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# Subnormal Composition Factors of Infinite Groups.

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The theory of composition factors of finite groups (or more generally of groups with a composition series) has its origin in the work of Jordan and Hölder and has been developed significantly by Wielandt. (We assume throughout that all composition series have *finite* length.) The main aim of the present work is to show that one of Wielandt's major theorems for groups with a composition series ((3) below) holds quite generally for arbitrary groups.

## 1. Notation and statement of results.

Let $X$ be a subnormal subgroup of an arbitrary group $G$. Thus there is a series of finite length from $X$ to $G$:

$$X = X_0 \lhd X_1 \lhd \ldots \lhd X_n = G .$$

If $Y \lhd X$ and $X/Y$ is a simple group, then we call $X/Y$ a *subnormal composition factor of* $G$. Denote by $\Sigma(G)$ the set (without multiplicities) of all abstract simple groups which occur as subnormal composition factors of $G$. (In passing, we mention that it appears to be unknown whether $G \neq 1$ always implies $\Sigma(G) \neq \emptyset$.) It is well-known that if $G$ has a composition series, then $G$ has only finitely many sub-

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normal subgroups ([4], p. 219). In this case the join $J$ of any (necessarily finite) collection of subnormal subgroups $H_\lambda$ $(\lambda \in \Lambda)$ of $G$ is also subnormal in $G$ ([4], Satz 7). It follows that $J$ has a composition series and it is an easy exercise (using the Jordan-Hölder theorem and induction on composition length) to show that

$$\Sigma(J) = \bigcup_\lambda \Sigma(H_\lambda)\,. \tag{1}$$

The generalisation of (1) to arbitrary groups is also well-known:

*If $G$ is generated by subnormal subgroups $H_\lambda$ $(\lambda \in \Lambda)$, then*

$$\Sigma(G) = \bigcup_\lambda \Sigma(H_\lambda)\,. \tag{2}$$

For a proof one can use a transfinite version of the proof of (1). Thus a class $\mathfrak{X}$ of groups closed with respect to forming normal products is also closed under taking subnormal joins ([7], Theorem 28, p. 246). Then let $\mathfrak{X}$ be the class of groups all of whose subnormal composition factors belong to $\bigcup_\lambda \Sigma(H_\lambda)$. Since two ascending series of subnormal subgroups have isomorphic refinements, (2) follows without difficulty.

If $H$ is a subgroup of a group $G$, we denote by $\Sigma(G{:}H)$ the set (again without multiplicities) of all abstract simple groups which are isomorphic to some $X/Y$ where $X$ is subnormal in $G$ and

$$H \leqslant Y \lhd X \leqslant G\,.$$

Then we shall obtain (in Theorem A) a much sharper result than (2), motivated by the following theorem of Wielandt's ([4], Sätze 6 and 9):

*If $G$ has a composition series and $H$, $K$ are subnormal subgroups of $G$, then*

$$\Sigma(H{:}H \cap K) = \Sigma(\langle H, K\rangle{:}K)\,. \tag{3}$$

Indeed we shall show that the hypothesis that $G$ has a composition series is redundant. The inclusion $\subseteq$ in (3) is quite straightforward and was generalized in another theorem of Wielandt's ([6], 8.5):

*Let $H_1, H_2, \ldots, H_n$ be finitely many subnormal subgroups of an* *arbitrary group* $G$. *Then for each* $i$

$$\Sigma(H_i:D) \subseteq \bigcup_{j\neq i} \Sigma(G:H_j)\,, \tag{4}$$

*where* $D = H_1 \cap \ldots \cap H_n$.

As an obvious corollary we have

$$\Sigma(G:D) = \bigcup_{i=1}^{n} \Sigma(G:H_i)\,. \tag{5}$$

Wielandt pointed out that (4) does not extend to infinitely many subnormal subgroups. For example, let $G = \langle g \rangle$ be an infinite cyclic group and let $H_n = \langle g^{2^n} \rangle$, for all $n \geqslant 1$. Then $D = \bigcap_{n\geqslant 1} H_n = 1$ and (4) clearly fails.

In order to obtain the relation $\supseteq$ in (3) Wielandt needed a more complicated argument. For ease of comparison with our methods in section 2, we recall the details briefly. Thus suppose that $G$ has a composition series and that $H$, $K$ are subnormal subgroups of $G$. Since $\langle H, K \rangle$ is subnormal in $G$, we may assume that $G = \langle H, K \rangle$. We proceed by induction on the composition length $l$ from $H$ to $G$ to show that

$$\Sigma(H:H \cap K) \supseteq \Sigma(G:K)\,. \tag{6}$$

We may assume that $l \geqslant 1$. Choose a maximal normal subgroup $G_1$ of $G$ containing $H$ and put $G_1 \cap K = K_1$.

*Case* (i). Suppose $K_1 = H \cap K$. Then $H \cap K \lhd K$. If $H \lhd HK$, (6) follows without difficulty. Therefore suppose that $H^K \neq H$. Then there is an element $k$ in $K$ such that

$$L = \langle H, H^k \rangle > H\,.$$

We note that $L \leqslant G_1$ and $L$ is subnormal in $G$. Now $G = \langle L, K \rangle$ and so, by induction on $l$,

$$\Sigma(G:K) \subseteq \Sigma(L:H \cap K) = \Sigma(L:H) \cup \Sigma(H:H \cap K)\,.$$

Again by induction, $\Sigma(L:H) \subseteq \Sigma(H^k:H \cap K) = \Sigma(H:H \cap K)$ and therefore we have (6).

*Case* (ii). Suppose that $K_1 > H \cap K$. Then $M = \langle H, K_1 \rangle > H$, $M \leqslant G_1$ and $M$ is subnormal in $G$. Now $G = \langle M, K \rangle$ and so induction gives

$$\Sigma(G:K) \subseteq \Sigma(M:M \cap K).$$

Also $\Sigma(M:M \cap K) \subseteq \Sigma(M:K_1) \subseteq \Sigma(H:H \cap K)$, by induction applied to $M = \langle H, K_1 \rangle$. Then again we have (6).

It is difficult to see how the above argument can be made to apply to arbitrary groups, first because composition lengths do not exist in general (and cannot be replaced in any obvious way by subnormal defects) and secondly because a join of subnormal subgroups is not always subnormal ([7], p. 235, exercise 23; [1], Theorems 6.1 and 6.2; [3], Theorem E). Nevertheless we shall prove

THEOREM A. *Let $G$ be generated by a (possibly infinite) set of subnormal subgroups $H_\lambda$ ($\lambda \in \Lambda$). Then, for each $\lambda$,*

$$\Sigma(G:H_\lambda) \subseteq \bigcup_{\mu \neq \lambda} \Sigma(H_\mu:H_\lambda \cap H_\mu).$$

In conjunction with (4) this gives

THEOREM B. *If $G$ is generated by finitely many subnormal subgroups $H_\lambda$ ($\lambda \in \Lambda$), then*

$$\bigcup_\lambda \Sigma(G:H_\lambda) = \bigcup_\lambda \Sigma(H_\lambda:D),$$

*where* $D = \bigcap_\lambda H_\lambda$.

The special case of Theorem B when $|\Lambda| = 2$ shows that (3) holds in arbitrary groups, i.e.

*if $H$, $K$ are subnormal subgroups of any group, then*

$$\Sigma(H:H \cap K) = \Sigma(\langle H, K \rangle:K).$$

We also obtain (2) as a corollary of Theorem A. For, if $G$ is generated by subnormal subgroups $H_\lambda$ ($\lambda \in \Lambda$), then for any $\mu \in \Lambda$,

$$\Sigma(G) = \Sigma(H_\mu) \cup \Sigma(G:H_\mu) \subseteq \bigcup_\lambda \Sigma(H_\lambda).$$

The reverse inclusion is trivial.

## 2. Proof of Theorem A.

We shall need some well-known results. First (see [1], Corollary 2.4)

LEMMA 1. *Let $H$, $K$ be subnormal subgroups of $G$. If $\langle H, K\rangle = HK$ (i.e. $HK = KH$), then $HK$ is subnormal in $G$.*

A special case of Theorem A of [3] says that if $H$, $K$ are subnormal subgroups of a group $G$, then every finitely generated subgroup $F$ of $J = \langle H, K\rangle$ is contained in some subnormal subgroup $L$ of $G$ with $F \leqslant L \leqslant J$. A simple induction argument then yields

THEOREM 1. *Let $H_1, H_2, \ldots, H_n$ be finitely many subnormal subgroups of a group $G$. Then every finitely generated subgroup $F$ of $J = \langle H_1, H_2, \ldots, H_n\rangle$ lies in some subnormal subgroup $L$ of $G$ with $F \leqslant L \leqslant J$.*

If $H$ and $K$ are arbitrary subgroups of any group, then, according to [3], the *permutizer* $P_K(H)$ of $H$ in $K$ is defined to be the unique largest subgroup of $K$ whose product with $H$ is a subgroup. The following result is proved in [3], Lemma 3:

LEMMA 2. *If $H$ and $K$ are subnormal subgroups of $G$, then $P_K(H)$ is also subnormal in $G$.*

Also concerning the permutizer we shall need

LEMMA 3 ([2], Corollary B1). *If $H$ and $K$ are subnormal subgroups of a group, then there is an integer $d$ ($\geqslant 0$) such that $P_K(H)$ contains the $d$-th term $K^{(d)}$ of the derived series of $K$.*

Finally, from the same paper of Roseblade's we require part of the main Theorem, viz.

THEOREM 2. *If $G$ is a group generated by subnormal subgroups $H_1, H_2, \ldots, H_n$ and if $d_1, d_2, \ldots, d_n$ are integers ($\geqslant 0$), then there is an integer $d \geqslant 0$ such that*

$$G^{(d)} \leqslant H_1^{(d_1)} H_2^{(d_2)} \ldots H_n^{(d_n)} .$$

PROOF OF THEOREM A. We have a group $G$ generated by subnormal subgroups $H_\lambda$ ($\lambda \in \Lambda$). Let $X/Y$ be a subnormal composition factor of $G$ with

$$H_\lambda \leqslant Y \lhd X \leqslant G ,$$

for some $\lambda$. We have to prove that $X/Y \in \Sigma(H_\mu : H_\lambda \cap H_\mu)$, for some $\mu \neq \lambda$. First we show that

(7) *we may assume that $\Lambda$ is a finite set.*

Thus choose an element $x$ from $X \setminus Y$. Then there are elements $\lambda_1, \lambda_2, \ldots, \lambda_n$ in $\Lambda$ such that

$$x \in \langle H_{\lambda_1}, H_{\lambda_2}, \ldots, H_{\lambda_n} \rangle = J ,$$

say, and we may suppose that $H_{\lambda_1} = H_\lambda$. To simplify notation put $H_i = H_{\lambda_i}$, $i = 1, 2, \ldots, n$. Now, by Theorem 1, there is a subnormal subgroup $L$ of $G$ such that

$$x \in L \leqslant J .$$

Then $X \cap L$ is subnormal in $G$ and so Lemma 1 implies that $(X \cap L)Y$ is also subnormal in $G$. Since $x \in X \cap L$ and $x \notin Y$, we must have $(X \cap L)Y = X$. Hence

$$(X \cap J)\, Y = X$$

and therefore $X/Y \cong (X \cap J)/(Y \cap J)$, a subnormal composition factor of $J$ above $H_1$ $(= H_\lambda)$. Replacing $G$ by $J$, (7) follows.

Thus we have $G = \langle H_1, H_2, \ldots, H_n \rangle$ with each $H_i$ subnormal in $G$ and $X/Y$ is a subnormal composition factor of $G$ with $H_1 \leqslant Y \lhd X \leqslant G$. It now suffices to prove the theorem in the case when

(8) *$G$ is soluble.*

For, let $P_2 = P_{H_2}(H_1)$, the permutizer of $H_1$ in $H_2$. Then $H_1 \cap H_2 \leqslant P_2$. Put $Q_2 = H_1 P_2$. By Lemma 2, $P_2$ is subnormal in $G$ and therefore $Q_2$ is also subnormal in $G$, by Lemma 1. Thus there are subnormal series

$$H_1 \lhd \ldots \lhd Q_2 \lhd \ldots \lhd G ,$$

$$H_1 \lhd \ldots \lhd Y \lhd X \lhd \ldots \lhd G .$$

According to Schreier's Theorem, these series have isomorphic refinements and so we may assume that either

(9) $$H_1 \leqslant Y \lhd X \leqslant Q_2$$

or

(10) $$Q_2 \leqslant Y \lhd X \leqslant G\,.$$

It is very easy to see that $\Sigma(Q_2 : H_1) = \Sigma(P_2 : H_1 \cap P_2)$ and so in case (9)

$$X/Y \in \Sigma(P_2 : H_1 \cap P_2) \subseteq \Sigma(H_2 : H_1 \cap H_2)\,,$$

as required. Therefore suppose that (10) holds. By Lemma 3 there is an integer $d_2 \geqslant 0$ such that $H_2^{(d_2)} \leqslant P_2$. There is no loss of generality in taking $H_1 = Q_2$ and so we may assume that $H_2^{(d_2)} \leqslant H_1$. Repeating this process with $H_3, \ldots, H_n$ in turn replacing $H_2$, we obtain

$$H_i^{(d_i)} \leqslant H_1\,,$$

$2 \leqslant i \leqslant n$, for certain integers $d_i$. Now by Theorem 2 there is an integer $d$ such that

$$G^{(d)} \leqslant H_1 H_2^{(d_2)} \ldots H_n^{(d_n)} \leqslant H_1\,.$$

Let $N = G^{(d)}$. Since $H_1 \cap H_i N = (H_1 \cap H_i)N$ and

$$\Sigma(H_i N : (H_1 \cap H_i)N) \subseteq \Sigma(H_i : H_1 \cap H_i)$$

for all $i$, it follows that we may replace $G$ by $G/N$, i.e. we may assume that $G$ is soluble. We have therefore established (8).

It is now clear that $X/Y \cong Z_p$, a cyclic group of prime order $p$. The next step is to show that, in this situation,

(11) *we may assume that $G$ is finitely generated.*

For choose $x \in X \setminus Y$. Then $x \in \langle K_1, K_2, \ldots, K_n \rangle = M$, say, where $K_i$ is a finitely generated subgroup of $H_i$, for each $i$. So $M$ is finitely generated. Let

$$H_i^* = H_i \cap M\,.$$

Thus $H_i^* \geqslant K_i$, $M = \langle H_1^*, \ldots, H_n^* \rangle$ and $H_i^*$ is subnormal in $M$. Put $X_1 = X \cap M$, $Y_1 = Y \cap M$. Since $x \in X_1 \setminus Y_1$, we have

$$X_1/Y_1 \cong X_1 Y/Y = X/Y$$

and hence $Z_p \in \Sigma(M:H_1^*)$. If the Theorem holds for finitely generated groups, then

$$(12) \qquad Z_p \in \Sigma(H_j^*:H_1^* \cap H_j^*)$$

for some $j \geqslant 2$. Let

$$H_1 \cap H_j = A_0 \lhd A_1 \lhd \ldots \lhd A_m = H_j$$

be a series with all $A_{i+1}/A_i$ abelian. Intersecting with $M$, we obtain

$$H_1^* \cap H_j^* = A_0 \cap M \lhd A_1 \cap M \lhd \ldots \lhd A_m \cap M = H_j^*$$

and

$$(A_{i+1} \cap M)/(A_i \cap M) \cong (A_{i+1} \cap M)A_i/A_i \leqslant A_{i+1}/A_i\,.$$

By (12), $Z_p \in \Sigma\big((A_{i+1} \cap M)/(A_i \cap M)\big)$, for some $i$, and so

$$Z_p \in \Sigma(A_{i+1}/A_i) \subseteq \Sigma(H_j:H_1 \cap H_j)\,,$$

showing that the theorem will hold for all groups. Therefore (11) follows.

Now $G$ is a finitely generated soluble group. Let

$$H_1 = B_0 \lhd B_1 \lhd \ldots \lhd B_l = G$$

be a series with each factor $B_{i+1}/B_i$ abelian. Choose $i$ as large as possible such that $B_{i+1}/B_i$ has $Z_p$ as a subnormal composition factor. Then for all $j \geqslant i+1$, $B_{j+1}/B_j$ must be periodic. Since $G$ is finitely generated, induction over $j$ decreasing shows that $|G:B_{i+1}|$ is finite and $B_{i+1}$ is finitely generated. Therefore we may assume that $X = B_{i+1}$ and $Y = H_1$. Hence $|G:H_1|$ is finite. We have already seen (while proving (8)) that it is in order to replace $G$ by $G$ factored by any normal subgroup contained in $H_1$. Thus we may assume that $H_1$ is core-free in $G$ and so $G$ is finite.

Finally, for $1 \leqslant i \leqslant n$, let $G_i = \langle H_1, H_2, \ldots, H_i \rangle$. Since all the subgroups $G_i$ are subnormal in $G$, Schreier's refinement theorem gives

$$X/Y \in \Sigma(G_{i+1}:G_i)$$

for some $i \geqslant 1$. Therefore by Wielandt's theorem ((3) above) applied to the finite group $G_{i+1}$ generated by the subnormal subgroups $H_{i+1}$ and $G_i$,

$$X/Y \in \Sigma(H_{i+1}:G_i \cap H_{i+1}) \subseteq \Sigma(H_{i+1}:H_1 \cap H_{i+1})$$

as required.

This completes the proof of Theorem A.

## 3. The connection with residuals.

Let $G$ be a finite group and let $\mathcal{S}$ be a set of abstract simple groups. By (5) there is a unique subnormal subgroup $G^*$ of $G$ minimal subject to

$$\Sigma(G:G^*) \subseteq \mathcal{S} ;$$

and clearly $G^* \lhd G$. Now suppose that $G$ (still finite) is generated by subnormal subgroups $H_1, H_2, \ldots, H_n$. Then

$$H_i/(H_i \cap G^*) \cong H_i G^*/G^*$$

and hence $\Sigma(H_i:H_i \cap G^*) \subseteq \mathcal{S}$. Therefore $H_i^* \leqslant H_i \cap G^*$ and so

$$\langle H_1^*, H_2^*, \ldots, H_n^* \rangle \leqslant G^* .$$

Conversely let $\langle H_1^*, \ldots, H_n^* \rangle \leqslant Y \lhd X$, where $X$ is subnormal in $G$ and $X/Y$ is simple. Then $G = \langle H_1, \ldots, H_n, Y \rangle$ and thus, by Theorem A,

$$X/Y \in \bigcup_{i=1}^{n} \Sigma(H_i:H_i \cap Y) , \tag{13}$$

$$\subseteq \bigcup_{i=1}^{n} \Sigma(H_i:H_i^*) \subseteq \mathcal{S} .$$

Hence $G^* \leqslant \langle H_1^*, \ldots, H_n^* \rangle$. Consequently

$$G^* = \langle H_1^*, \ldots, H_n^* \rangle \, . \tag{14}$$

This result is due to Wielandt ([5], Satz 2.4). Clearly by far the greater part of the argument lies in establishing (13), i.e. Theorem A. However, Theorem A does not require $G$ or the number of generating subnormal subgroups to be finite, and so we can obtain, as our final result, what is essentially a generalisation of (14) to arbitrary groups:

THEOREM C. *Let $G$ be generated by subnormal subgroups $H_\lambda$ ($\lambda \in \Lambda$). If $X/Y$ is a subnormal composition factor of $G$, then*

$$X/Y \in \bigcup_\lambda \Sigma(H_\lambda : H_\lambda \cap Y) \, .$$

PROOF. Since $G = \langle Y, H_\lambda | \lambda \in \Lambda \rangle$, we may take $H_\lambda$ in Theorem A to be $Y$ here.

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