perp}{\alpha}} = T^{\cdots}_{\cdots\beta}\overset{\perp}{g}{}^{\beta}{}_{\alpha} \qquad (\text{whence } T^{\cdots}_{\cdots\overset{\perp}{\alpha}}u^\alpha = 0)\,.$$

The index α of $T^{\cdots}_{\cdots\alpha}$ is said to be *spatial* if $T^{\cdots}_{\cdots\alpha}u^\alpha = 0$.

B) Einstein gravitation and conservation equations for materials capable of heat conduction.

Let $\mathcal{T}_E$ be C. Eckart's theory of relativistic thermodynamics——cf. [9]— in the general version presented in [3], but in absence of electromagnetic phenomena and couple stress. Assume that c is the speed of light in vacuum, $k[\varrho]$ is conventional mass [gravitational mass (in energy units)] per unit proper volume—cf. [3], p. 54—, q_α is the (spatial) heat flux (vector), $Q_{\alpha\beta}$ is Eckart's thermodynamic tensor, and $X^{\alpha\beta}$ is the (completely spatial) Eulerian stress tensor. Then [3]

$$(2.5)\qquad Q_{\alpha\beta} = 2u_{(\alpha}q_{\beta)}\,, \qquad q^\alpha u_\alpha = 0\,, \qquad u_\alpha X^{\alpha\beta} = 0 = X^{\alpha\beta}u_\beta;$$

furthermore the continuity equation and definition of the (actual) internal energy w per unit reference mass read—cf. [3]:

$$(2.6)\qquad (ku^\alpha)_{/\alpha} = 0\,, \qquad \varrho = k(c^2 + w)\,.$$

Denoting by $A_{\alpha\beta}$ $(= A_{\beta\alpha})$ and h Levi Civita's tensor and Cavendish's constant respectively, in the framework of $\mathcal{T}_E$ *Einstein gravita-*

[3] $2T_{(\alpha\beta)} = T_{\alpha\beta} + T_{\beta\alpha}\,, \quad 2T_{[\alpha\beta]} = T_{\alpha\beta} - T_{\beta\alpha}\,.$

tion equations read

$$(2.7)\qquad A_{\alpha\beta}+\frac{8\pi h}{c^4}\mathcal{U}_{\alpha\beta}=0$$

$$\text{with } \mathcal{U}_{\alpha\beta}=\varrho u_\alpha u_\beta+X_{\alpha\beta}+Q_{\alpha\beta},\ \text{hence } X_{[\alpha\beta]}=0\,.$$

Of course the consequence $(2.7)_3$ of $(2.7)_{1,2}$, $(2.5)_1$, and the symmetry of $A_{\alpha\beta}$, constitute the relativistic version of the 2nd Cauchy equation for non polar continuous media. The spatial and temporal part of conservation equations—which constitute the consequence $\mathcal{U}^{\alpha\beta}{}_{/\beta}=0$ of $(2.7)_1$—can be put into the respective forms—cf. [3], p. 62:

$$(2.8)\quad \begin{cases} \varrho A_\alpha=-\overset{\perp}{g}_{\alpha\gamma}(X^{\gamma\beta}+Q^{\gamma\beta})_{/\beta} \\ \qquad\qquad \text{where } \overset{\perp}{g}_{\alpha\gamma}Q^{\gamma\beta}{}_{/\beta}=k\left(\overset{\perp}{g}_{\alpha\varrho}\dfrac{D}{Ds}\dfrac{q^\varrho}{k}+u_{\alpha/\sigma}\dfrac{q^\sigma}{k}\right), \\ k\dfrac{Dw}{Ds}+\dfrac{\delta l^{(i)}}{Ds}=c^{-1}kq_{\text{ass}} \\ \qquad\qquad \text{where } kq_{\text{ass}}=cu_\alpha Q^{\alpha\beta}{}_{/\beta} \text{ and } \dfrac{\delta l^{(i)}}{Ds}=X^{\alpha\beta}u_{\alpha/\beta}\,. \end{cases}$$

Assume that $\mathcal{E}\in\mathcal{W}_{\mathcal{C}}$, the world tube of $\mathcal{C}$, and that the frame (x) is natural and proper at $\mathcal{E}$, i.e.

$$(2.9)\quad g_{\alpha\beta,\gamma}=0\,,\qquad g_{\alpha r}=\delta_{\alpha r}\,,\qquad g_{00}=-1\,,\qquad u^\alpha=\delta^\alpha_0\qquad (f_{,\gamma}=\partial f/\partial x^\gamma)$$

hold there. Then, up to terms of order c^{-2}, equations $(2.8)_{1,3}$ equal the 1st Cauchy dynamic equation for continuous media and the first principle, written within classical physics in a Euclidean frame that is locally freely gravitating and non-rotating with respect to Galileian frames. Hence they constitute acceptable relativistic versions of those laws.

C) Second principle of thermodynamics.

Let T (>0) be the absolute temperature of $\mathcal{C}$ at the typical event point $\mathcal{E}\in\mathcal{W}_{\mathcal{C}}$, measurable by observer locally joined to matter. The 2nd principle of thermodynamics reads in relativity theory substantially in the same way as in classical physics:

For every material point P^* of $\mathcal{C}$ there is a function η of the local intrinsic physical state of $\mathcal{C}$ at P^*—called the entropy function—such *that along avery possible physical process we have*—cf. [3 (25.1)]—

$$k\frac{D\eta}{D\tau} \geqslant \frac{q_{\text{ass}}}{T} \qquad (s = c\tau)\,. \tag{2.10}$$

On the other hand, in classical physics, Clausius-Duhem inequality

$$\int_V k\dot{\eta}\, dv \geqslant \int_{\partial V} \frac{\bar{q}^i}{T} da_i \qquad (\bar{q}^i = cq^i)\,, \tag{2.11}$$

constitutes a much used version of the 2nd principle. It yields

$$k\dot{\eta} \geqslant -\left(\frac{\bar{q}^i}{T}\right)_{,i}\,. \tag{2.12}$$

This local form is relativized into

$$k\frac{D\eta}{Ds} \geqslant -\left(\frac{q^\alpha}{T}\right)_{/\alpha}\,. \tag{2.13}$$

Thus the classical divergence $(T^{-1}\bar{q}^i)_{,i}$ is relativized into a space-time divergence (and not e.g. into $(T^{-1}q^\alpha)_{/\underset{\alpha}{\perp}}$) in harmony with Cattaneo's point of view—cf. [4]—justified by Bressan in [2] by means of kinematic considerations.

If $\mathcal{C}$ is capable of only *reversible processes*, i.e. processe that render (2.10) an equality in $\mathcal{W}_{\mathcal{C}}$, then by $(2.8)_4$ and $(2.5)_1$, (2.13) yields

$$q^\alpha\theta_\alpha \leqslant 0 \qquad \text{where } \theta_\alpha = T_{/\underset{\alpha}{\perp}} + TA_\alpha\,. \tag{2.14}$$

By considerations involving pocket temperature, θ_α appears to be the relativistic analogue of the classical temperature gradient $T_{,i}$ —cf. [3], §§ 25, 45. Then (2.14) is a natural relativization of the classical relation $\bar{q}^iT_{,i} \leqslant 0$, assumed to hold for general processes and materials. Therefore is, besides (2.10), inequality (2.14) postulated (4).

(4) This formulation constitutes substantially a relativistic version of the classical version of the 2nd principle in [13]: $\gamma_{\text{loc}} \geqslant 0$, $\gamma_{\text{con}} \geqslant 0$ where $\gamma_{\text{loc}} = \dot{\eta} + (kT)^{-1}\bar{q}^i_{,i}$, $\gamma_{\text{con}} = -k^{-1}T^{-2}\bar{q}^iT_{,i}$.

D) Explicit form of the heat flux vector. On some viscous fluids capable of heat conduction.

The following theorem is proved in $\mathcal{C}_E$—cf. [3], Theor. 25.1, p. 66:

Let q^α be a function of the position gradient α^ϱ_L, T, $T_{/\overset{\perp}{\alpha}}$, and A_α, that is linear in $T_{/\overset{\perp}{\alpha}}$ and A_α. Then inequality (2.14) implies, in relativity, the version

$$(2.15)\qquad q^\alpha = -\varkappa^{\alpha\beta}(T_{/\beta} + TA_\beta) \qquad (\varkappa^{\alpha\beta}\zeta_\alpha\zeta_\beta > 0)$$

of Fourier's law, with $\varkappa^{\alpha\beta}$ spatial and depending at most on α^ϱ_L and T. Since for fluids Eckart proposed the special version

$$(2.16)\qquad q_\alpha = -\varkappa(T_{/\overset{\perp}{\alpha}} + TA_\alpha) \qquad (\varkappa > 0)$$

of the relation (2.15), this is often called the Fourier-Eckart law of heat conduction.

A. Bressan proved—cf. [3], Theor. 25.3—that $\varkappa^{[\alpha\beta]} = 0$ iff q_{ass} complies with the *principle of material frame indifference*—cf. [3], §§ 80-82—or more simply iff q_{ass} is rotationally objective.

In [3] the viscous fluids are considered for which w, η, and $X^{\alpha\beta}$ are functions of k, T, $u_{\alpha/\overset{\perp}{\beta}}$, and also N unspecified physical parameters ξ_1 to ξ_N; after setting

$$(2.17)\qquad \psi = w - T\eta = \breve{\psi}(k, T, u_{\varrho/\overset{\perp}{\sigma}}, \xi_1, \ldots, \xi_N)\,, \qquad X^{\alpha\beta} = k^2\frac{\partial\breve{\psi}}{\partial k}\overset{\perp}{g}{}^{\alpha\beta} + X^{\alpha\beta}_{(\text{irr})}\,,$$

so that ψ is the free energy, the constitutive relations

$$(2.18)\qquad \eta = -\frac{\partial\breve{\psi}}{\partial T}\,, \quad \frac{\partial\breve{\psi}}{\partial u_{\varrho/\overset{\perp}{\sigma}}} = 0 = \frac{\partial\breve{\psi}}{\partial\xi_i}\,, \qquad i = 1, \ldots, N\,, \; X^{\alpha\beta}_{(\text{irr})}u_{\alpha/\beta} \leqslant 0$$

are proved there with a procedure of the Coleman-Noll type. Set

$$(2.19)\qquad p = k^2\frac{\partial\breve{\psi}(k, T)}{\partial k}\,, \qquad \text{whence } X^{\alpha\beta} = p\overset{\perp}{g}{}^{\alpha\beta} + X^{\alpha\beta}_{(\text{irr})}\,.$$

As is well known by Helmoltz's postulate and (2.18-19) the expressions $T = \tilde{T}(k, \eta)$ and $w = \tilde{w}(k, \eta)$ can be specified and the classical Gibbs's relation

$$p = k^2 \frac{\partial \tilde{w}(k, \eta)}{\partial k}, \qquad T = \frac{\partial \tilde{w}(k, \eta)}{\partial \eta}, \qquad T\, d\eta = dw + p\, d\frac{1}{k} \tag{2.20}$$

can be deduced.

In the sequel a (perhaps non-linearly) viscous fluid $\mathcal{F}$ is considered. Let it be described by the constitutive equations (2.16) and (2.20) where—cf. $(2.18)_4$—

$$X^{\alpha\beta}_{(\mathrm{irr})} = \tilde{X}^{\alpha\beta}_{(\mathrm{irr})}(k, \eta, u_{(\varrho/\overset{\perp}{\sigma})}), \qquad \tilde{X}^{\alpha\beta}_{(\mathrm{irr})}(k, \eta, 0) = 0 . \tag{2.21}$$

3. Generalization of pocket temperature. Expression of this temperature in thermodynamics terms.

The conditions for classical thermodynamic equilibrium ($\bar{q}^i \equiv 0$, $v^i{}_{,j} \equiv 0 \equiv v^i$) involve a rigid motion. Therefore it is natural to extend this notion of equilibrium to general relativity by means of a definition such as the following

DEFINITION 3.1. *The body* $\mathcal{C}$ *is said* (*in* $\mathcal{C}_{\mathrm{E}}$) *to be in* (*or to undergo a process of*) *e-equilibrium in the region* $\mathcal{R} \subseteq \mathcal{W}_{\mathcal{C}}$, *if in* $\mathcal{R}$ *we have*

$$q^\alpha \equiv 0 \equiv u_{(\varrho/\overset{\perp}{\sigma})} . \tag{3.1}$$

Remark that if (a) $\mathcal{C}$ has a co-moving frame (x) which is stationary ($g_{\alpha\beta,0} \equiv 0$) or in particular static ($g_{\alpha\beta,0} \equiv 0 \equiv g_{0r}$) in $\mathcal{R}$, and (b) no heat conduction takes place there, then (c) $\mathcal{C}$ is in e-equilibrium in $\mathcal{R}$. Indeed $u_{(\varrho/\overset{\perp}{\sigma})} = (-g_{00})^{-\frac{1}{2}}(g_{\varrho\sigma} - g_{\varrho 0} g_{\sigma 0}/g_{00})_{,0}$ so that (a) yields $(3.1)_2$. The converse is usually false in that (c) generally fails to imply (a).

THEOREM 3.1. *Let the viscous fluid* $\mathcal{F}$—cf. (2.16), (2.20), *and* (2.21)—*be in e-equilibrium in* $\mathcal{R}$ ($\subseteq \mathcal{W}_{\mathcal{F}}$). *Then we have there*

$$\frac{Dk}{Ds} \equiv 0 , \qquad \frac{D\eta}{Ds} \equiv 0 , \qquad A_\alpha = -\frac{p_{/\overset{\perp}{\alpha}}}{\varrho + p} . \tag{3.2}$$

PROOF. By $(2.6)_1$ and $(3.1)_2$, $(3.2)_1$ holds. By $(2.8)_{3,4,5}$, (2.20), and $(2.21)_2$, (3.1) yield $(3.2)_2$; and $(3.2)_3$ follows from (3.1) by $(2.8)_{1,2}$, $(2.19)_2$ and $(2.21)_2$. q.e.d.

By (3.2) and (2.20) in a region of e-equilibrium

$$(3.3)\qquad \frac{Dg}{Ds} \equiv 0 \qquad \text{for } g = \hat{g}(k, \eta, T, p)\,.$$

In particular this holds for Gibbs's function (or free enthalpy)

$$(3.4)\qquad \phi = w + \frac{p}{k} - T\eta = \breve{\phi}(p, T)\,.$$

As is well known

$$(3.5)\qquad \frac{1}{k} = \frac{\partial \breve{\phi}}{\partial p}\,, \qquad \eta = -\frac{\partial \breve{\phi}}{\partial T}\,.$$

* * *

Let us now consider any process $\mathfrak{T}$ for $\mathfrak{C}$ in $\mathcal{C}_E$, for which the (field of the) intrinsic acceleration A_α is *lamellar*—cf. [10], p. 824—in the space time region Δ, i.e. for some scalar field φ

$$(3.6)\qquad A_\alpha(x) = \varphi(x)_{/\overset{\perp}{\alpha}}\,, \qquad \forall x \in \Delta\,, \qquad \text{where } \Delta \subseteq \mathcal{W}_{\mathfrak{C}}\,.$$

In this case can q^α be given a useful expression.

THEOREM 3.2. *In the process* $\mathfrak{T}$ *for* $\mathfrak{C}$ *let* (3.6), (2.15), *and the definitions*

$$(3.7)\qquad \Theta = Te^{\varphi}\,, \qquad \varkappa'^{\alpha\beta} = \varkappa^{\alpha\beta} e^{-\varphi}$$

hold. Then

$$(3.8)\qquad q^\alpha = -\varkappa'^{\alpha\beta}\Theta_{/\beta}\,.$$

The proof is obvious. Remark that Θ *is a natural extension to lamellar fields of Tolman and Ehrenfest's pocket temperature* $T_{(p)}$—cf. [11], [12]—which notion was introduced for the first time in 1930, in connection with the equilibrium of a black body with respect to a static frame:

$$(3.9)\qquad T_{(p)} = T\sqrt{-g_{00}}\,.$$

Indeed if the co-moving frame (for $\mathcal{C}$) is stationary (or static),

$$A_\alpha = u_{\alpha/\beta}u^\beta = \frac{g_{00,\alpha}}{2g_{00}} = (\ln\sqrt{-g_{00}})_{,\alpha}\,, \tag{3.10}$$

so that, for $\varphi = \ln\sqrt{-g_{00}}$ we have

$$\Theta = Te^\varphi = Te^{\ln\sqrt{-g_{00}}} = T\sqrt{-g_{00}} = T_{(p)}\,. \tag{3.11}$$

Therefore Θ will be called *pocket temperature* in the sequel.

Obviously A_α generally fails to be lamellar for a body $\mathcal{C}$ (in $\mathcal{C}_E$). In spite of this we can prove the following

THEOREM 3.3. *Let $\mathcal{F}$ is a viscous fluids capable of heat conduction, defined by* (2.16), (2.20), *and* (2.21); *furthermore let $\mathcal{F}$ be in e-equilibrium in $\mathfrak{R}$* ($\subseteq \mathcal{W}_\mathcal{F}$). *Then, under definition* (3.4), *in $\mathfrak{R}$ we have*

$$\begin{cases} A_\alpha = -\left[\ln\left(1+\dfrac{\phi}{c^2}\right)\right]_{/\alpha}, \\ \Theta = \dfrac{T}{1+\phi/c^2}, \qquad \Theta_{/\alpha} \equiv 0\,. \end{cases} \tag{3.12}$$

PROOF. By Theor. 3.1 $(3.2)_3$ holds; furthermore by (3.4) and $(2.6)_2$ $\varrho + p = k(\phi + T\eta + c^2)$. Hence the first of the relations

$$\begin{aligned}(\phi + T\eta + c^2)A_\alpha &= -\frac{\partial\breve{\phi}}{\partial p}p_{/\overset{\perp}{\alpha}} = -\phi_{/\overset{\perp}{\alpha}} + \frac{\partial\breve{\phi}}{\partial T}T_{/\overset{\perp}{\alpha}} = \\ &= -(\phi + c^2)_{/\overset{\perp}{\alpha}} - \eta T_{/\overset{\perp}{\alpha}}\end{aligned} \tag{3.13}$$

holds by $(3.5)_1$, while $(3.13)_{2,3}$ follow from (3.4) and $(3.5)_2$: By (2.16), for $\varkappa \neq 0$, $q^\alpha \equiv 0$ implies $\eta T A_\alpha = -\eta T_{/\overset{\perp}{\alpha}}$, so that (3.13) yields

$$A_\alpha = -[\ln(\phi + c^2)]_{/\overset{\perp}{\alpha}} = -\left[\ln\left(1+\frac{\phi}{c^2}\right)\right]_{/\overset{\perp}{\alpha}}\,. \tag{3.14}$$

In addition by (3.3) $D\phi/Ds \equiv 0$ in $\mathfrak{R}$. Then $(3.12)_1$ holds. This is

(3.6) for $\varphi = -\ln(1 + c^{-2}\phi)$. Hence by Theor. 3.2 we have $(3.7)_1$, i.e. $(3.12)_2$, and (3.8). Since $\varkappa'$, as well as $\varkappa$, is strictly positive definite, (3.8) and $(3.1)_1$ yield $\Theta_{/\overset{\perp}{\alpha}} \equiv 0$ in $\mathfrak{R}$; and the above definition of φ together with (3.3) yields $D\Theta/Ds = 0$ there. Then $(3.12)_3$ holds. q.e.d.

4. Expression of g_{00} in the stationary case of e-equilibrium. Comparison with the case of a weak (classical) gravitational field.

Let $\mathcal{F}$ be in e-equilibrium in $\mathfrak{R}$ ($\subseteq \mathcal{W}_{\mathcal{F}}$) and let any co-moving frame of it be stationary (in $\mathfrak{R}$). Then, by (3.11), for some constant K,

$$\text{(4.1)} \qquad \ln\frac{1}{K} + \ln\sqrt{-g_{00}} = \varphi = -\ln\left(1 + \frac{\phi}{c^2}\right)$$

$$\text{hence } g_{00} = -\frac{K^2}{(1 + \phi/c^2)^2} \,.$$

The thermodynamic expression $(4.1)_3$ of g_{00} holds also in a strong gravitational field provided q^α has a linear expression in $T_{/\overset{\perp}{\alpha}}$ and A_α. Let us now show that if the gravitational field is weak $(4.1)_3$ becomes the well known relation

$$\text{(4.2)} \qquad g_{00} \simeq -\left(1 - \frac{2U}{c^2}\right)$$

up to an additive constant, where U is the newtonian potential of the gravitational field. Indeed the classical equations for the thermodynamic equilibrium of a viscous fluid in the above gravitational field read $kU_{,i} = p_{,i}$ and $T_{,i} = 0$. By (3.5) they yield

$$\text{(4.3)} \qquad U_{,i} = \frac{\partial \breve{\phi}}{\partial p} p_{,i} = \phi_{,i}\,, \qquad \text{hence } U = \phi + K_1 \ (K_1 = \text{const})\,.$$

Since $c^{-2}|\phi| \ll 1$, $(1 + c^{-2}\phi)^{-2} \simeq 1 - 2c^{-2}\phi = 1 - 2c^{-2}U +$ const. Hence $(4.1)_3$ becomes (4.2), up to the constant $2c^{-2}K_1$.

Now remark that since $(4.1)_3$ must be equivalent with (4.2), $(4.1)_3$ holds for $K = 1$ and the determination of ϕ that fulfils condition $(4.3)_3$ with $K_1 = 0$.

5. An alternative to Eckart's choice of the heat flux vector that leads to a nonequivalent but equally acceptable relativistic thermodynamics for heat conducting fluids.

I want to show that *in every process for a possibly viscous fluid* $\mathcal{F}$*—defined by* (2.16), (2.20) *and* (2.21)—

$$(5.1)\qquad \overset{P}{q}{}^{\alpha} = [1 + O(c^{-2})]\overset{E}{q}{}^{\alpha}\,, \quad \textit{where } \overset{P}{q}_{\alpha} = -\bar{\varkappa}\Theta_{/\overset{\perp}{\alpha}} \textit{ and } \bar{\varkappa} = \frac{(c^2+\phi)^2 k}{c^2(\varrho + p)}\varkappa$$

($\overset{E}{q}{}^{\alpha}$ is the Fourier-Eckart heat flux vector q^{α} defined in (2.16)), *and that*

(*a*) $\overset{P}{q}_{\alpha}$ *vanishes in any e-equilibrium process.*

Since the magnitude Θ is the pocket temperature—cf. N. 3—, I shall call $\overset{P}{q}_{\alpha}$ the *pocket heat flux.*

Let $\overset{E}{\mathcal{U}}_{\alpha\beta}$ be Eckart's energy-momentum tensor for viscous fluids and let $\overset{P}{\mathcal{U}}_{\alpha\beta}$ result from it by replacing $\overset{E}{q}_{\alpha}$ with $\overset{P}{q}_{\alpha}$. By $(5.1)_1$

$$(5.2)\qquad \overset{P}{\mathcal{U}}_{\alpha\beta} = [1 + O(c^{-2})]\overset{E}{\mathcal{U}}_{\alpha\beta}\,,$$

$$\text{where } \overset{P}{\mathcal{U}}_{\alpha\beta} = \varrho u_{\alpha}u_{\beta} + X_{\alpha\beta} + 2\overset{P}{q}_{(\alpha}u_{\beta)}\,.$$

(*b*) *For non-viscous fluids one can strengthen* $(5.1)_1$ *and* $(5.2)_1$ *into*

$$(5.3)\qquad \overset{P}{q}{}^{\alpha} = [1 + O(c^{-4})]\overset{E}{q}{}^{\alpha}\,, \qquad \overset{P}{\mathcal{U}}_{\alpha\beta} = [1 + O(c^{-4})]\overset{E}{\mathcal{U}}_{\alpha\beta}\,.$$

The use of $\overset{P}{\mathcal{U}}_{\alpha\beta}$ as the energy-momentum tensor in special or general relativity is in agreement with experiments by (5.1). By (*a*) this use also complies with the considerations made to support Eckart's proposal q^{α}. Indeed these considerations substantially say that the relation $T_{/\overset{\perp}{\alpha}} = -TA_{\alpha}$ must hold rigorously in thermodynamic equilibrium—cf. [3], § 45.

In order to prove (5.1) we first write an explicit expression of $d\Theta$ for any viscous fluid. Now I consider $\phi = \hat{\phi}(k, T) = \check{\phi}[\hat{p}(k, T), T]$

where—cf. N. 2—$\hat{p}(k, T) = k^2\,\partial\breve{\psi}(k, T)/\partial k$ and $\hat{w}(k, T)$ and $\hat{\eta}(k, T)$ are defined by $(2.17)_1$ and $(2.18)_1$. By $(3.12)_2$ one easily obtains

$$(5.4)\qquad d\hat{\Theta}(k, T) = \frac{c^2}{(c^2+\hat{\phi})^2 k}\left(\hat{\varrho}+\hat{p}-T\frac{\partial\hat{p}}{\partial T}\right)\cdot \left(dT - T\frac{\partial\hat{p}/\partial k}{\hat{\varrho}+\hat{p}-T(\partial\hat{p}/\partial T)}\,dk\right).$$

On the other hand

$$(5.5)\qquad \frac{\partial\hat{\Theta}(k, T)}{\partial T} = \frac{1}{1+\hat{\phi}/c^2} - \frac{T}{c^2}\,\frac{(\partial\breve{\phi}/\partial p)(\partial\hat{p}/\partial T)+\partial\breve{\phi}/\partial T}{(1+\hat{\phi}/c^2)^2} = \frac{c^2}{(c^2+\hat{\phi})^2 k}\left(\hat{\varrho}+\hat{p}-T\frac{\partial\hat{p}}{\partial T}\right),$$

whence

$$(5.6)\qquad d\hat{\Theta}(k, T) = \frac{\partial\hat{\Theta}(k, T)}{\partial T}\left(dT - T\frac{\partial\hat{p}/\partial k}{\hat{\varrho}+\hat{p}-T(\partial\hat{p}/\partial T)}\,dk\right).$$

Now let us eliminate A_α from the expression (2.16) of $\overset{\mathrm{E}}{q}_\alpha$ in connection with the above typical viscous fluid. By (2.16) and $(2.8)_1$ we obtain

$$\overset{\mathrm{E}}{q}_\alpha = -\varkappa\left[T_{/\overset{\perp}{\alpha}} - T\,\frac{p_{/\overset{\perp}{\alpha}} + \overset{\perp}{g}_{\alpha\varrho}(X^{\varrho\sigma}_{(\mathrm{irr})}+Q^{\varrho\sigma})_{/\sigma}}{\varrho+p}\right],$$

hence, for $p = \hat{p}(k, T)$ and $\varrho = \hat{\varrho}(k, T) = k[\hat{w}(k, T) + c^2]$,

$$(5.7)\qquad \overset{\mathrm{E}}{q}_\alpha = -\frac{\hat{\varrho}+\hat{p}-T(\partial\hat{p}/\partial T)}{\hat{\varrho}+\hat{p}}\,\varkappa\cdot \left[T_{/\overset{\perp}{\alpha}} - T\frac{\partial\hat{p}/\partial k}{\hat{\varrho}+\hat{p}-T(\partial\hat{p}/\partial T)}\,k_{/\overset{\perp}{\alpha}} - T\frac{\overset{\perp}{g}_{\alpha\varrho}(X^{\varrho\sigma}_{(\mathrm{irr})}+Q^{\varrho\sigma})_{/\sigma}}{\hat{\varrho}+\hat{p}-T(\partial\hat{p}/\partial T)}\right].$$

By (5.4), under definitions $(5.1)_{2,3}$, (5.7) becomes

$$(5.8)\qquad \overset{\mathrm{E}}{q}_\alpha = \overset{\mathrm{P}}{q}_\alpha + \frac{\varkappa T}{\hat{\varrho}+\hat{p}}\,\overset{\perp}{g}_{\alpha\varrho}(X^{\varrho\sigma}_{(\mathrm{irr})}+Q^{\varrho\sigma})_{/\sigma}\,.$$

Since, in units of ordinary sizes, $\bar{q}^\alpha = cq^\alpha$ and $u^\alpha = c^{-1}v^\alpha = c^{-1} \cdot$ $\cdot Dx^\alpha/D\tau$, the members of $(2.8)_2$ are $O(c^{-2})$ and $\varkappa$ is $O(c^{-1})$ (with respect to ordinary size magnitudes). Hence

$$(5.9) \qquad \frac{\varkappa T}{\varrho + p} \overset{\perp}{g}_{\alpha\varrho} X^{\varrho\sigma}_{(\mathrm{irr})/\sigma} \approx O(c^{-3}) \,, \qquad \frac{\varkappa T}{\varrho + p} \overset{\perp}{g}_{\alpha\varrho} Q^{\varrho\sigma}{}_{/\varrho} \approx O(c^{-5}) \,.$$

Then by (5.8) and (5.9) we have $(5.1)_1$. By $(2.5)_1$ and $(2.7)_2$ this yields $(5.2)_1$ and (5.3) when $X^{\alpha\beta}_{(\mathrm{irr})} \equiv 0$.

Lastly in any e-equilibrium process $\Theta_{/\alpha} \equiv 0$—cf. N. 3. Hence $(5.1)_2$ yields (a). q.e.d.

6. Comparison of $\mathcal{C}_{\mathrm{E}}$ with Alts and Müller's theory $\mathcal{C}_{\mathrm{AM}}$.

A) Comparison of $\mathcal{C}_{\mathrm{E}}$ and $\mathcal{C}_{\mathrm{AM}}$ in equilibrium processes.

In [1] a relativistic thermodynamic theory $\mathcal{C}_{\mathrm{AM}}$ is presented by Alts and Müller. In this theory a magnitude ϑ, called *empirical temperature* (*or heat potential*) is introduced. This temperature cannot be identified with the absolute one—as the deductions in [1] show—and in the general case it lacks any operative physical interpretation; furthermore *E-equilibrium* is defined in $\mathcal{C}_{AM}$ by means of the condition $\vartheta_{/\alpha} \equiv 0$.

Along E-equilibrium processes for (simple) non viscous fluids capable of heat conduction the validity of Gibbs's differential relation $(2.20)_3$ is proved (5), so that the corresponding well known two-parameter thermodynamics holds. In this case the deductions made in [1] to differentiate ϑ thought of as a function of k and T, lead to a result which, with the present notations, reads

$$(6.1) \qquad d\vartheta = \frac{\partial\vartheta}{\partial T}\left(dT - T\,\frac{\partial p/\partial k}{\varrho + p - T(\partial p/\partial T)}\,dk\right)$$

—cf. [1 (5.19)]—where p and ϱ are suitable functions that can express the pressure and density of gravitational mass (in energy units) in terms of k and T.

(5) The analogue for viscous fluids is not done in $\mathcal{C}_{\mathrm{AM}}$.

By comparing the relation (5.6), deduced in $\mathcal{C}_E$, with the result (6.1) of $\mathcal{C}_{AM}$, we see that the condition

$$T_{/\alpha} = T \frac{\partial p / \partial k}{\varrho + p - T(\partial p / \partial T)} k_{/\alpha} \tag{6.2}$$

which characterizes E-equilibria in $\mathcal{C}_{AM}$, also holds in e-equilibria (in $\mathcal{C}_E$); and in them it is equivalent to the only condition on thermodynamic fields present in the definition of e-equilibria.

B) Determination of all choice for the equilibrium empirical temperature $\breve{\vartheta}(k, T)$ in terms of Gibbs's function.

Remark that, while in [1] the usual integrability conditions of (6.1)—cf. [1 (7.8)]—are made explicit, here the analysis of e-equilibrium and in particular (5.6), where the definition $(3.12)_2$ is presupposed, allows us to solve the differential condition (6.1) in the unknown function $\breve{\vartheta}(k, T)$. It suffices to set

$$\breve{\vartheta}(k, T) = f(\Theta) \qquad (\Theta = T/(1 + c^{-2}\phi)) \tag{6.3}$$

or in particular $\vartheta = \Theta$. Hence *the empirical temperature ϑ can be identified with the pocket temperature Θ as far as equilibrium is concerned.*

Conservely (5.6) and (6.1) imply $\partial(\Theta, \vartheta)/\partial(k, T) = 0$, which yields (6.3) for some differential function f. Thus (6.3) *is the general solution of* (6.1). So the empirical temperature ϑ is determined up to a change of the metric on the possible values of an arbitrarily prefixed choice of empirical temperature.

C) Comparison of $\mathcal{C}_E$ and $\mathcal{C}_{AM}$ in the non equilibrium case.

In order to compare $\mathcal{C}_E$ with $\mathcal{C}_{AM}$ in non equilibrium cases remark that, on the one hand, a choice of constitutive equations in $\mathcal{C}_{AM}$, that express $w, p, \eta, \chi, \varphi$, and Q in terms of the magnitudes k and ϑ (among which $\vartheta_{/\alpha}$ does not appear) is compatible with the restrictions due to the entropy principle—cf. [1], N. 3—up to $O(c^{-4})$ [6].

[6] To realize this directly, consider the fluids in $\mathcal{C}_{AM}$ that are capable of heat conduction and are defined by a sixtuple of constitutive functions that express $w, p, \eta, \chi, \varphi$, and Q in terms of k and ϑ (but not of $\vartheta_{/\alpha}$ as in general cases);

Hence in this case the heat flux has a linear expression $\overset{\mathrm{L}}{q}_\alpha$ in $\vartheta_{/\alpha}$:

$$(6.4) \qquad \overset{\mathrm{L}}{q}_\alpha = -\chi \frac{\partial\vartheta}{\partial T}\left(T_{/\overset{\perp}{\alpha}} - T\frac{\partial p/\partial k}{\varrho + p - T(\partial p/\partial T)} k_{/\overset{\perp}{\alpha}}\right) = -\chi\vartheta_{/\overset{\perp}{\alpha}} ,$$

—cf. [1 (5.21)]—and here χ, ϑ, ϱ, and p are thought of as functions of k and T.

On the other hand in $\mathcal{T}_{\mathrm{E}}$ the heat flux for non-viscous fluids has an expression, $\overset{\mathrm{P}}{q}_\alpha$—cf. $(5.1)_2$ and $(5.3)_1$—, which differs from $\overset{\mathrm{E}}{q}_\alpha$ to $O(c^{-4})$.

Lastly Alts and Müller conclude in [1] that $\mathcal{T}_{\mathrm{AM}}$, which dealts only with non-viscous fluids, is in good agreement with Chernikov's relativistic kinetic theory—cf. [5] to [7]—, say $\mathcal{T}_{\mathrm{C}}$, in that a certain expression $\overset{\mathrm{C}}{q}_\alpha$ for the heat flux obtained in $\mathcal{T}_{\mathrm{C}}$ is suitably identifiable with the expression (6.4) for $\overset{\mathrm{L}}{q}_\alpha$ (hence with $\overset{\mathrm{P}}{q}_\alpha$ too). Therefore I can conclude that, since for same fluids $\overset{\mathrm{P}}{q}_\alpha$ [$\overset{\mathrm{L}}{q}_\alpha$] complies with $\mathcal{T}_{\mathrm{E}}$'s [$\mathcal{T}_{\mathrm{AM}}$'s] axioms up to $O(c^{-4})$, *$\mathcal{T}_{\mathrm{C}}$ agrees with $\mathcal{T}_{\mathrm{E}}$ at the same approximation order as with $\mathcal{T}_{\mathrm{AM}}$*.

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remark that for them the restriction relations $(4.12)_{2,3}$ and $(4.13)_2$ in [1] become

$$ARQ = ARQ/2 , \qquad AQ = -AR\chi c^{-2} , \qquad AQ/(2k) = 0 .$$

where $R = (\chi + Q\dot{\vartheta})(\varrho + p - \chi\dot{\vartheta}c^{-2})^{-1}$; and $R \approx O(c^{-2})$ because χ, the counterpart of $c\varkappa$ in $\mathcal{T}_{\mathrm{E}}$, is an ordinary size magnitude. (The magnitudes φ and Q have no counterparts in $\mathcal{T}_{\mathrm{E}}$). Hence, for χ and A not vanishing (in E-equilibria A^{-1} is the absolute temperature T), a suitable choice of the above constitutive functions is compatible with the axioms of $\mathcal{T}_{\mathrm{AM}}$ if $Q(k, \vartheta) = 0$ and $Rc^{-2}(\approx O(c^{-4}))$ is regarded as negligible.

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